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Preface

This material is prepared based on the lecture for Topics in Electricity and Magnetism I and II (PHYS 4310 & 4330) at Middle Tennessee State University.
Introduction
Part I

Electricity and Magnetism I
Chapter 1

Vector Analysis

This is the first chapter of Griffiths. This chapter covers vector algebra, differential calculus, integral calculus, spherical coordinates, and cylindrical coordinates. All these topics have been covered in either of the two consecutive courses in Mathematical methods for theoretical physics (PHYS 3150 or PHYS 3160). Since I am not going to cover this part, I strongly advise you to revise these materials. If you are interested my lecture notes for PHYS 3150 and 3160 which is available on my website [http://capone.mtsu.edu/derenso/ Theoretical_Physics_1_Mathematical_Methods.pdf]. If you have any questions, please do not hesitate to ask.
Chapter 2

Electrostatics

The theory of electricity and magnetism generally based on charges that are stationary or moving relative to some reference frame. We begin with electrostatic that teaches us the theories describing the properties of stationary charges (static charges).

2.1 Coulomb’s Law

Consider two stationary point charges, \( q \) and \( Q \), are positioned in a free space (vacuum) described by the position vectors, \( \mathbf{r}' \) and \( \mathbf{r} \), respectively [see Fig.2.1]. Coulomb’s law states that there is an electrostatic force that could be an attractive or repulsive between these two charges. The magnitude of the force on any of these charges is directly proportional to the magnitude of the charges and inversely proportional to the square of the separation distance. Vectorial this force can be expressed as

\[
\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{|\mathbf{r}-\mathbf{r}'|^2} \mathbf{R},
\]

where

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N\cdot m^2}
\]

is the electrical permittivity of a free space and \( \mathbf{R} \) is the unit vector along the direction of the vector, \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \). As you already know from Theoretical Physics I, this unit vector is given by

\[
\mathbf{R} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.
\]

Thus the Coulomb’s force in Eq. (2.1) can be expressed as

\[
\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{|\mathbf{r}-\mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}').
\]
CHAPTER 2. ELECTROSTATICS

In general the force line of action is along the line joining the two charges. This force would be an attractive if the two charges are opposite otherwise it would be repulsive.

The principle of Superposition: the interaction of two charges does not affected by the presence of other charges. For example, suppose if there are \( n \) charges \((q_1, q_2, q_3 \ldots q_n)\) in the vicinity of the test charge \(Q\); the force, \(\vec{F}_1\), (both the magnitude and direction) exerted by charge \(q_1\) on the test charge \(Q\) remain unchanged whether the charges \(q_2, q_3 \ldots q_n\) do or do not exist. We can then apply the principle of superposition to find the net force, \(\vec{F}_{net}\), on the test charge \(Q\) due to the charges \((q_1, q_2, q_3 \ldots q_n)\),

\[
\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots \vec{F}_n = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\vec{r} - \vec{r}_i|} \left(\vec{r} - \vec{r}_i\right)
\]

(2.5)

**Example 1.1A** Consider two identical point particles carrying a same positive charge, \(+q\). The point particles are separated by a distance, \(d\). Find

(a) Find the electric force (Magnitude and direction) on a positive charge \(Q\) placed a distance \(a\) above the midpoint of the line connecting the point particles.

(b) Repeat part (a), only this time make the right-side charge \(-q\) instead of \(+q\)

Solution:

(a) Consider the origin of the Cartesian coordinate system be at midpoint between the two charges with the two source charges on the y-axis and the test charge on the z-axis shown in the Fig. 2.2.
2.1. COULOMB’S LAW

Figure 2.2: The interaction of two positive point charges

The position vectors for the two source charges ($\vec{r}_1$ for the charge on the left side, $\vec{r}_2$ for the charge on the right side) and the test charge ($\vec{r}$), using the Cartesian unit vectors ($\hat{x}, \hat{y}, \hat{z}$), can be expressed as

$$\vec{r}_1 = -\frac{d}{2} \hat{y}, \quad \vec{r}_2 = \frac{d}{2} \hat{y}, \quad \vec{r} = a \hat{z}. \quad (2.6)$$

so that

$$\vec{r} - \vec{r}_1 = a \hat{z} - \frac{d}{2} \hat{y} = \frac{d}{2} \hat{y} + a \hat{z} \Rightarrow |\vec{r} - \vec{r}_1| = \sqrt{a^2 + \frac{d^2}{4}} \quad (2.7a)$$

$$\vec{r} - \vec{r}_2 = a \hat{z} - \frac{d}{2} \hat{y} = \frac{d}{2} \hat{y} + a \hat{z} \Rightarrow |\vec{r} - \vec{r}_2| = \sqrt{a^2 + \frac{d^2}{4}}. \quad (2.7b)$$

The net force on the test charge is the vector sum of the two forces due to the two source charges (Principle of Superposition). Using equation (2.4), we may write the net force as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 = \frac{1}{4\pi \varepsilon_0} \frac{qQ}{|\vec{r} - \vec{r}_1|^3} (\vec{r} - \vec{r}_1) + \frac{1}{4\pi \varepsilon_0} \frac{qQ}{|\vec{r} - \vec{r}_2|^3} (\vec{r} - \vec{r}_2) \quad (2.8)$$

From Eqs. (2.7a) and (2.7b), we note that $\vec{r} - \vec{r}_1 \neq \vec{r} - \vec{r}_2$ but $|\vec{r} - \vec{r}_1| = |\vec{r} - \vec{r}_2|$. This means $\vec{F}_1$ and $\vec{F}_2$ have different directions but same magnitudes. Therefore, the net force will be

$$\vec{F}_{\text{net}}^\prime = \vec{F}_1 + \vec{F}_2$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{qQ}{(a^2 + \frac{d^2}{4})^{\frac{3}{2}}} \left(\frac{d}{2} \hat{y} + a \hat{z}\right) + \frac{1}{4\pi \varepsilon_0} \frac{qQ}{(a^2 + \frac{d^2}{4})^{\frac{3}{2}}} \left(-\frac{d}{2} \hat{y} + a \hat{z}\right) \quad (2.9)$$
and this can easily simplified into

\[ \vec{F}_{net} = \frac{qQ}{2\pi\varepsilon_0} \frac{a}{\left(a^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} \hat{z}, \]  

(2.10)

which is the net force acting on the charge \( Q \).

(b) Replacing the right-hand charge \( q \) by \(-q\) produces an attractive force on the test charge instead of a repulsive force as shown in Fig. 2.3. This changes the second term in Eqs. (2.8) and (2.9) negative,

\[
\vec{F}_{net} = \frac{qQ}{4\pi\varepsilon_0} \left( \frac{d}{2} \hat{y} + a\hat{z} \right) - \frac{qQ}{4\pi\varepsilon_0} \left( -\frac{d}{2} \hat{y} + a\hat{z} \right)
\]  

(2.11)

\[
\Rightarrow \vec{F}_{net} = \frac{qQ}{4\pi\varepsilon_0} \frac{d}{\left(a^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} \hat{y}
\]  

(2.12)

**Example 1.1B** Re-do Example 1.4, chapter one, in *Theoretical Physics I-Mathematical Methods in Theoretical Physics* (PHYS 3150)

## 2.2 The electric field

Consider an isolated particle that can be treated as a point particle. Suppose this particle is positioned at a point described by the position vector \( \vec{r} \) and
2.2. THE ELECTRIC FIELD

carries a charge $q$. The electric field due to this charge at a position $\vec{r}'$ is defined as the force per unit charge that would be exerted on a positive test charge $Q$ if this test charge is placed at the position $\vec{r}$,

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r} - \vec{r}'|^2} \hat{R}. \quad (2.13)$$

The direction of the electric field depends on whether the charge is positive or negative. It is directed outward for a positive (Fig. 2.4) and inward for a negative (Fig. 2.5) charge.

Figure 2.4: The electric field direction of a positive point charge.

*Discreet point charges:* One can also find the net electric field at position $\vec{r}'$ due to several point charges, $q_1, q_2, q_3, \ldots q_N$ positioned at different points described by the position vectors $\vec{r}'_1, \vec{r}'_2, \vec{r}'_3, \ldots \vec{r}'_N$, respectively [Fig. 2.6]. This field is just the net force per unit charge on a positive test charge positioned at the point $\vec{r}$. Applying the Principle of superposition, this electric field can be expressed as

$$\vec{E} = \frac{\vec{F}_{\text{net}}}{Q} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}'_i|^2} \hat{R}_i, \quad (2.14)$$

where we used Eq. (2.5) for the net force.

*Continuous charge:* when the charge is distributed continuously over some region, we may divide the charge into an infinitesimal charges, $\Delta q'_i$ so that the net electric field can be approximated using

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i} \frac{\Delta q'_i}{|\vec{r} - \vec{r}'_i|^2} \hat{R}_i = \frac{1}{4\pi\varepsilon_0} \sum_{i} \frac{\Delta q'_i}{|\vec{r} - \vec{r}'_i|^3} \left(\vec{r} - \vec{r}'_i\right), \quad (2.15)$$
where the summation is taken over all charges in the region. The exact electric field can be determined by making the infinitesimal charge smaller and smaller. In other words we can write

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i'}{|\vec{r} - \vec{r}_i'|^3} \left(\vec{r} - \vec{r}_i'\right).
\] (2.16)

From integral calculus in the limit the charge becomes vanishingly small ($\Delta q_i' \to 0$) the summation is replaced by integration

\[
\lim_{\Delta q_i' \to 0} \sum_i \Delta q_i' \to \int dq'.
\]

The limits of integration is determined by the geometry and size of the charge distribution. Therefore, Eq. (2.16) can be expressed as

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\left(\vec{r} - \vec{r}'\right) dq'}{|\vec{r} - \vec{r}'|^3}.
\] (2.17)

For a line charge distribution with linear charge density (charge-per-unit length), $\lambda(\vec{r}')$, we have

\[
dq = \lambda(\vec{r}')dl'
\]

so that the electric field in Eq. (2.17) becomes

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{line} \frac{\left(\vec{r} - \vec{r}'\right) \lambda(\vec{r}') dl'}{|\vec{r} - \vec{r}'|^3}.
\] (2.18)
Similarly for surface charge, with surface charge density \( \sigma(\vec{r}') \) and volume charge with a volume charge density \( \rho(\vec{r}') \), using

\[
 dq' = \begin{cases} 
 \sigma(\vec{r}')d\alpha', & \text{surface} \\
 \rho(\vec{r}')d\tau', & \text{Volume} 
\end{cases}
\]

we can express the electric field as

\[
 \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{surface}} \frac{(\vec{r} - \vec{r}') \sigma(\vec{r}')d\alpha'}{|\vec{r} - \vec{r}'|^3} \quad (2.19)
\]

and

\[
 \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{(\vec{r} - \vec{r}') \rho(\vec{r}')d\tau'}{|\vec{r} - \vec{r}'|^3}, \quad (2.20)
\]

respectively. Note that \( d\alpha' \) (\( d\tau' \)) is the infinitesimal area (volume) occupied by the charge \( dq \).

**Example 1.2**

(a) Find the electric field (Magnitude and direction) at a distance \( a \) above the midpoint between two equal charges, \( q \), a distance \( d \) apart. Check that your result is consistent with what you would expect when \( a \gg d \).

**Solution:** In example 2.1 part (a) we have found the force on a test charge \( Q \) a distance \( a \) above the two charges midpoint to be

\[
 \vec{F}_{net} = \frac{qQ}{2\pi\epsilon_0} \frac{a}{(a^2 + \frac{d^2}{4})^{\frac{3}{2}}} \hat{z}. \quad (2.21)
\]
Recalling that the electric field at a given point is just the force per unit charge that would be exerted on a test charge, if we were to place one at that point, we can simply write the electric field as

\[
\vec{E} = \frac{\vec{F}_{\text{net}}}{Q} = \frac{q}{2\pi\varepsilon_0} \left( \frac{a}{a^2 + \frac{d^2}{4}} \right)^{\frac{3}{2}} \hat{z} \Rightarrow \vec{E} = \frac{q}{2\pi\varepsilon_0 a^2} \left( \frac{1}{1 + \frac{d^2}{4a^2}} \right)^{\frac{3}{2}} \hat{z}.
\] (2.22)

For \( a \gg d \), since \( 1 + \frac{d^2}{4a^2} \approx 1 \), we find

\[
\vec{E} \approx \frac{q}{2\pi\varepsilon_0 a^2} \hat{z} = \frac{q'}{4\pi\varepsilon_0 a^2} \hat{z}, \text{ where } q' = 2q.
\] (2.23)

Which means as we go far away from the two charges the electric field is just like the electric field of a point charge \( q' = 2q \) placed at the origin.

(b) Repeat part (a), only this time make the right-hand charge \(-q\) instead of \(+q\).

**Solution:** Similarly, using the result in Example 2.1 part (b) Eq. (2.12) we can express the electric field in this case as

\[
\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{d}{(a^2 + \frac{d^2}{4})^{\frac{3}{2}}} \hat{y} = \frac{qd}{4\pi\varepsilon_0 a^3} \frac{1}{(1 + \frac{d^2}{4a^2})^{\frac{3}{2}}} \hat{y}
\] (2.24)

For \( a \gg d \), we still have \( 1 + \frac{d^2}{4a^2} \approx 1 \) and this gives

\[
\vec{E} \approx \frac{qd}{4\pi\varepsilon_0 a^3} \hat{y} = \frac{\vec{p}}{4\pi\varepsilon_0 a^3}
\] (2.25)
2.2. THE ELECTRIC FIELD

2.2.1 Electric Field of a Dipole

Figure 2.8: Electric field line of two opposit point charges located in a close proximity.

where \( \vec{p} = qd\vec{y} \), as we will study later, is know as the dipole moment. Eq. (2.25) represents an electric field of a dipole at a distance \( a \) from the dipole. Unlike the electric field of a point charge, for a dipole the electric field far away from the dipole, is inversely proportional to the cube of the distance.

Example 1.3 Find the electric field a distance \( a \) above the midpoint of a straight line segment of length \( 2L \) which carries a uniform line charge \( \lambda \).

Solution: Let’s assume the charged line segment lies along the y-axis with its midpoint at the origin as shown in Fig. 2.9. We want to find the electric field a distance \( a \) from the midpoint of this charged line segment. The infinitesimal charge \( dq = \lambda dl' \) corresponding to the infinitesimal length \( dl' \) produces an infinitesimal electric field \( d\vec{E} \) as shown in the Fig. 2.9. The total electric field is the integral of these infinitesimal electric fields which is given by Eq. (2.18)

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{line} \frac{(\vec{r} - \vec{r'}) \lambda(\vec{r'})dl'}{|\vec{r} - \vec{r'}|^3}.
\] (2.26)

From the figure above, we note that

\[
\vec{r'} = l'\hat{x}, \vec{r} = a\hat{z} \implies \vec{r} - \vec{r'} = -l'\hat{x} + a\hat{z} \implies |\vec{r} - \vec{r'}| = \sqrt{l'^2 + a^2}
\] (2.27)

and the limit of integration is \( l' = -L \) to \( l' = L \) we write Eq. (2.26) as

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{(-l'\hat{x} + a\hat{z}) \lambda dl'}{(l'^2 + a^2)^{3/2}}.
\] (2.28)
Since the charge distribution is uniform we can take $\lambda$ out of the integral and write the above expression as

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0} \left[ \int_{-L}^{L} \frac{l' \, dl'}{(l'^2 + a^2)^{3/2}} \hat{\hat{x}} + \int_{-L}^{L} \frac{a \, dl'}{(l'^2 + a^2)^{3/2}} \hat{\hat{z}} \right]. \quad (2.29)$$

Since the function

$$f_1(l') = \frac{l'}{(l'^2 + a^2)^{3/2}}$$

is an odd function (i.e. $f_1(l') \neq f_1(-l') = -f_1(l')$),

$$f_2(l') = \frac{a}{(l'^2 + a^2)^{3/2}}$$

is an even function (i.e. $f_2(l') = f_2(-l')$),

we have

$$\int_{-L}^{L} f(l') \, dl' = \left\{ \begin{array}{ll} 0, & \text{for odd function} \\ \int_{0}^{L} 2f(l') \, dl', & \text{for even function} \end{array} \right. \quad (2.32)$$

so that Eq. (2.29) reduces to

$$\vec{E} = \frac{2\lambda a}{4\pi\epsilon_0} \int_{0}^{L} \frac{dl'}{(l'^2 + a^2)^{3/2}} \hat{\hat{z}}. \quad (2.33)$$

To evaluate the integral

$$I = \int \frac{dl'}{(l'^2 + a^2)^{3/2}} \quad (2.34)$$
we introduce the transformation of variables defined by
\[ l' = a \tan \theta \Rightarrow \sin \theta = \frac{l'}{\sqrt{l'^2 + a^2}}, dl' = a \sec^2 \theta d\theta. \] (2.35)

Substituting Eq. (2.35) into Eq. (2.34), we find
\[ I = \int \frac{a \sec^2 \theta d\theta}{(a^2 \tan^2 \theta + a^2)^{3/2}} = \frac{1}{a^2} \int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^{3/2}}. \] (2.36)

Noting that
\[ \tan^2 (\theta) + 1 = \sec^2 \theta \] (2.37)
we may write
\[ I = \frac{1}{a^2} \int \frac{d\theta}{\sec \theta} = \frac{\cos \theta d\theta}{a^2} \Rightarrow I = \frac{1}{a^2} \sin \theta = \frac{l'}{a^2 \sqrt{l'^2 + a^2}}. \] (2.38)

Using this result the expression for the electric field in Eq. (2.33) becomes
\[ \vec{E} = \frac{2\lambda}{4\pi \varepsilon_0} \frac{l'}{a^2 \sqrt{a^2 + L^2}} \frac{L}{\hat{z}} \Rightarrow \vec{E} = \frac{\lambda}{2\pi \varepsilon_0} \frac{L}{a^2 \sqrt{a^2 + L^2}} \hat{z}. \] (2.39)

Note: If you understood the result we obtained in Example 2.2 you should know that to find the electric field we may just simply start from
\[ \vec{E} = \frac{1}{4\pi \varepsilon_0} \int_0^L \frac{\lambda dl'}{(l'^2 + a^2)^{3/2}} \hat{z}. \]

and multiply the result by two (see Fig. 2.2). That is what Griffiths did. Do you know why? Do you think this will work if the line charge distribution is not uniform? Why?

Example 1.4 Find the electric field a distance \( a \) above the center of a flat circular disk of radius \( R \) as shown in Fig. 2.10 which carries a uniform surface charge \( \sigma \). What does your formula give in the limit \( R \to \infty \)? Also check the case \( a \gg R \).

Solution: For a surface charge distribution the electric field is given by
\[ \vec{E} = \frac{1}{4\pi \varepsilon_0} \int_{\text{surface}} \frac{(\vec{r} - \vec{r}')}{(\vec{r} - \vec{r}')^3} \sigma(\vec{r}') da'. \] (2.40)

Consider an infinitesimal area \( da' \) at a distance \( r' \) from the center of the disk and located at angle \( \varphi' \) measured from the positive x axis. We can express the position vector for this charge, using Cartesian unit vectors as,
\[ \vec{r}' = r' \cos \varphi' \hat{x} + r' \sin \varphi' \hat{y}. \] (2.41)
The infinitesimal area $d\alpha'$ in terms of the variable $r'$ and $\varphi'$ can also be expressed as

$$d\alpha' = r'dr'd\varphi. \quad (2.42)$$

The position vector where we need to evaluate the electric field is given by

$$\vec{r} = a\hat{z}. \quad (2.43)$$

Using Eqs. (2.41) and (2.43), we may write

$$\vec{r} - \vec{r}' = -r' \cos \varphi' \hat{x} - r' \sin \varphi' \hat{y} + a \hat{z} \Rightarrow |\vec{r} - \vec{r}'| = \sqrt{r'^2 + a^2} \quad (2.44)$$

so that substituting Eqs. (2.42) and (2.44) into Eq. (2.40) and noting that the surface charge distribution is uniform over the entire surface, the electric field can be expressed as

$$\vec{E} = \frac{\sigma}{4\pi \epsilon_0} \int_0^R \int_0^{2\pi} \frac{r'}{(r'^2 + a^2)^{3/2}} \left[ -r' \cos \varphi' \hat{x} - r' \sin \varphi' \hat{y} + a \hat{z} \right] dr'd\varphi \quad (2.45)$$

$$\vec{E} = \frac{\sigma}{4\pi \epsilon_0} \left[ \int_0^R \frac{r'dr'}{(r'^2 + a^2)^{3/2}} \int_0^{2\pi} (-r' \cos \varphi' \hat{x} - r' \sin \varphi' \hat{y}) d\varphi \right. \right.$$ 

$$+ \left. \int_0^R \frac{r'dr'}{(r'^2 + a^2)^{3/2}} \int_0^{2\pi} a d\varphi \hat{z} \right]. \quad (2.46)$$

Noting that

$$\int_0^{2\pi} \cos \varphi' d\varphi = \int_0^{2\pi} \sin \varphi' d\varphi = 0, \int_0^{2\pi} d\varphi = 2\pi, \quad (2.47)$$
we find

\[ \vec{E} = \frac{a\sigma}{2\epsilon_0} \int_0^R \frac{r\,dr'}{(r'^2 + a^2)^{3/2}} \hat{z} = \frac{a\sigma}{2\epsilon_0} \left( \frac{1}{a} - \frac{1}{\sqrt{R^2 + a^2}} \right) \hat{z} \]

\[ \Rightarrow \vec{E} = \frac{a\sigma}{2\epsilon_0} \left( \frac{1}{a} - \frac{1}{\sqrt{R^2 + a^2}} \right) \hat{z}. \quad (2.48) \]

When the charged disk becomes bigger (\( R \to \infty \)), from Eq. (2.48), the electric field becomes

\[ \vec{E} \simeq \frac{\sigma}{2\epsilon_0} \hat{z}, \quad (2.49) \]

which is a constant. This is the electric field of an infinite plate of uniform charge distribution \( \sigma \). On the other hand when \( a \gg R \), Eq. (2.48) may be approximated as

\[ \vec{E} = \frac{a\sigma}{2\epsilon_0} \left( \frac{1}{a} - \frac{1}{\sqrt{R^2 + a^2}} \right) \hat{z} = \frac{a\sigma}{2\epsilon_0} \left( \frac{1}{a} - \frac{1}{a\sqrt{1 + \left( \frac{R}{a} \right)^2}} \right) \hat{z} \]

\[ \simeq \frac{a\sigma}{2\epsilon_0} \left( \frac{1}{a} - \frac{1}{a} \left( 1 - \frac{1}{2} \left( \frac{R}{a} \right)^2 \right) \right) \hat{z} \Rightarrow \vec{E} \simeq \frac{\sigma R^2}{4\epsilon_0 a^2} \hat{z}. \quad (2.50) \]

In terms of the total charge on the disk, the surface charge density is given by

\[ \sigma = \frac{Q}{\pi R^2} \quad (2.51) \]
so that the electric field in Eq. (2.50) can be expressed as

\[ \vec{E} \simeq \frac{Q}{4\pi\varepsilon_0 a^2} \hat{z}. \]  

(2.52)

This shows that at a distance far from the disk, the electric field behaves like the electric field of a point charge.

**Example 1.5** Find the field inside and outside a sphere of radius \( R \), which carries a uniform volume charge density \( \rho \).

**Solution:** Let’s consider a point outside the sphere along the z-axis which is a distance \( a \) from the center of the sphere as shown in the Fig. 2.11.

![Figure 2.11: Spherical (volume) charge distribution](image)

The position vector of an infinitesimal volume \( d\tau' \)

\[ \vec{r}' = r' \sin \theta' \cos \varphi' \hat{x} + r' \sin \theta' \sin \varphi' \hat{y} + r' \cos \theta' \hat{z} \]  

(2.54)

then

\[ \vec{r} - \vec{r}' = -r' \sin \theta' \cos \varphi' \hat{x} - r' \sin \theta' \sin \varphi' \hat{y} + (a - r' \cos \theta') \hat{z}, \]  

(2.55)
2.2. THE ELECTRIC FIELD

\[ |\mathbf{r} - \mathbf{r}'| = \sqrt{r'^2 + a^2 - 2ar' \cos \theta'} \]  \hspace{1cm} (2.56)

The infinitesimal volume \(dr'\) can be expressed, in terms of the variables \(r', \theta'\) and \(\varphi'\) as

\[ dr' = r'^2 \sin \theta' \, dr' \, d\theta' \, d\varphi'. \]  \hspace{1cm} (2.57)

Then, using the results above in Eqs. (2.55) and (2.57) in the expression for the electric field of a volume charge distribution Eq. (2.20), we find

\[ \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \int_0^R \int_0^\pi \int_0^{2\pi} \frac{\rho(r')r'^2 \sin \theta' \, dr' \, d\theta' \, d\varphi'}{(r'^2 + a^2 - 2ar' \cos \theta')^{3/2}} \times \left[ -r' \sin \theta' \cos \varphi' \hat{x} - r' \sin \theta' \sin \varphi' \hat{y} + (a - r' \cos \theta') \hat{z} \right]. \]  \hspace{1cm} (2.58)

In view of the relation in Eq. (2.47), integration over \(\varphi\) gives zero. This eliminates the \(x\)- and \(y\)-components of the electric field which leads to

\[ \mathbf{E} = \frac{2\pi \rho}{4\pi \varepsilon_0} \int_0^R \int_0^\pi \frac{r'^2 \sin \theta' \, dr' \, d\theta'}{(r'^2 + a^2 - 2ar' \cos \theta')^{3/2}} (a - r' \cos \theta') \hat{z}, \]  \hspace{1cm} (2.59)

which can be rewritten as

\[ \mathbf{E} = \frac{\rho}{2\varepsilon_0} \int_0^R r'^2 \, dr' \int_0^\pi \frac{(a - r' \cos \theta') \sin \theta' \, d\theta'}{(r'^2 + a^2 - 2ar' \cos \theta')^{3/2}} \hat{z}. \]  \hspace{1cm} (2.60)

We first evaluate the integral with respect to \(\theta'\) which we write as

\[ I = \int_0^\pi \frac{(a - r' \cos \theta') \sin \theta' \, d\theta'}{(r'^2 + a^2 - 2ar' \cos \theta')^{3/2}}. \]  \hspace{1cm} (2.61)

To this end, we introduce the transformation of variable defined by

\[ u = r'^2 + a^2 - 2ar' \cos \theta' \Rightarrow \frac{du}{2ar'} = \sin \theta' \, d\theta', \quad r' \cos \theta' = \frac{r'^2 + a^2 - u}{2a} \]  \hspace{1cm} (2.62)

so that we may write the integral as

\[ I = \int_0^\pi \frac{(a - \frac{r'^2 + a^2 - u}{2a})}{u^{3/2}} \frac{du}{2ar'} \Rightarrow I = \frac{1}{4a^2 r'} \int_0^\pi \frac{(a^2 - r'^2 + u)}{u^{3/2}} \, du \]  \hspace{1cm} (2.63)

\[ I = \frac{a^2 - r'^2}{4a^2 r'} \int_0^\pi \frac{du}{\sqrt{u}} + \frac{1}{4a^2 r'} \int_0^\pi \frac{du}{\sqrt{u}} = \frac{1}{4a^2 r'} \left[ \frac{-2(a^2 - r'^2)}{\sqrt{u}} + 2\sqrt{u} \right]_0^\pi \]  

\[ \Rightarrow I = \frac{1}{2a^2 r'} \left[ u - a^2 + r'^2 \right]_0^\pi \]  \hspace{1cm} (2.64)
Substituting the explicit expression \( u = r'^2 + a^2 - 2ar' \cos \theta' \), we find

\[
I = \frac{1}{a^2} \frac{r' - a \cos \theta'}{\sqrt{r'^2 + a^2 - 2ar' \cos \theta'}} |^\pi_0 = \frac{1}{a^2} \left[ \frac{a + r'}{\sqrt{(a + r')^2}} - \frac{r' - a}{\sqrt{(a - r')^2}} \right] \\
\Rightarrow I = \frac{1}{a^2} \left[ \frac{a + r'}{a + r'} + \frac{a - r'}{|a - r'|} \right] = \frac{1}{a^2} \left[ 1 + \frac{a - r'}{|a - r'|} \right]
\] (2.65)

Using the result in Eq. (2.65), the expression in Eq. (2.60) becomes

\[
\vec{E} = \frac{\rho}{2\varepsilon_0 a^2} \int_0^R r'^2 dr' \left\{ 1 + \frac{a - r'}{|a - r'|} \right\} \hat{z}.
\] (2.66)

In the region outside the sphere \( a > r' \geq R \), we note that

\[
\frac{a - r'}{|a - r'|} = 1
\] (2.67)

and the electric field becomes

\[
\vec{E} = \frac{\rho R^3}{3\varepsilon_0 a^2} \hat{z}.
\] (2.68)

However, inside the sphere, for \( a < R \), since

\[
\frac{a - r'}{|a - r'|} = \begin{cases} -1, & \text{for } a < r' < R \\ 1, & \text{for } 0 < r' < a \end{cases}
\] (2.69)

we need to split the integral into two parts

\[
\vec{E} = \frac{\rho}{2\varepsilon_0 a^2} \left[ \int_0^a 2r'^2 dr' + \int_a^R (-1 + 1)r'^2 dr' \hat{z} \right]
\] (2.70)

and the electric field becomes

\[
\vec{E} = \frac{\rho a}{3\varepsilon_0} \hat{z}.
\] (2.71)

### 2.3 Electric field flux and Gauss’s Law

In Theoretical Physics I (Mathematical methods in physics) you have introduced to the concept of flux. Flux is a quantity used to measure the strength of a vector field passing through a unit area normal to the vector. Consider the vector field \( \vec{F} \) shown by the blue curved lines crossing through the surface as shown in Fig. (2.12). The direction of the vector field at the point where it crosses the infinitesimal area \( da \) is shown by the yellow tangent vector to the vector field that can be decomposed into normal \( (F_\perp) \) and parallel \( (F_\parallel) \) to the area \( da \).

The infinitesimal flux, \( d\phi \) through the infinitesimal area \( da \) is defined as

\[
d\phi = F_\perp da.
\] (2.72)
Consider the outward unit vector, \( \hat{n} \), normal to the infinitesimal area \( da \). Let the direction of the vector field \( \vec{F} \) at the point it crosses the surface \( da \) is described by the angle \( \theta \) measured from the unit vector \( \hat{n} \). Using the normal component of the vector field, \( F_\perp = F \cos(\theta) \), one can express

\[
d\phi = F da \cos(\theta) = \vec{F} \cdot d\hat{a}.
\]

where

\[
d\hat{a} = \hat{n} da.
\]

Then the total flux of the vector field, \( \vec{F} (\vec{r}) \) over an area, \( \vec{A} \), becomes

\[
\phi = \int_A F (\vec{r}) \cdot d\hat{a}.
\]

Now let’s consider the electric field vector, \( \vec{E} (r) \), due to some charge distribution and the resulting electric field flux over a closed surface area \( A \), which we express as

\[
\phi_E = \oint_A \vec{E} (r) \cdot d\hat{a}.
\]

Suppose this field is due to a positive point charge, \( q \), positioned at the origin (Fig.2.3). This electric field, at a point in space described by the position vector (in spherical coordinates),

\[
\vec{r} = r \sin(\theta) \cos(\phi) \hat{x} + r \sin(\theta) \sin(\phi) \hat{y} + r \cos(\theta) \hat{z}
\]

can be expressed as

\[
\vec{E} = \frac{q}{4\pi \varepsilon_0 r^2} \hat{r}.
\]
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Then the electric flux due to this charge over a closed surface, $A$, becomes

$$\phi_E = \oint_A \vec{E}(r) \cdot d\vec{a} = \frac{q}{4\pi \epsilon_0} \oint_A \frac{\hat{r} \cdot d\vec{a}}{r^2} = \frac{q}{4\pi \epsilon_0} \int_A \frac{\hat{r} \cdot d\vec{a}}{r^2}.$$  \hspace{1cm} (2.78)

We note that the quantity

$$\frac{1}{4\pi \epsilon_0} \oint_A \frac{\hat{r} \cdot d\vec{a}}{r^2} = \text{constant for a given closed surface.}$$

This means that the electric flux through any closed surface is the measure of the total charge inside.

Let’s consider a spherical surface of radius $r$ that encloses this point charge. The infinitesimal area vector, $d\vec{a}$, can then be expressed as

$$d\vec{a} = r^2 \sin(\theta) d\theta d\varphi \hat{r},$$

where $\hat{r}$ is the outward unit vector normal to the infinitesimal area $da$ on the spherical surface of radius $r$. The electric flux over this closed surface becomes

$$\phi_E = \oint_A \vec{E}(r) \cdot d\vec{a} = \frac{q}{4\pi \epsilon_0} \oint_A \frac{\hat{r} \cdot r^2 \sin(\theta) d\theta d\varphi \hat{r}}{r^2} = \frac{q}{4\pi \epsilon_0} \int_0^{2\pi} \int_0^\theta \sin(\theta) d\theta d\varphi$$

$$\Rightarrow \phi_E = \oint_A \vec{E}(r) \cdot d\vec{a} = \frac{q}{\epsilon_0}.$$ \hspace{1cm} (2.79)

Note that the result in Eq. (2.79) shows that the electric field at any point on the surface depends on that charge $q$ inside the surface. A charge outside the surface will contribute nothing to the total flux, since its field lines pass one side ($d\vec{a}_1 = d\vec{a}$) and out the other ($d\vec{a}_2 = -d\vec{a}$)

For $N$ point charges where the electric field is the vector sum of the electric field of each charges

$$\vec{E} = \sum_{i=1}^N \vec{E}_i (r) = \frac{1}{4\pi \epsilon_0} \sum_{i=1}^N \frac{q_i}{|\vec{r} - \vec{r}_i|^3} \left(\vec{r} - \vec{r}_i\right),$$ \hspace{1cm} (2.80)
2.3. ELECTRIC FIELD FLUX AND GAUSS’S LAW

the electric flux over a closed surface

\[ \phi_E = \oint_{A} \vec{E}(r) \cdot d\vec{a} = \oint_{A} \sum_{i=1}^{N} \vec{E}_i(r) \cdot d\vec{a} = \sum_{i=1}^{N} \oint \vec{E}_i \cdot d\vec{a} \]  

(2.81)

using the result for a point charge in Eq.(2.79), we find

\[ \phi_E = \oint_{A} \vec{E}(r) \cdot d\vec{a} = \sum_{i=1}^{N} \frac{q_i}{\varepsilon_0} \]  

(2.82)

Gauss’s law relates the electric flux over a closed surface with the charge enclosed. It can be expressed as an Integral or differential form.

Integral form: For any closed surface

\[ \phi_E = \oint_{A} \vec{E}(r) \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc} \]  

(2.83)

\( Q_{enc} \) is the the total charge enclosed.

Differential form: For a volume charge distribution described by a charge density, \( \rho(\vec{r}) \), the total enclosed charge, \( Q_{enc} \), can be expressed as

\[ Q_{enc} = \int_{V} \rho(\vec{r}) \, d\tau. \]  

(2.84)

Applying the Divergence theorem, we have

\[ \oint_{A} \vec{E}(\vec{r}) \cdot d\vec{a} = \int_{V} \left[ \nabla \cdot \vec{E}(\vec{r}) \right] \, d\tau \]  

(2.85)

so that upon substituting Eqs. (2.84) and (2.85) into (2.83), one finds

\[ \phi_E = \int_{V} \left[ \nabla \cdot \vec{E}(\vec{r}) \right] \, d\tau = \int_{V} \rho(\vec{r}) \, d\tau \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}. \]  

(2.86)

This is the differential form of Gauss’s law.

The is useful tool for calculating the electric field whenever the charge distributions has a some kind of symmetry such that the magnitude of the field is a constant over the entire surface enclosing that charge. The following imaginary surface ("Gaussian Surface") with the corresponding charge distribution symmetry are useful to find the electric field.

<table>
<thead>
<tr>
<th>Charge distribution symmetry</th>
<th>Gaussian surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td>Concentric sphere</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>Coaxial cylinder</td>
</tr>
<tr>
<td>Plane</td>
<td>Pillbox</td>
</tr>
</tbody>
</table>

Spherical Gaussian Surfaces: such surfaces are useful to find the electric field of a charge distribution that depends only the radial distance \( r \) not on \( \theta \) and \( \varphi \).
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Figure 2.13: Spherical Gaussian surfaces for a charge distribution with spherical symmetry.

Figure 2.14: Cylindrical Gaussian surfaces for a charge distribution with cylindrical symmetry.

Cylindrical surfaces: we use these surfaces when charge distribution depends on the distance \( r \) from the z-axis not on \( \varphi \) and \( z \) in spherical coordinates.

Pillbox surfaces: such surfaces are useful when there is a plane or disk charge distribution.

The differential form of Gauss’s Law is useful when the electric field due to some charge distribution is known and we are interested in finding the charge distribution.

Example 1.6 Find the field inside and outside a uniform charged sphere of radius \( R \) and total charge \( q \)

Solution: The charge distribution is uniform and has a spherical symmetry. We can then use a concentric spherical Gaussian surface of radius \( r \) shown in the figure below. First, let’s determine the field outside the sphere. Using the integral form of Gauss’s law, Eq. (2.83), and noting that the total
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Figure 2.15: Pillbox Gaussian surfaces for a charge distribution with a plane symmetry.

Figure 2.16: Spherical Gaussian surfaces for a charge distribution with spherical symmetry.

charge enclosed by the Gaussian surface is \( q \), we can write

\[
\oint_S \vec{E} \cdot d\vec{a} = \frac{q}{\varepsilon_0}. \tag{2.87}
\]

The charge distribution is uniform and it has a spherical symmetry. From this we know that the magnitude of the electric field is constant over the entire Gaussian surface and is directed in the radial direction. Hence,

\[
\oint_S |\vec{E}| \hat{r} \cdot d\vec{a} = \frac{q}{\varepsilon_0} \Rightarrow |\vec{E}| \oint_S \hat{r} \cdot d\vec{a} = \frac{q}{\varepsilon_0}. \tag{2.88}
\]

In spherical coordinates the infinitesimal area vector, \( d\vec{a} \) is given by

\[
d\vec{a} = d\alpha \hat{r} = r^2 \sin \theta d\theta d\phi \hat{r} \tag{2.89}
\]
so that Eq. (2.88) leads to
\[
\left| \vec{E} \right| \oint_S \hat{r} \cdot d\vec{a} = \left| \vec{E} \right| r^2 \int_0^\pi \sin \theta d\theta \int_0^\pi d\varphi \hat{r} \cdot \hat{r} = \frac{q}{\epsilon_0}
\]
\[
\left| \vec{E} \right| = \frac{q}{4\pi\epsilon_0 r^2} \Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \quad (2.90)
\]

The field outside the sphere is exactly the same as it would have been if all the charge has been concentrated at the center. To find the electric field inside the sphere we still consider a concentric spherical Gaussian surface of radius \( r \). However, in this case \( r < R \). This changes the charge enclosed by the Gaussian surface. Since the total charge \( q \) is distributed uniformly over the entire sphere of radius \( R \), the fraction of the charge enclosed by the Gaussian surface will be
\[
Q_{\text{enclosed}} = \frac{4}{3} \pi r^3 = \frac{q}{4\pi R^3} \frac{4}{3} \pi r^3 \Rightarrow q = \frac{r^3}{R^3} \quad (2.91a)
\]

Therefore, the electric field inside the sphere is given by
\[
\left| \vec{E} \right| \oint_S \hat{r} \cdot d\vec{a} = \frac{1}{\epsilon_0} \left( q \frac{r^3}{R^3} \right) \Rightarrow \left| \vec{E} \right| = \frac{qr}{4\pi\epsilon_0 R^3} \Rightarrow \vec{E} = \frac{qr}{4\pi\epsilon_0 R^3} \hat{r} \quad (2.92)
\]

If we express \( Q_{\text{enclosed}} \) in terms of the charge density \( \rho \) (Eq. (2.91), we see that the electric field is given by
\[
\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{r} \quad (2.93)
\]

Eq. (2.93) is exactly the same as the result in Eq. (2.71) that we obtained in Example 2.5 except that here we used \( r \) instead of \( a \) to represent the distance from the center of the sphere to the point where we determined the electric field. Did you notice how much Gauss’s law makes life easy when it comes to calculating Electric field? But unfortunately Gauss’s law is limited by symmetry of the charge distribution. So we continue looking for other techniques useful to find the electric field when there is no symmetry in the charge distribution.

**Example 1.7** A long cylinder (see figure below) carries a charge density that is proportional to the distance from the axis: \( \rho = ks \), for some constant \( k \). Find the electric field inside the cylinder.

**Solution:** Though the charge distribution is not uniform, it has cylindrical symmetry and we can consider a coaxial cylindrical Gaussian surface (see figure above). The Gaussian surface has length \( l \) and radius \( s \). The total charge in volume enclosed by the Gaussian,
\[
Q_{\text{enc}} = \int_V \rho d\tau = \int_{-l/2}^{l/2} dz \int_0^{2\pi} d\varphi \int_0^s ks^2 ds = \frac{2\pi kl}{3} s^3 \quad (2.94)
\]
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where we have assumed the origin of the cylindrical coordinate system to be at the center of the Gaussian surface and used $d\tau = sdsd\phi dz$. Applying the integral form of Gauss’s law (Eq. (2.83)) and Eq. (2.94) we can write

$$\int_S \vec{E} \cdot d\vec{a} = \frac{2\pi kl}{\varepsilon_0} s^3. \quad \text{(2.95)}$$

From the cylindrical symmetry of the charge distribution we can see that the electric field has only radial component. If we define the unit vectors $\hat{s}, \hat{\phi},$ and $\hat{z}$ in cylindrical coordinates, we write the electric field as

$$\vec{E} = E_s \hat{s}, \quad (E_\phi = E_z = 0). \quad \text{(2.96)}$$

We split the integral over the Gaussian surface in Eq. (2.95) into two parts. The first one is integration over the curved surface of the cylinder and the second one is over the two plane surfaces at the two ends of the
Gaussian surface

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\text{curved}} \vec{E} \cdot d\vec{a} + \int_{\text{plane}} \vec{E} \cdot d\vec{a}. \quad (2.97)$$

Using $d\vec{a} = da \hat{s}$ for the curved surface, $d\vec{a} = da \hat{z}$ for the right end plane surface, $d\vec{a} = -da \hat{z}$ for the left end plane surface, and Eq. (2.96) we get

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\text{curved}} E_s \hat{s} \cdot da \hat{s} + \int_{\text{right}} E_s \hat{s} \cdot da \hat{z} - \int_{\text{left}} E_s \hat{s} \cdot da \hat{z} \quad (2.98)$$

The last two terms in Eq. (2.98) are zero since $\hat{s} \cdot \hat{z} = 0$. The charge density does not depend on $\varphi$ and $z$, as long as $s$ is fixed, the electric field remains constant. It changes whenever $s$ is changing. Hence, in the first term we can take $E_s$ out of the integral,

$$\oint_S \vec{E} \cdot d\vec{a} = E_s \int_{\text{curved}} da = E_s \int_{l=1/2}^{l=1/2} \int_{0}^{2\pi} s dz d\varphi \Rightarrow \oint_S \vec{E} \cdot d\vec{a} = E_s 2\pi sl \quad (2.99)$$

Therefore, from Eq. (2.95) and (2.99) we find

$$E_s 2\pi sl = \frac{2\pi kl}{3\varepsilon_0} s^3 \Rightarrow E_s = \frac{k}{3\varepsilon_0} s^2 \Rightarrow \vec{E} = E_s \hat{s} = \frac{k}{3\varepsilon_0} s^2 \hat{s} \quad (2.100)$$

**Example 1.8** An infinite plane carries a uniform surface charge $\sigma$. Find its electric field.

**Solution:** Here we have a plane charge distribution and we consider a pillbox Gaussian surface (see figure below). The a square pillbox of dimension $l$ is centered at the plane carrying a surface charge density $\sigma$. From the symmetry of the charge distribution we have an electric field pointing out of the normal to the plane surface. Therefore, the total flux for the pillbox

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\text{bottom}} \vec{E} \cdot d\vec{a} + \int_{\text{top}} \vec{E} \cdot d\vec{a} + \int_{\text{sides}} \vec{E} \cdot d\vec{a} \quad (2.101)$$
2.3. ELECTRIC FIELD FLUX AND GAUSS’S LAW

The flux across the side surfaces is zero since the electric field is parallel to these surfaces. For the bottom and top surfaces the electric field is normal and is also constant. Therefore,

$$\int_S \vec{E} \cdot d\vec{a} = \vec{E} \cdot \int_{\text{bottom}} d\vec{a} + \vec{E} \cdot \int_{\text{top}} d\vec{a}.$$  \hspace{1cm} (2.102)

For the bottom surface $\vec{E} = -E\hat{z}$ and for the top surface $\vec{E} = E\hat{z}$, the infinitesimal areas for the two surfaces $d\vec{a} = -da\hat{z}$ (bottom surface) $d\vec{a} = da\hat{z}$ (top surface)

$$\int_S \vec{E} \cdot d\vec{a} = 2El^2 = 2EA.$$  \hspace{1cm} (2.103)

where $A = l^2$ is the area. Part of the infinite plane area enclosed by the pillbox contains a total charge of

$$Q_{\text{encl}} = \sigma A.$$  \hspace{1cm} (2.104)

Now applying Gauss’s law we find

$$\int_S \vec{E} \cdot d\vec{a} = \frac{Q_{\text{encl}}}{\varepsilon_0} \Rightarrow 2EA = \sigma A.$$  
\Rightarrow E = \frac{\sigma}{2\varepsilon_0} \Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{n}.$$  \hspace{1cm} (2.105)

here $\hat{n}$ is a unit vector normal to the surface.

Important Note: The corresponding field lines for example for the two negative charges

![Electric field for two point charges](image)

Figure 2.18: Electric field for two point charges (a) Opposite, (b) positive (c) negative

Example 1.9 Suppose the electric field in some region is found to be $E = kr^3\hat{r}$ in spherical coordinates ($k$ is some constant).

(a) Find the charge density $\rho$

(b) Find the total charge contained in a sphere of radius $R$, centered at the origin. (Do it in two different ways)
Figure 2.19: Electric field lines: (a) Two negative charges and (b) Two opposite charges.

**Solution:**

(a) From Gauss’s Law in a differential form (Eq. (2.83)) we know that

\[ \rho = \varepsilon_0 \left( \nabla \cdot \vec{E} \right) \quad (2.106) \]

If we are given a vector \( \vec{E} \) expressed in terms of spherical coordinates \( r, \theta, \) and \( \phi \), as

\[ \vec{E} = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi}, \]

its divergence in spherical coordinates is given by

\[ \nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 E_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta E_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi}. \quad (2.107) \]

The electric field has component only in the radial direction (i.e. \( E_r = kr^3, E_\theta = 0, E_\phi = 0 \)),

\[ \nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 kr^3 \right) = \frac{k}{r^2} 5kr^4 = 5kkr^2. \quad (2.108) \]

The charge density will then be

\[ \rho = \varepsilon_0 \left( \nabla \cdot \vec{E} \right) = 5\varepsilon_0 kr^2. \quad (2.109) \]

(b) One way of finding the charge is to apply Gauss’s Law (Eq. (2.83)),

\[ \oint \vec{E} \cdot \vec{dA} = \frac{1}{\varepsilon_0} Q_{\text{enc}} \Rightarrow Q_{\text{enc}} = \varepsilon_0 \oint \vec{E} \cdot \vec{dA} \quad (2.110) \]

Considering a spherical surface that just encloses the charge (i.e. a sphere with a radius \( R \)), the electric field on this surface is \( \vec{E} = kR^3 \hat{r} \). Using an infinitesimal area vector on this surface, \( \vec{dA} = (R^2 \sin \theta d\theta d\phi) \hat{\phi} \), we can then write the total charge encloses as

\[ Q_{\text{enc}} = \varepsilon_0 \oint kR^3 \hat{r} \cdot (R^2 \sin \theta d\theta d\phi) \hat{\phi} = \varepsilon_0 kR^5 \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi (\hat{\phi} \cdot \hat{r}) \]

\[ \Rightarrow Q_{\text{enc}} = 4\pi \varepsilon_0 kR^5 \]
The second way of finding the total charge is to integrate the charge density obtained in part (a) over the total volume (Eq. (2.84))

\[
Q_{\text{enc.}} = \int_V \rho d\tau = \int_0^R \int_0^{2\pi} \int_0^\pi 5\epsilon_0 kr^2 r^2 dr d\theta d\varphi = 5\epsilon_0 k \int_0^R r^4 dr \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \Rightarrow Q_{\text{enc.}} = 4\pi \epsilon_0 k R^5
\]

(2.111)

**Example 1.10** Two infinite parallel planes each carry a surface charge density \(\sigma_1\) and \(\sigma_2\) as shown in Fig. (??)

![Figure 2.20: Two Infinite parallel conducting plates.](image)

(a) Find the electric field in the three regions shown in the figure (i.e. in the regions labeled as A, B, and C)

(b) Consider the case in which the two surfaces carry equal but opposite uniform charge densities \((\sigma_1 = \sigma, \ \text{and} \ \sigma_2 = -\sigma)\).

### 2.4 Divergence and Curl of an electric field

It is important to recall some relations from theoretical physic I. The following relations are useful to study the divergence and curl of the electric field.

(a) The three-dimensional Delta function:

\[
\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z) = \begin{cases} 
\infty, & \text{for } \vec{r} = (0,0,0) \\
0, & \text{for } \vec{r} \neq (0,0,0)
\end{cases}
\]

(2.112)
(b) Integration involving three-dimensional Delta function
\[ \int_{all\ space} \delta^3(\vec{r})d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1. \] (2.113a)
\[ \int_{all\ space} f(\vec{r})\delta^3(\vec{r} - \vec{a})d\tau = f(\vec{a}). \] (2.113b)
(c) Differentiation involving the position vector
\[ \nabla \cdot \left( \frac{\vec{r}}{r^3} \right) = 4\pi \delta^3(\vec{r}), \quad \nabla \left( \frac{1}{r} \right) = -\frac{\vec{r}}{r^2} \] (2.113c)
\[ \nabla \cdot \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = 4\pi \delta^3(\vec{r} - \vec{r}'). \] (2.113d)
(d) The curl and the divergence of a vector

**Cartesian coordinates:**
\[ \vec{A}(\vec{r}) = A_x(x, y, z) \hat{x} + A_y(x, y, z) \hat{y} + A_z(x, y, z) \hat{z} \]
\[ \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \quad \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \]
\[ = \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \] (2.114)

**Spherical coordinates:**
\[ \vec{A} = A_r(r, \theta, \varphi) \hat{r} + A_\theta(r, \theta, \varphi) \hat{\theta} + A_\varphi(r, \theta, \varphi) \hat{\varphi} \]
\[ \nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A_r \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin(\theta) A_\theta \right) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \varphi} (A_\varphi) \]
\[ \nabla \times \vec{A} = \frac{\hat{r}}{r \sin(\theta)} \left( \frac{\partial}{\partial \theta} \left( \sin(\theta) A_\varphi \right) - \frac{\partial A_\theta}{\partial \varphi} \right) + \frac{\hat{\theta}}{r} \left( \frac{1}{\sin(\theta)} \frac{\partial A_r}{\partial \varphi} - \frac{\partial A_\varphi}{\partial r} \right) + \frac{\hat{\varphi}}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right). \] (2.115)

**Cylindrical coordinates:**
\[ \vec{A} = A_s(s, \varphi, z) \hat{s} + A_\varphi(s, \varphi, z) \hat{\varphi} + A_z(s, \varphi, z) \hat{z} \]
\[ \nabla \cdot \vec{A} = \frac{1}{s} \frac{\partial}{\partial s} \left( s A_s \right) + \frac{1}{s} \frac{\partial}{\partial \varphi} (A_\varphi) + \frac{\partial}{\partial z} (A_z), \] (2.117)
\[ \nabla \times \vec{A} = \hat{s} \left( \frac{1}{s} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) + \hat{\varphi} \left( \frac{\partial A_z}{\partial s} - \frac{\partial A_s}{\partial z} \right) + \hat{z} \left( \frac{1}{s} \frac{\partial}{\partial s} (s A_\varphi) - \frac{\partial A_s}{\partial \varphi} \right). \] (2.118)
(e) The Divergence and Stoke’s theorem:

The divergence theorem:
\[
\iint_{\text{Volume}} \left( \nabla \cdot \vec{A} \right) \, d\tau = \iint_{\text{Closed Surface}} \vec{A} \cdot d\vec{a}
\] (2.119)

Stokes’ theorem:
\[
\iint_{\text{Surface}} \left( \nabla \times \vec{A} \right) \cdot d\vec{a} = \oint_{\text{Path}} \vec{A} \cdot d\vec{l}
\] (2.120)

Identity:
\[
\iint_{\text{Volume}} f \left( \nabla \cdot \vec{A} \right) \, d\tau = - \iint_{\text{Volume}} \vec{A} \cdot (\nabla f) \, d\tau + \oint_{\text{Path}} f \vec{A} \cdot d\vec{l}
\]

The divergence of the electric field: The divergence of the electric field for a volume charge distribution (Eq. (2.20))
\[
\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{Volume}} \nabla \cdot \frac{\left( \vec{r} - \vec{r}' \right)}{\left| \vec{r} - \vec{r}' \right|^3} \rho(\vec{r}') \, d\tau'
\]
\[
= \frac{1}{4\pi\epsilon_0} \int_{\text{All Space}} \nabla \cdot \frac{\left( \vec{r} - \vec{r}' \right)}{\left| \vec{r} - \vec{r}' \right|^3} \rho(\vec{r}') \, d\tau',
\] (2.121)
using Eqs. (2.113d)
\[
\nabla \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{All Space}} 4\pi\delta^3(\vec{r} - \vec{r}') \rho(\vec{r}') \, d\tau' = \frac{1}{\epsilon_0} \int_{\text{All Space}} \delta^3(\vec{r} - \vec{r}) \rho(\vec{r}) \, d\tau',
\] (2.122)
and applying Eq. (2.113b) we find
\[
\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0},
\] (2.123)
which is the differential form of Gauss’s law. Integrating Eq. (2.123) over the volume and applying the divergence theorem (Eq. (2.85)), we find
\[
\int_{V} \nabla \cdot \vec{E} \, d\tau = \oint_{S} \vec{E} \cdot d\vec{a} = \int_{V} \frac{\rho(\vec{r})}{\epsilon_0} \, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}} \Rightarrow \oint_{S} \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},
\] (2.124)
which is the integral form Gauss’s law.

Curl of the electric field: The Curl of the electric field measures how much the \( \vec{E} \) field rotates (Curls) around at the point in question. Generally, in Electrostatic, the electric field is irrotational. The electric field vectors shown in
CHAPTER 2. ELECTROSTATICS

Figure 2.21: Irrotational vectors (Vectors with zero curl)

Fig. 2.21 (the electric field of a point charge and a uniform electric field) are examples of vectors with zero curl. There are vector fields that are rotational. You have introduced to one of these, the magnetic field vector, $\vec{B}$ in introductory physics. For example, the magnetic field created by a long straight wire carrying a current $I$, curls around the wire as shown in Fig. 2.22.

Figure 2.22: Magnetic field of long wires carrying a current $I$ out of and into the page.

Next we show the electric field is irrotational ($\nabla \times \vec{E}(\vec{r}) = 0$) using the electric field of a point charge. Consider the electric field of a point charge positioned at the origin as shown in Fig. 2.23. For this charge the electric field is given by

$$\vec{E} = \frac{q}{4\pi \varepsilon_0 r^2} \hat{r} = \frac{q}{4\pi \varepsilon_0 r^3} \vec{r}$$

(2.125)

Integrate the electric field along the path shown in the figure from point $a$ to $b$

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \frac{q}{4\pi \varepsilon_0 r^2} \hat{r} \cdot d\vec{l}$$

(2.126)

using $d\vec{l} = d\hat{r}$ and $\hat{r} = \frac{\vec{r}}{r}$

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \frac{q}{4\pi \varepsilon_0 r^2} \frac{\vec{r}}{r} \cdot d\vec{l} = \frac{q}{4\pi \varepsilon_0} \int_a^b \frac{dr}{r^2}$$

$$= - \frac{q}{4\pi \varepsilon_0} \int_a^b \frac{dr}{r^2} = - \frac{q}{4\pi \varepsilon_0 r} \bigg|_a^b \Rightarrow \int_a^b \vec{E} \cdot d\vec{l} = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{r_a} - \frac{1}{r_b} \right]$$
For a closed curve, \( r_a = r_b \)

\[
\oint \vec{E} \cdot d\vec{l} = 0.
\]  

(2.127)

Applying Stokes’ theorem

\[
\int_s \left( \nabla \times \vec{E} \right) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l} = 0,
\]  

(2.128)

that leads to

\[
\nabla \times \vec{E} = 0.
\]  

(2.129)

For any static charge distribution the curl of the electric field is always zero. We have proved this for more general case in Example 1.11.

**Example 1.11** Using the expression for the electric field for charge distribution in a free space described by a volume charge density \( \rho(\vec{r}) \),

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} \left( \vec{r} - \vec{r}' \right) d\tau' 
\]  

(2.130)

show that the electric field is irrotational.

**Solution:** We want to show that \( \nabla \times \vec{E}(\vec{r}) = 0 \). To this end, we note that

\[
\nabla \times \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \left\{ \nabla \times \left[ \frac{1}{|\vec{r} - \vec{r}'|^3} \left( \vec{r} - \vec{r}' \right) \right] \right\} \rho(\vec{r'})d\tau'.
\]  

(2.131)

Applying the relation

\[
\nabla \times \left( f(\vec{r}) \vec{A}(\vec{r}) \right) = f(\vec{r}) \left[ \nabla \times \left( \vec{A}(\vec{r}) \right) \right] - \vec{A}(\vec{r}) \times (\nabla f(\vec{r})),
\]  

(2.132)
we may write

\[
\nabla \times \left[ \frac{1}{|\vec{r} - \vec{r}'|^3} \left( \vec{r} - \vec{r}' \right) \right] = \frac{1}{|\vec{r} - \vec{r}'|^3} \nabla \times \left( \vec{r} - \vec{r}' \right) - \left( \vec{r} - \vec{r}' \right) \times \nabla \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right)
\]

so that using Eq. (2.133),

\[
\nabla \times \left( \vec{r} - \vec{r}' \right) = \nabla \times \vec{r} = \hat{x} \left( \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \hat{y} \left( \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \hat{z} \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = 0.
\]

one finds

\[
\nabla \times \left[ \frac{1}{|\vec{r} - \vec{r}'|^3} \left( \vec{r} - \vec{r}' \right) \right] = - \left( \vec{r} - \vec{r}' \right) \times \nabla \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right).
\]

Using the magnitude of the vector \(|\vec{r} - \vec{r}'|\)

\[
|\vec{r} - \vec{r}'| = \sqrt{\left( x - x' \right)^2 + \left( y - y' \right)^2 + \left( z - z' \right)^2},
\]

it can be shown that

\[
\nabla \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right) = - \frac{3}{|\vec{r} - \vec{r}'|^4} \nabla \left( \sqrt{\left( x - x' \right)^2 + \left( y - y' \right)^2 + \left( z - z' \right)^2} \right)
\]

\[
= - \frac{3}{|\vec{r} - \vec{r}'|^4} \frac{x - x'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \hat{x} + \frac{y - y'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \hat{y} + \frac{z - z'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \hat{z} = \frac{-3}{|\vec{r} - \vec{r}'|^4} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^4}
\]

\[
\Rightarrow \nabla \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right) = - \frac{3}{|\vec{r} - \vec{r}'|^5} \left( \vec{r} - \vec{r}' \right).
\]

Now substituting Eq. (2.137) into Eq. (2.135), we have

\[
\nabla \times \left[ \frac{1}{|\vec{r} - \vec{r}'|^3} \left( \vec{r} - \vec{r}' \right) \right] = \left( \vec{r} - \vec{r}' \right) \times \frac{3}{|\vec{r} - \vec{r}'|^5} \left( \vec{r} - \vec{r}' \right).
\]

so that curl of the electric field becomes

\[
\nabla \times \vec{E} = \frac{3}{4\pi\varepsilon_0} \int \frac{\left( \vec{r} - \vec{r}' \right) \times \left( \vec{r} - \vec{r}' \right)}{|\vec{r} - \vec{r}'|^5} \rho(\vec{r'}) d\tau' = 0,
\]

as the cross product of a vector with itself is zero.
2.5 The electric field and potential

In Example 1.11 we have shown that the curl of the electric field vector for any static charge distribution is zero which indicates electric field is an irrotational field. This has significant impact in electrostatic problems. In theoretical physics I we were introduced to conservative vector fields. We recall that for a vector field, \( \vec{F}(\vec{r}) \), if

(a) the line integral over a closed curve is zero independent of the path,

\[
\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0,
\]

or (b) the curl of the vector field is zero,

\[
\nabla \times \vec{F}(\vec{r}) = 0,
\]

or (c) the vector field can be expressed as a gradient of some scalar function \( V(x, y, z) \),

\[
\vec{F}(\vec{r}) = \nabla V(\vec{r}),
\]

then the vector field is a conservative field. For any vector proved to be a conservative vector by any one of these conditions, it is true that the other two conditions are also satisfied (i.e. \( \vec{F} \) satisfies one if and only if it satisfies all the others). For example, if (a) or (b) is satisfied Stokes theorem easily proves that (b) or (a) is also satisfied.

Figure 2.24: Two different paths (i) and (ii) does not change the result of the integral.
The electric field is irrotational, \( \nabla \times \vec{E}(\vec{r}) = 0 \), and it is a conservative field. Therefore, in electrostatic independent of the path of integration the electric field satisfies the condition

\[
\oint_{C} \vec{E}(\vec{r}) \cdot d\vec{r} = 0, \quad (2.143)
\]

and it can also be expressed as

\[
\vec{E}(\vec{r}) = -\nabla V(\vec{r}), \quad (2.144)
\]

where \( V(\vec{r}) \) is a scalar function referred as the electrostatic potential. The electric potential at a given position, \( \vec{r} \), can then be expressed in terms of the electric field as

\[
V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E}(\vec{r}).d\vec{r}. \quad (2.145)
\]

where \( O \) is some reference point. SI units of electric potential is volt (V)

\[
Volt(V) = \frac{Newton}{meter.coulomb}.
\]

The following are important notes about the electric potential in electrostatic:

(a) The potential difference between two points \( a \) and \( b \):

\[
V(\vec{r}_b) - V(\vec{r}_a) = -\int_{O}^{\vec{r}_b} \vec{E}.d\vec{r} - \left( -\int_{O}^{\vec{r}_a} \vec{E}.d\vec{r} \right) = -\int_{O}^{\vec{r}_b} \vec{E}.d\vec{r} - \int_{\vec{r}_a}^{O} \vec{E}.d\vec{r}
\]

\[
\Rightarrow V(\vec{r}_b) - V(\vec{r}_a) = -\left( \int_{\vec{r}_a}^{O} \vec{E}.d\vec{r} + \int_{O}^{\vec{r}_b} \vec{E}.d\vec{r} \right) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}.d\vec{r} \quad (2.146)
\]

Changing the reference point does not change the potential difference between two points

\[
V'(\vec{r}) = -\int_{O'}^{\vec{r}} \vec{E}.d\vec{r} = -\int_{O}^{\vec{r}} \vec{E}.d\vec{r} - \int_{O}^{\vec{r}} \vec{E}.d\vec{r} + K.
\]

\[
\Rightarrow V'(\vec{r}_b) = -\int_{O}^{\vec{r}_b} \vec{E}.d\vec{r} + K, \quad V'(\vec{r}_a) = -\int_{O}^{\vec{r}_a} \vec{E}.d\vec{r} + K \quad (2.147)
\]

\[
\Rightarrow V'(\vec{r}_b) - V'(\vec{r}_a) = -\int_{O}^{\vec{r}_b} \vec{E}.d\vec{r} - \int_{O}^{\vec{r}_a} \vec{E}.d\vec{r} \Rightarrow V'(\vec{r}_b) - V'(\vec{r}_a)
\]

\[
\Rightarrow V'(\vec{r}_b) - V'(\vec{r}_a) = V(\vec{r}_b) - V(\vec{r}_a) \quad (2.148)
\]

(b) Infinity is always used as a reference point.

(c) The electric potential obeys the superposition principle, that means

\[
V = V_1 + V_2 + V_3 + \ldots V_n = \sum_{i=1}^{n} V_i \quad (2.149)
\]
(d) **Equipotential surface**: a surface over which the potential is constant.

**Example 1.12** Find the potential inside and outside a spherical shell of radius $R$ (Fig. 2.25), which carries a uniformly distributed surface charge, $q$. Set the reference point at infinity.

![Figure 2.25: A spherical shell with a charge $q$.](image)

**Solution**: The electric field inside ($r < R$) and outside ($r > R$) the spherical shell, using Gauss’s law, can be shown to be

$$
\vec{E} = \begin{cases} 
0 & r < R \\
\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} & r > R
\end{cases}
$$

(2.150)

First let’s find the potential outside the shell $(r > R)$

$$
V(\vec{r}) = -\int_\infty^r \vec{E} \cdot d\vec{r} = -\frac{q}{4\pi\varepsilon_0} \int_\infty^r \frac{1}{r^2} \hat{r} \cdot d\vec{r} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r} \bigg|_\infty^r \Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r}
$$

(2.151)

Inside the shell $(r < R)$ the potential is given by

$$
V(\vec{r}) = -\int_R^r \vec{E} \cdot d\vec{r} - \int_\infty^R \vec{E} \cdot d\vec{r} = -\int_R^r 0 \cdot d\vec{r} - \frac{q}{4\pi\varepsilon_0} \int_\infty^R \frac{1}{r^2} \hat{r} \cdot d\vec{r}
$$

$$
= \frac{q}{4\pi\varepsilon_0} \bigg|_R^\infty \Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R}
$$

(2.152)

The potential inside the shell is a constant but not zero although the electric field is zero.
Example 1.13 Find the potential inside and outside a uniformly charged solid sphere whose radius is \( R \) and whose total charge is \( q \). Use infinity as your reference point. Compute the gradient of \( V \) in each region, and check that it yields the correct field. Sketch \( V(r) \)

**Solution:** In Example 1.6 we have seen that the electric field inside and outside the sphere is given by

\[
\vec{E} = \begin{cases} 
\frac{qr}{4\pi\epsilon_0 R^2} \hat{r} & r < R \\
\frac{q}{4\pi\epsilon_0 r^2} \hat{r} & r > R 
\end{cases}
\]  

(2.153)

then the electric potential outside the sphere \((r > R)\)

\[
V(\vec{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = -\frac{q}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{1}{r^2} \hat{r} \cdot d\vec{r} = \frac{q}{4\pi\epsilon_0} \frac{1}{r} \left|_{\infty}^{r} \right.
\]

\[
\Rightarrow V(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{r}.
\]  

(2.154)

and inside the sphere \((r < R)\)

\[
V(\vec{r}) = -\int_{R}^{r} \vec{E} \cdot d\vec{r} - \int_{r}^{\infty} \vec{E} \cdot d\vec{r} = -\int_{R}^{r} \frac{qr}{4\pi\epsilon_0 R^2 r^2} \hat{r} \cdot d\vec{r} - \int_{r}^{\infty} \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot d\vec{r}
\]

\[
= -\frac{q}{4\pi\epsilon_0 R^3} \frac{r^2}{2} \left|_{R}^{r} \right. + \frac{q}{4\pi\epsilon_0} \frac{1}{r} \left|_{r}^{\infty} \right. = -\frac{qr^2}{8\pi\epsilon_0 R^3} + \frac{q}{8\pi\epsilon_0 R} + \frac{q}{4\pi\epsilon_0} R
\]

\[
\Rightarrow V(r) = \frac{3q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3}.
\]  

(2.155)

Unlike the potential inside a spherical shell where the potential is a constant, here the potential changes with \( r \)

2.6 Laplace’s and Poisson’s equations

In the previous section we have seen that in electrostatic the electric field is irrotational \((\nabla \times \vec{E}(\vec{r}) = 0)\) and the field can be expressed as a gradient of a scalar function \((\vec{E}(\vec{r}) = -\nabla V(\vec{r}))\). The change in the electric field in space depends the charge distribution in the region \((\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0})\), Gauss’s law. Replacing the electric field by the gradient of the electric potential results in Poisson’s equation

\[
\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \Rightarrow \nabla \cdot (-\nabla V(\vec{r})) = \frac{\rho(\vec{r})}{\epsilon_0} \Rightarrow \nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}.
\]  

(2.158)

If there is no charge in the region \((\rho(\vec{r}) = 0)\), one finds

\[
\nabla^2 V = 0,
\]  

(2.159)

which is Laplace’s equation for the electric potential.
2.6. LAPLACE’S AND POISSON’S EQUATIONS

(a) A point charge: Using the electric field for a single point charge, \( q \), and taking the reference point at infinity, one can easily show that the electric potential is found to be

\[
V(\vec{r}) = \frac{q}{4\pi \varepsilon_0} \frac{1}{|\vec{r} - \vec{r}'|}.
\] (2.160)

(b) \( N \) point charges: For a collection of \( N \) point charges with charge \( q_1, q_2, q_3...q_N \) positioned at \( \vec{r}_1', \vec{r}_2', \vec{r}_3'...\vec{r}_N' \), respectively, the electric potential at a point in free space described by the position vector \( \vec{r}' \), can be determined using

\[
V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i'|}.
\] (2.161)
CHAPTER 2. ELECTROSTATICS

(c) Continuous charge: for a charge distributed continuously on a given line, surface, or volume, the electric potential can be obtained using

\[ V(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{dq'}{|\vec{r} - \vec{r}'|}. \]  

(2.162)

For line charge with charge density \( (\lambda) \), surface charge with charge density \( (\sigma) \), and volume charge with charge density \( (\rho) \), one can express Eq. (2.162) as

\[ V(\vec{r}) = \begin{cases} 
\frac{1}{4\pi \epsilon_0} \int \frac{\rho(\vec{r}') \, d\vec{r}'}{|\vec{r} - \vec{r}'|}, & \text{volume charge} \\
\frac{1}{4\pi \epsilon_0} \int \frac{\sigma(\vec{r}') \, d\vec{a}'}{|\vec{r} - \vec{r}'|}, & \text{surface charge} \\
\frac{1}{4\pi \epsilon_0} \int \frac{\lambda(\vec{r}') \, d\vec{l}'}{|\vec{r} - \vec{r}'|}, & \text{line charge}
\end{cases} \]  

(2.163)

Example 1.14 Find the potential of a charge \( q \) that is uniformly distributed over spherical shell of radius \( R \). (i.e. \( \sigma = q/4\pi R^2 \))

Solution: For a surface charge density the potential is given by

\[ V(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{\sigma(\vec{r}') \, d\vec{a}'}{|\vec{r} - \vec{r}'|} \]  

(2.164)

For a uniform charge distribution, \( \sigma(\vec{r}') = \sigma \) is a constant. the infinitesimal area in spherical coordinates is given by

\[ d\vec{a}' = R^2 \sin \theta' \, d\theta' \, d\phi'. \]  

(2.165)
If we consider the point $p$ where we want to evaluate the electric potential be on the $z-$axis as shown in the figure, we can write

$$|\vec{r} - \vec{r}'| = \sqrt{z^2 + R^2 - 2zR\cos\theta'}.$$  

Then the electric potential

$$V(\vec{r}) = \frac{\sigma}{4\pi \varepsilon_0} \int_0^\pi \int_0^{2\pi} \frac{R^2 \sin\theta' \, d\theta' \, d\varphi''}{\sqrt{z^2 + R^2 - 2zR\cos\theta'}}$$

$$= \frac{\sigma 2\pi R^2}{4\pi \varepsilon_0} \int_0^\pi \frac{\sin\theta' \, d\theta'}{\sqrt{z^2 + R^2 - 2zR\cos\theta'}} = \frac{\sigma R^2}{2\varepsilon_0} \int_{-1}^1 \frac{du}{\sqrt{z^2 + R^2 - 2zRu}}$$

$$= -\frac{\sigma R^2}{2\varepsilon_0} \frac{2\sqrt{z^2 + R^2 - 2zR} - 2zR}{2zR} \biggr|_{-1}^1 = \frac{\sigma R}{2\varepsilon_0 z} \left[ \sqrt{z^2 + R^2 + 2zR} - \sqrt{z^2 + R^2 - 2zR} \right]$$

$$\Rightarrow V(\vec{r}) = \frac{\sigma R}{2\varepsilon_0 z} \left( \sqrt{(z + R)^2} - \sqrt{(z - R)^2} \right)$$

(2.166)

If point $p$ is outside the spherical shell we have $z > R$, and we find

$$V(\vec{r}) = \frac{\sigma R}{2\varepsilon_0 z} [z + R - (z - R)] = \frac{\sigma R^2}{\varepsilon_0 z}$$

(2.167)

and if it is inside $z < R$

$$V(\vec{r}) = \frac{\sigma R}{2\varepsilon_0 z} [z + R - (R - z)] = \frac{\sigma R}{\varepsilon_0}$$

(2.168)
2.7 The electric field and potential at a boundary

Consider two regions (region 1 and region 2) shown in Fig. ???. These two regions are considered to be a free space (for now) with electrical permittivity $\varepsilon_0$. Let’s consider that at the boundary separating the two regions there is some sort of surface charge with density $\sigma (\vec{r})$. Suppose there is some electric field created by some charge some where. Let this electric field in region 1 be $\vec{E}_1 (\vec{r})$ and in region 2 be $\vec{E}_2 (\vec{r})$.

Let the electric fields can be decomposed into a normal ($\hat{n}$) and tangential ($\hat{t}$) components at a given point on the surface as

$$\vec{E}_1 (\vec{r}) = E_1^\perp (\vec{r}) \hat{n} + E_1^\parallel (\vec{r}) \hat{t},$$

$$\vec{E}_2 (\vec{r}) = E_2^\perp (\vec{r}) \hat{n} + E_2^\parallel (\vec{r}) \hat{t}.$$ (2.169a)

Consider the Pillbox Gaussian surface with area $\Delta A$ and thickens $T$ shown in Fig. 2.31. Applying Gauss’s law for the pillbox shown in this figure, one can write

$$\oint_S \vec{E}.d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{\Delta A} \sigma (\vec{r}) d\vec{a}$$

$$\Rightarrow \int_{\text{Below}} \vec{E}_1 (\vec{r}).d\vec{a}_1 + \int_{\text{Above}} \vec{E}_2 (\vec{r}).d\vec{a}_2 + \int_{\text{side}} \vec{E} (\vec{r}).d\vec{a} = \frac{1}{\varepsilon_0} \int_{\Delta A} \sigma (\vec{r}) d\vec{a}$$ (2.170)

We are interested in the behavior of the electric field at the boundary. So we consider the case where the thickness of the pillbox becomes zero. In this limit
2.7. THE ELECTRIC FIELD AND POTENTIAL AT A BOUNDARY

condition \fig.

Figure 2.31: A cylindrical pillbox Gaussian surface of area $\Delta A$ and thickness $T$.

$T \to 0$, the flux contribution from the side is zero

$$\int_{\text{side}} \vec{E} \cdot d\vec{a} = 0.$$  \hspace{1cm} (2.171)

Moreover, in this limit, from Fig. 2.31 we note that

$$d\vec{a}_1 = -\hat{n}da, d\vec{a}_2 = \hat{n}da.$$  \hspace{1cm} (2.172)

Now substituting Eqs. (2.171), (2.172), and (2.162) into Eq. (2.170), we have

$$\int_{\Delta A} \left( E_2^+ (\vec{r}) \hat{n} + E_1^+ (\vec{r}) \hat{t} \right) \cdot \hat{n} da + \int_{\Delta A} \left( E_1^+ (\vec{r}) \hat{n} + E_1^+ (\vec{r}) \hat{t} \right) \cdot (-\hat{n} da)

= \frac{1}{\epsilon_0} \int_{\Delta A} \sigma (\vec{r}) \cdot da \Rightarrow \int_{\Delta A} E_2^+ (\vec{r}) \cdot da - \int_{\Delta A} E_1^+ (\vec{r}) \cdot da = \frac{1}{\epsilon_0} \int_{\Delta A} \sigma (\vec{r}) \cdot da.

\Rightarrow \int_{\Delta A} \left( E_2^+ (\vec{r}) da - E_1^+ (\vec{r}) da \right) = \int_{\Delta A} \frac{\sigma (\vec{r})}{\epsilon_0} da.$$  \hspace{1cm} (2.173)

There follows that

$$E_2^+ (\vec{r}) da - E_1^+ (\vec{r}) da = \frac{\sigma (\vec{r})}{\epsilon_0}.$$  \hspace{1cm} (2.174)

The normal component of the electric field is discontinuous at the boundary of the two regions if there is a surface charge at the interface of the two regions.

In order to see what happens to the tangential component of the electric field we will apply the irrotational nature of the electric field and Stoke’s theorem (i.e. the electric field is a conservative vector field),

$$\nabla \times \vec{E} = 0 \Rightarrow \int_{\text{surface}} \left( \nabla \times \vec{E} \right) \cdot d\vec{a} = \oint_{\text{closed curve}} \vec{E} \cdot d\vec{l} = 0.$$  \hspace{1cm} (2.175)

Let’s consider the rectangular closed curve shown in Fig. 2.31. The rectangle has
length, $L$ (parallel to the surface) and height, $H$ (perpendicular to the surface). The line integral over this rectangle can be expressed as

$\int \vec{E} \cdot d\vec{l} = \int_{(1)} \vec{E}_1 \cdot d\vec{l}_1 + \int_{(2)} \vec{E}_2 \cdot d\vec{l}_2 + \int_{(a)} \vec{E}_a \cdot d\vec{l}_a + \int_{(b)} \vec{E}_b \cdot d\vec{l}_b = 0. \quad (2.176)$

Note that the we labeled the electric field on the sides of rectangle labeled (a) and (b) as $\vec{E}_a$ and $\vec{E}_b$. But it would be $\vec{E}_1$ or $\vec{E}_2$ depending whether the part of these sides of the rectangle are in region 1 or region 2, respectively. In terms of the unit vectors $\hat{t}$ and $\hat{n}$, one can write

$\vec{d}l_1 = -dL \hat{t}, \vec{d}l_2 = dL \hat{t}, \vec{d}l_a = dH \hat{n},$ and $\vec{d}l_b = -dH \hat{n}, \quad (2.177a)$

$\vec{E}_1 (\vec{r}) = E_1^1 (\vec{r}) \hat{n} + E_1^1 (\vec{r}) \hat{t}, \vec{E}_2 (\vec{r}) = E_2^1 (\vec{r}) \hat{n} + E_2^1 (\vec{r}) \hat{t}, \quad (2.177b)$

$\vec{E}_a (\vec{r}) = E_a^1 (\vec{r}) \hat{n} + E_a^1 (\vec{r}) \hat{t}, \vec{E}_b (\vec{r}) = E_b^1 (\vec{r}) \hat{n} + E_b^1 (\vec{r}) \hat{t}, \quad (2.177c)$

where we considered the closed line integral be in a counterclockwise direction. Substituting the expressions in Eq. (2.177a) into Eq. (2.176), we find

$- \int_{(1)} \left( E_1^1 (\vec{r}) \hat{n} + E_1^1 (\vec{r}) \hat{t} \right) \cdot dL \hat{t} + \int_{(2)} \left( E_2^1 (\vec{r}) \hat{n} + E_2^1 (\vec{r}) \hat{t} \right) \cdot dL \hat{t}$

$+ \int_{(a)} \left( E_a^1 (\vec{r}) \hat{n} + E_a^1 (\vec{r}) \hat{t} \right) \cdot dH \hat{n} - \int_{(b)} \left( E_b^1 (\vec{r}) \hat{n} + E_b^1 (\vec{r}) \hat{t} \right) \cdot dH \hat{n} = 0$

$\Rightarrow - \int_{(1)} E_1^1 (\vec{r}) dL + \int_{(2)} E_2^1 (\vec{r}) dL + \int_{(a)} E_a^1 (\vec{r}) dH - \int_{(b)} E_b^1 (\vec{r}) dH = 0$

$\int_{0}^{L} E_1^1 (\vec{r}) dL + \int_{0}^{L} E_2^1 (\vec{r}) dL + \int_{0}^{H} E_a^1 (\vec{r}) dH - \int_{0}^{H} E_b^1 (\vec{r}) dH = 0 \quad (2.178)$

At the boundary we have to let $H \to 0$. Thus at the interface of the two regions

$\int_{0}^{H} E_a^1 (\vec{r}) dH = \int_{0}^{H} E_b^1 (\vec{r}) dH = 0 \quad (2.179)$
and one finds
\[ \int_0^L \left[ E_2^\parallel (\vec{r}) - E_1^\parallel (\vec{r}) \right] dL = 0. \tag{2.180} \]

There follows that
\[ \left[ E_2^\parallel (\vec{r}) = E_1^\parallel (\vec{r}) \right] \tag{2.181} \]

which shows that the tangential component of the electric field is continuous at the boundary of two regions independent of the surface charge density at the interface. Combining Eqs. (2.174) and (2.181), we can write for the electric field at the boundary of the two regions
\[ \vec{E}_2 (\vec{r}) - \vec{E}_1 (\vec{r}) = \sigma (\vec{r}) \frac{\hat{n}}{\epsilon_0} \tag{2.182} \]

which shows that \textit{Generally the electric field is discontinuous at a boundary.} \textit{Note that} \( \hat{n} \) \textit{is the unit vector normal to the interface of the two regions where the surface charge is distributed and must be directed from region 1 to region 2 as shown in Fig. ?? so that Eq. (2.182) to be true.}

The electric potential is continuous at the boundary of the two regions. It is independent of the surface charge at the interface of the two regions. This can be easily shown using the relation
\[ V_2 (\vec{r}) - V_1 (\vec{r}) = - \int_{(1)}^{(2)} \vec{E} (\vec{r}) \cdot d\vec{l} \tag{2.183} \]

along the line joining point (1) and point (2) in the two regions shown in Fig.2.7. At the boundary we have to let the length between these two point be zero. In this limit,
\[ V_2 (\vec{r}) - V_1 (\vec{r}) = \int_{(1)}^{(2)} \vec{E} (\vec{r}) \cdot d\vec{l} = 0. \tag{2.184} \]

that leads to
\[ V_2 (\vec{r}) = V_1 (\vec{r}) \tag{2.185} \]

at the boundary of the two regions. Note that the gradient of \( V \) is discontinuous since \( \vec{E} = -\nabla V \) which we proved to be discontinuous.
2.8 Electrical work and energy

In the previous sections we know that for \( N \) point charges with charge \( q_1, q_2, q_3, \ldots q_N \) positioned at \( \vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_N \), respectively, the electric field and the electric potential at a point in free space described by the position vector \( \vec{r} \), given by

\[
V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|}.
\]  
(2.186)

and

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i).
\]  
(2.187)

We now want to know how much electrical work should be done to move a charge from one position to another in the region this electric field exist. Consider a positive test charge \( Q \) initially positioned at \( \vec{r}_1 \) is moved along the path shown in Fig. (2.32) to a new position, \( \vec{r}_2 \).

The work that one must do to move this charge is

\[
W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r}.
\]  
(2.188)

Since this work done is against the repulsive electrical force, the force must balance the electrostatic force on the test charge \( Q \) (i.e. \( \vec{F}(\vec{r}) = -\vec{F}_e(\vec{r}) \)). The
2.8. ELECTRICAL WORK AND ENERGY

electrical force, \( \vec{F}_e (\vec{r}) \), can be expressed in terms of the total electric field by all point charges, as

\[
\vec{F} (\vec{r}) = -Q \vec{E} (\vec{r}).
\]  
(2.189)

The work done can then be put in the form

\[
W = -Q \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} (\vec{r}) \cdot d\vec{r}.
\]  
(2.190)

From the relationship between electrical field and potential in electrostatics, we note that

\[
\vec{E} (\vec{r}) = \nabla V (\vec{r}) \Rightarrow V (\vec{r}_2) - V (\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} (\vec{r}) \cdot d\vec{r}.
\]  
(2.191)

Therefore, the work done becomes

\[
W = Q (V (\vec{r}_2) - V (\vec{r}_1))
\]  
(2.192)

The work done to move a test charge from infinity, \( \vec{r}_1 = \infty \) (where the potential considered to be zero, \( V (\infty) = 0 \)) to a point described by a position vector, \( \vec{r}_2 = \vec{r} \) can then be expressed as

\[
W = Q (V (\vec{r}) - V (\infty)) = QV (\vec{r}).
\]  
(2.193)

Using the expression for the electrostatic potential \( V (\vec{r}) \) in Eq. (2.186), one can write

\[
W = QV (\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{Qq_i}{|\vec{r} - \vec{r}_i|},
\]  
(2.194)

**Electrostatic energy (discreet charge configuration):** The energy of a point charge distribution can be determined using

\[
W = \frac{1}{2} \sum_{i=1}^{N} q_i V (\vec{r}_i),
\]  
(2.195)

where \( V (\vec{r}_i) \) is the potential due to all other charges at position \( i \) except the charge located at \( i \) (\( q_i \)).

Next we want to derive Eq. (2.195). To this end, consider two-point-charge configuration, \( q_1 \) and \( q_2 \) located at positions \( \vec{r}_1 \) and \( \vec{r}_2 \), respectively. The energy of this configuration is the same as the magnitude of the work done required to put them in this configuration,

\[
W_2 = q_2 (V (\vec{r}_2) - V (\infty)) = q_2 V (\vec{r}_2) \Rightarrow W_2 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|},
\]  
(2.196)

which we may put it in the form

\[
W_2 = \frac{1}{2} \left[ \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_1}{|\vec{r}_1 - \vec{r}_2|} \right].
\]  
(2.197)
Then the energy for the two-point-charge configuration can be expressed as

$$W = W_2 = \frac{1}{2} \left[ \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{2} \sum_{j=1 \atop j \neq i}^{2} \frac{q_i q_j}{r_i^2 - r_j^2} \right]$$

(2.198)

The additional work that must be done if we want to add a third charge \( q_3 \) at position \( \vec{r}_3 \) (three-point-charge configuration) is

$$W_3 = q_3 (V(\vec{r}_{13}) + V(\vec{r}_{23})) \Rightarrow W_3 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|}$$

(2.199)

that can be rewritten as

$$\Rightarrow W_3 = \frac{1}{2} \left[ \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} + \frac{1}{4\pi\varepsilon_0} \frac{q_3 q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{1}{4\pi\varepsilon_0} \frac{q_3 q_2}{|\vec{r}_3 - \vec{r}_2|} \right].$$

(2.200)

Then the total work done (the energy of the three-point-charge configuration)
would be

\[ W = W_2 + W_3 + W_4 = \frac{1}{2} \left[ \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|r_2 - r_1|} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_1}{|r_1 - r_2|} \right. \\
\left. + \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{|r_2 - r_3|} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{|r_3 - r_2|} + \frac{1}{4\pi\epsilon_0} \frac{q_3 q_1}{|r_3 - r_1|} + \frac{1}{4\pi\epsilon_0} \frac{q_3 q_2}{|r_1 - r_3|} \right] \]

\[ \Rightarrow W = \frac{1}{2} \left[ \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} \frac{q_i q_j}{|r_i - r_j|} \right] \]  
(2.201)

If we add a fourth charge \( q_4 \) at position \( \vec{r}_4 \), the additional work that must be done

\[ W_4 = q_4 (V(\vec{r}_1) + V(\vec{r}_2) + V(\vec{r}_3)) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_4}{|r_1 - r_4|} + \frac{q_2 q_4}{|r_2 - r_4|} + \frac{q_3 q_4}{|r_3 - r_4|} \right\} \]

\[ \Rightarrow W_4 = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_4}{|r_1 - r_4|} + \frac{q_2 q_4}{|r_2 - r_4|} + \frac{q_3 q_4}{|r_3 - r_4|} \right\} . \]  
(2.202)

Then the energy for four-point-charge configuration becomes

\[ W = W_2 + W_3 + W_4 = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_1 q_2}{|r_2 - r_1|} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_1}{|r_1 - r_2|} + \frac{q_1 q_3}{|r_2 - r_3|} + \frac{q_2 q_3}{|r_3 - r_2|} + \frac{q_1 q_4}{|r_3 - r_1|} + \frac{q_3 q_4}{|r_3 - r_1|} + \frac{q_2 q_4}{|r_1 - r_4|} + \frac{q_3 q_2}{|r_3 - r_1|} \right\} , \]  
(2.203)

which can be put in the form

\[ W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{4} \sum_{j=1, j\neq i}^{4} \frac{q_i q_j}{|r_i - r_j|} = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{|r_i - r_j|} . \]  
(2.204)

Noting that

\[ V(\vec{r}_i) = \sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{|r_i - r_j|} \]  
(2.205)

is the potential at position \( \vec{r}_i \) due to all charges except the charge \( q_i \) located at this position, we may write the energy for \( N \)-point-charge configuration as

\[ W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i) . \]  
(2.206)
Continuous charge distribution: The expression for the energy for a number of point charges can be generalized for a continuous charge distribution as

\[ W = \frac{1}{2} \int dq V(r) \quad (2.207) \]

where

\[ dq = \begin{cases} \rho d\tau, & \text{volume charge} \\ \sigma d\alpha, & \text{surface charge} \\ \lambda dl, & \text{line charge} \end{cases} \quad (2.208) \]

Let’s consider the energy for a volume charge density

\[ W = \frac{1}{2} \int \rho V d\tau. \quad (2.209) \]

Using the differential form of Gauss’s law

\[ \rho = \varepsilon_0 \nabla \cdot \vec{E} \quad (2.210) \]

one can express Eq. (2.209) as

\[ W = \frac{\varepsilon_0}{2} \int V (\nabla \cdot \vec{E}) d\tau. \quad (2.211) \]

Applying the divergence theorem, we may write

\[ \int_{\text{volume}} \nabla \cdot (V \vec{E}) d\tau = \oint_{\text{surface}} V \vec{E} \cdot d\vec{a} \quad (2.212) \]

and applying the product rule for differentiation of two functions, the volume integral can put in the form

\[ \int_{\text{volume}} \nabla \cdot (V \vec{E}) d\tau = \int_{\text{volume}} V (\nabla \cdot \vec{E}) d\tau + \int_{\text{volume}} \vec{E} \cdot (\nabla V) d\tau. \quad (2.213) \]

Taking the relation between the electric field and potential in electrostatic, have

\[ \vec{E} = -\nabla V \Rightarrow \vec{E} \cdot (\nabla V) = \vec{E} \cdot \vec{E} = -E^2 \quad (2.214) \]

so that Eq. (2.213) can be rewritten as

\[ \int_{\text{volume}} \nabla \cdot (V \vec{E}) d\tau = \int_{\text{volume}} V (\nabla \cdot \vec{E}) d\tau - \int_{\text{volume}} E^2 d\tau \]

\[ \Rightarrow \int_{\text{volume}} V (\nabla \cdot \vec{E}) d\tau = \int_{\text{volume}} E^2 d\tau + \oint_{\text{surface}} V \vec{E} \cdot d\vec{a} \quad (2.215) \]
Upon substituting Eq. (2.215) into Eq. (2.211), one finds another form of expression for the electrical energy

$$ W = \frac{\varepsilon_0}{2} \left[ \int_{\text{volume}} E^2 d\tau + \oint_{\text{surface}} V \vec{E} \cdot d\vec{a} \right]. $$

(2.216)

In Eq. (2.216) the volume integral is carried out over the region occupied by the charge with density $\rho(r)$ and the surface integral is also carried out over a surface enclosing the volume charge. The result of this expression would not change if the integrals are carried out over any space of volume and closed surface greater than the space occupied by the charge as long as the corresponding values for the potential and the electric fields are used. For a localized charge distribution, for a point in space far away from the charge we note that $V \sim 1/r$, $E \sim 1/r^2$, and Area $\sim r^2$, the surface integral become

$$ \oint_S V \vec{E} \cdot d\vec{a} \sim \frac{1}{r} \frac{1}{r^2} r^2 = \frac{1}{r}. $$

(2.217)

Thus for an infinite spherical space with infinite radius ($r = \infty$), the surface integral becomes zero, and we can express the electrical energy in terms of the electric field as

$$ W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d\tau. $$

(2.218)

Any one of Eqs. (2.209), (2.216), and (2.218) can be used to determine electrical energy but commonly Eq. (2.218) is used. From this form of the expression for the electrical energy one can write

$$ \frac{dW}{d\tau} = \frac{\varepsilon_0}{2} E^2, $$

(2.219)

which is the electrical energy density. We often see Eq. (2.218) in electromagnetic waves theory as it can be used to determine electromagnetic wave intensity in space. (see Example 1.18)

**Example 1.16**

(a) Three charges are situated at the corners of a square (side $a$), as shown in figure below. How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corner?

(b) How much work does it take to assemble the whole configuration of four charges?

**Solution:**
### (a) The work required to bring a fourth charge to the empty corner of the square can be easily obtained if we first determine the potential at this corner of the square. It is given by

\[
V(r) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{3} \frac{q_j}{|r - \vec{r}_j|} = \frac{1}{4\pi\varepsilon_0} \left( -\frac{q}{a} + \frac{q}{a\sqrt{2}} - \frac{q}{a} \right)
\]

\[
\Rightarrow \quad V(r) = -\frac{q}{4\pi\varepsilon_0a} \left( \frac{2\sqrt{2} - 1}{\sqrt{2}} \right) \quad (2.220)
\]

Then the work done to bring the charge +q from infinity to the fourth corner of the square is given by

\[
W = qV(r) \Rightarrow W = -\frac{q^2}{4\pi\varepsilon_0a} \left( \frac{2\sqrt{2} - 1}{\sqrt{2}} \right) \quad (2.221)
\]

### (b) The work required to assemble all the charges into the square configuration shown in the figure is given by

\[
W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i) \quad (2.222)
\]

where \(V(\vec{r}_i)\) is the potential at position \(\vec{r}_i\), where charge \(q_i\) is residing due to the presence of the remaining charges. If we call the four charges \(q_1 = -q, q_2 = +q, q_3 = -q,\) and \(q_4 = +q\), then we may write

\[
W = \frac{1}{2} \left[ -qV(\vec{r}_1) + qV(\vec{r}_2) - qV(\vec{r}_3) + qV(\vec{r}_4) \right]. \quad (2.223)
\]

We note that

\[
V(\vec{r}_1) = V(\vec{r}_3) = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{a} - \frac{q}{a\sqrt{2}} + \frac{q}{a} \right) = \frac{q}{4\pi\varepsilon_0a} \left( \frac{2\sqrt{2} - 1}{\sqrt{2}} \right) \quad (2.224)
\]

\[
V(\vec{r}_2) = V(\vec{r}_4) = \frac{1}{4\pi\varepsilon_0} \left( -\frac{q}{a} + \frac{q}{a\sqrt{2}} - \frac{q}{a} \right) = -\frac{q}{4\pi\varepsilon_0a} \left( \frac{2\sqrt{2} - 1}{\sqrt{2}} \right) \quad (2.225)
\]
The work required will then be

\[ W = -\frac{1}{2} \frac{q}{4\pi \varepsilon_0 a} \left( \frac{2\sqrt{2} - 1}{\sqrt{2}} \right) (q + q + q) \Rightarrow W = -\frac{q^2}{2\pi \varepsilon_0 a} \left( \frac{2\sqrt{2} - 1}{\sqrt{2}} \right) \]  

(2.226)

Example 1.17 Find the energy stored in a uniformly charged solid sphere of radius \( R \) and charge \( q \), using

(a)

\[ W = \frac{1}{2} \int \rho V d\tau \]  

(2.227)

(b)

\[ W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d\tau. \]  

(2.228)

(c)

\[ W = \frac{\varepsilon_0}{2} \left[ \int_{\text{vol}} E^2 d\tau + \int_S V E\, d\mathbf{a} \right]. \]  

(2.229)

Take a spherical volume of radius \( a \). What happens as \( a \to \infty \).

Solution:

(a) In example 2.12 we have determined the electric potential for a volume charge distribution (Eqs. (2.155) and (2.157)) to be

\[ V(\hat{r}) = \begin{cases} 
\frac{q}{8\pi \varepsilon_0 R} \left( -\frac{1}{3} r^2 \right) & r > R \\
\frac{3q}{8\pi \varepsilon_0 R} - \frac{qr^2}{8\pi \varepsilon_0 R^3} & r < R 
\end{cases} \]  

(2.230)

since \( W = \frac{1}{2} \int \rho V d\tau \) should be integrated over the region where there is charge, we then use \( V(\hat{r}) \) in the region \( r < R \). Hence

\[ W = \frac{1}{2} \int \rho V d\tau = \frac{1}{2} \int \frac{q}{3\pi R^3} \left( \frac{3q}{8\pi \varepsilon_0 R} - \frac{qr^2}{8\pi \varepsilon_0 R^3} \right) d\tau \]

\[ = \frac{3q}{8\pi R^3} \left[ \int_0^R \int_0^{2\pi} \int_0^\tau \left( \frac{3q}{8\pi \varepsilon_0 R} - \frac{qr^2}{8\pi \varepsilon_0 R^3} \right) r^2 \sin \theta dr d\theta d\phi \right] \]  

(2.231)

After integrating over \( \theta \) and \( \varphi \) we find

\[ W = \frac{3q^2}{16\pi \varepsilon_0} \left[ \int_0^R \frac{3r^2}{R^4} dr - \int_0^R \frac{r^4}{R^6} dr \right] = \frac{3q^2}{16\pi \varepsilon_0} \left[ 1 - \frac{1}{5} \right] = \frac{1}{4\pi \varepsilon_0} \frac{3q^2}{5} \]  

(2.232)
(b) In Example 1.6 we found the magnitude of the electric field inside and outside a uniformly charged solid sphere is given by

\[ E = \begin{cases} \frac{qr}{4\pi\varepsilon_0 r^2} & r < R \\ \frac{q}{4\pi\varepsilon_0 r^2} & r > R \end{cases} \]  

then the energy stored can be expressed as

\[ W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d\tau = \frac{\varepsilon_0}{2} \left[ \int_0^R \int_0^\pi \int_0^{2\pi} E^2 r^2 d\tau \right] \]

\[ = \frac{\varepsilon_0}{2} \left( \frac{q}{4\pi\varepsilon_0} \right)^2 \left[ \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{R^6} + \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{r^4} \right] r^2 dr \sin \theta d\theta d\phi. \]  

(2.234)

Integration over \( \phi \) and \( \theta \) gives

\[ W = \frac{q^2}{8\pi\varepsilon_0} \left[ \int_0^R \frac{r^4 dr}{R^6} + \int_R^\infty \frac{dr}{r^2} \right] \]  

(2.235)

and then over \( r \)

\[ W = \frac{q^2}{8\pi\varepsilon_0} \left[ \frac{1}{5R} + 1 \right] = \frac{1}{4\pi\varepsilon_0} \left[ \frac{3q^2}{5} \right] \]  

(2.236)

(c) Considering a spherical surface of radius \( a \) the total energy

\[ W = \frac{\varepsilon_0}{2} \left[ \int_{\text{vol}} E^2 d\tau + \oint_S V \hat{E} \cdot d\hat{a} \right] \]  

(2.237)

can be put in the form

\[ W = \frac{\varepsilon_0}{2} \int_0^R \int_0^\pi \int_0^{2\pi} E^2 r^2 d\tau \sin \theta d\theta d\phi \]

\[ + \frac{\varepsilon_0}{2} \int_0^a \int_0^\pi \int_0^{2\pi} E^2 r^2 d\tau \sin \theta d\theta d\phi \]

\[ + \frac{\varepsilon_0}{2} \int_0^\pi \int_0^{2\pi} V (\vec{a}) \left( \vec{E} (\vec{a}) \cdot \hat{r} \right) a^2 \sin \theta d\theta d\phi. \]  

(2.238)

where we used \( d\hat{a} = \hat{r} \hat{a} \). Using the electric field in Eq. (2.233) and the electric potential in Eq. (2.155), upon integrating over \( \phi \), we find

\[ W = \frac{q^2}{8\pi\varepsilon_0} \left[ \int_0^R \frac{r^4 dr}{R^6} + \int_R^a \frac{r^2 dr}{r^4} \right] \]

\[ + \frac{q^2}{8\pi\varepsilon_0 a} \int_0^\pi \hat{r} \cdot \hat{\theta} \sin \theta d\theta \]  

(2.239)

and then integrating over \( r \) and \( \theta \)

\[ W = \frac{q^2}{4\pi\varepsilon_0} \left[ \frac{3}{5R} - \frac{1}{2a} \right] + \frac{q^2}{4\pi\varepsilon_0} \left[ \frac{1}{2a} \right] \Rightarrow W = \frac{1}{4\pi\varepsilon_0} \left[ \frac{3q^2}{5} \right]. \]  

(2.240)

The result is independent of \( a \) as it should be.
Example 1.18 Show that the intensity of an electromagnetic radiation in a free space is given by

\[ I = \frac{1}{2} \epsilon_0 c E^2 \]  

(assuming only the electric field contributes to the energy of the field), where \( c \) is the speed of an electromagnetic field radiation in free space.

Solution: Noting that the energy density (the electrical) can be expressed as

\[ \frac{dW}{d\tau} = \frac{1}{2} \epsilon_0 E^2 \]  

and for a plane electromagnetic field in free space

\[ d\tau = Adl = Ac dt \]  

we may write

\[ \frac{dW}{Ac dt} = \frac{1}{2} \epsilon_0 E^2 \Rightarrow \frac{dW}{Adt} = \frac{1}{2} \epsilon_0 E^2 \Rightarrow I = \frac{P}{A} = \frac{1}{2} \epsilon_0 E^2. \]  

The intensity is proportional to the square of electric field amplitude.

2.9 Electric field and conductors

Different materials respond to an external electric field differently. Based on electrical response materials are classified as conductors and insulators. As you have introduced in modern physics, the states of the electrons in materials form what is known as energy bands. These bands known as the valence and

![Figure 2.34: Electrons states in (a) insulators and (b) conductors.](image)

...
when it is tightly bind to the its respective atoms or molecules in the material. The conduction band represent the allowed states of the electrons are freed from the atom. In between the conduction and valence band there is the band-gap that represent the states that the electrons can not occupy. These band structures for conductors and insulators are shown in Fig. 2.34. Figure 2.34 (b) shows, for conductors, the band-gap is much smaller than for insulators. At room temperature, the electrons can gain enough thermal energy to surpass the band-gap, be in the conduction band, and be able to move freely in the material. Because of these free electrons, conductors respond in a very different way than insulators in an external electric field that give it some unique properties. The following are the basic Properties of conductors.

(a) **Inside a conductor the electric field is zero:** There are plenty of free electrons at room temperatures in conductors. When an external electric field \( \vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) \) is applied to a conductor, it instantaneously creates an electrical force that pushes the free electrons away from the positively charged atoms in a conductor. This separation instantaneously leads to a counter acting electric field inside directed from the positively charged to the negatively charged surface \( \vec{E}_i(\vec{r}) = -\vec{E}_0(\vec{r}) \). As a result the net electric field inside \( \vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) + \vec{E}_i(\vec{r}) = 0 \) as shown in Fig. (2.35).

![Figure 2.35: Electric field inside a conductor.](image)

(b) **Inside a conductor the volume charge density is zero:** Any free charge added into conductor can only resides on the surface and the volume charge density, \( \rho(\vec{r}) = 0 \). In fact this is justified by Gauss’s law the property we discussed in (a). Inside a conductor, \( \vec{E}_i(\vec{r}) = 0 \),

\[
\rho(\vec{r}) = \frac{1}{\epsilon_0} \left( \nabla \cdot \vec{E}(\vec{r}) \right) = 0. \tag{2.245}
\]

(c) **Induced Charges:** Equal but opposite surface charges are induced on the nearest side of the surface of a conductor when it is brought close to external charge.

(d) **The electric field has only normal component on the surface of a conductor:** Suppose \( \vec{E}_{out}(\vec{r}) \) is the electric field near outside a conductor. This electric
field could be due to some external free charges added and distributed on the surface of the conductor (e.g. Fig. 2.36 (a) & (b)) or due to some distant charges producing induced charges on the surface of the conductor. In either case there will be some kind of surface charge distribution, \( \sigma (\vec{r}) \).

Figure 2.36: Electric field lines near the surface of a conductor.

From section 2.7, at a boundary that separates two regions (region 1 and region 2), we know that the electric field is discontinuous,

\[
\vec{E}_2 (\vec{r}) - \vec{E}_1 (\vec{r}) = \frac{\sigma (\vec{r})}{\epsilon_0} \hat{n},
\]

where \( \hat{n} \) is the unit vector normal to the surface pointing from region 1 to region 2. Let region 1 be inside and region 2 be outside the conductor. From the property discussed in (a), the field inside a conductor is zero, \( \vec{E}_1 (\vec{r}) = 0 \). Thus at the boundary, just outside the conductor, \( \vec{E}_2 (\vec{r}) = \vec{E}_{out} (\vec{r}) \), one can easily then see that

\[
\vec{E}_{out} (\vec{r}) = \frac{\sigma (\vec{r})}{\epsilon_0} \hat{n},
\]

is normal to the surface of the conductor.
(e) A conductor forms an equipotential region of space: Inside a conductor the electric field is zero and therefore
\[ \vec{E}(\vec{r}) = -\nabla V(\vec{r}) = 0 \Rightarrow V(\vec{r}) = \text{Constant.} \tag{2.248} \]

(f) The electric field within any cavity inside a conductor is zero: For a conductor with any shape and size of cavity within that has no charge, the electric field is zero. This can be easily proven if we use the irrotational nature of the electric field and the fact that the electric field inside a conductor is zero,
\[ \oint \vec{E} \cdot d\vec{l} = \int \int \left( \nabla \times \vec{E} \right) \cdot d\vec{a} = 0, \quad \vec{E}(\vec{r}) = 0 \]

Using these two equations for the closed curve shown in yellow, we note that the electric field must be zero inside the cavity.

**Example 1.19** An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (see figure below). Somewhere within the cavity is a charge \( q \). What is the field inside and outside the conductor?

**Solution:** We use Gauss’s law to find the electric field. If we consider a spherical Gaussian surface inside the sphere
\[ \int_s \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \tag{2.249} \]
Due to the charge \( q \) inside the cavity there will be an induced negative charge \(-q\) at the inner surface of the sphere. As a result the net charge enclosed by the Gaussian surface would be zero and therefore the electric field inside the spherical conductor would be zero. However, outside the sphere the net charge is \( q \) and the electric field will be

\[
E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}.
\]  

(2.250)

**Example 1.20** A metal sphere of radius \( R \), carrying a charge \( q \), is surrounded by a thick concentric metal shell (inner radius \( a \), outer radius \( b \), as shown below). The shell carries no net charge.

(a) Find the surface charge density \( \sigma \) at \( R \), at \( a \), and at \( b \)

(b) Find the electric field in the regions \( r < R \), \( R < r < a \), \( a < r < b \), and \( r > b \)

(c) Find the potential at the center, using infinity as the reference point

(d) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a), (b), and (c) change?

**Solution:**

(a) The sphere is a conductor. If it carries any charge the charge is uniformly distributed over the surface of the sphere. Hence the surface charge density at is

\[
\sigma = \frac{q}{4\pi R^2}.
\]  

(2.251)

At a (inner surface of the spherical shell) there would be an induced charge \(-q\) due to the positive charge on the spherical conductor. This induced surface charge is also distributed on the inner surface of the shell. Therefore the surface charge density is

\[
\sigma_{in} = -\frac{q}{4\pi a^2}.
\]  

(2.252)
At \( b \) there would positive induced charge which is distributed uniformly on the outer surface of the shell. This surface charge density is
\[
\sigma_{in} = \frac{q}{4\pi b^2}. \tag{2.253}
\]

(b) Using Gauss’s Law we can easily find the electric field in these regions.

(i) In the region \( r < R \) there is no charge \( (Q_{enc} = 0) \)
\[
\oint_s \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E} = 0 \tag{2.254}
\]

(ii) In the region \( R < r < a \) due to the charge on the surface of the metallic sphere the enclosed total charge inside a Gaussian sphere of radius \( r \) is \( Q_{enc} = q \). Hence,
\[
\oint_s \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E} = \frac{q}{4\pi \varepsilon_0 r^2} \hat{r} \tag{2.255}
\]

(iii) In the region \( a < r < b \) due to the induced charge \( -q \) on the inner surface of the spherical shell the total charge enclosed inside a Gaussian surface of radius \( r \) is zero \( (Q_{enc} = 0) \). Therefore,
\[
\oint_s \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E} = 0 \tag{2.256}
\]

(iv) In the region \( r > b \) the total charge enclosed in a Gaussian sphere of radius \( r > b \) is \( Q_{enc} = q \). The electric field will then be
\[
\oint_s \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E} = \frac{q}{4\pi \varepsilon_0 r^2} \hat{r} \tag{2.257}
\]

(c) The potential at the center can be expressed as
\[
V(0) = -\int_\infty^0 \vec{E} \cdot d\vec{r} = -\int_0^R \vec{E} \cdot d\vec{r} - \int_0^R \vec{E} \cdot d\vec{r} - \int_a^b \vec{E} \cdot d\vec{r} - \int_b^1 \vec{E} \cdot d\vec{r} \tag{2.258}
\]
Using the electric field obtained in part (b) we see that the first and the third terms are zero.
\[
V(0) = -\int_0^R \vec{E} \cdot d\vec{r} + \int_0^b \vec{E} \cdot d\vec{r} - \int_0^R \frac{q}{4\pi \varepsilon_0 r^2} dr - \int_0^b \frac{q}{4\pi \varepsilon_0 r^2} \int_0^R \frac{dr}{r^2} + \frac{q}{4\pi \varepsilon_0} \int_0^b \frac{dr}{r^2}
\]
\[
= -\frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{a} - \frac{1}{R} - \frac{1}{b} \right]
\]
\[
\Rightarrow V(0) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{a} + \frac{1}{b} - \frac{1}{a} \right] \tag{2.259}
\]
The surface charge density at $R$ and $a$ is still the same. On the surface of the shell due to the grounding it becomes zero. As a result the electric field in the region $r < R$ and $R < r < a$ is going to be the same but in the region $r > b$ the electric field becomes zero since the net charge enclosed is zero. Therefore the potential

$$V(0) = -\int_{R}^{0} \vec{E} \cdot d\vec{r} - \int_{a}^{R} \vec{E} \cdot d\vec{r} - \int_{b}^{a} \vec{E} \cdot d\vec{r} - \int_{\infty}^{b} \vec{E} \cdot d\vec{r}$$

becomes

$$V(0) = -\int_{R}^{0} 0 \cdot d\vec{r} - \int_{a}^{R} \frac{q}{4\pi\varepsilon_{0}r^{2}} dr - \int_{b}^{a} 0 \cdot d\vec{r} - \int_{\infty}^{b} 0 \cdot d\vec{r} = -\int_{a}^{R} \frac{q}{4\pi\varepsilon_{0}r^{2}} dr$$

$$\Rightarrow V(0) = \frac{q}{4\pi\varepsilon_{0}} \left( \frac{1}{R} - \frac{1}{a} \right)$$

**Example 1.21** "The Execution Cavity": A guy designed the following device which he calls "Execution cavity" and claims it is more humane way of execution that is better than Lethal injection. It consists of a spherical cavity of radius $a$ surrounded by a conducting spherical shell as shown in the figure below. The spherical shell has an outer radius $b$. According to the guy if a living person is put in this cavity (assume there is enough oxygen to breath) and the outer surface of the conductor is set at a very high voltage, the person will be **Electrocuted** and die quickly with less pain. Is Electrocution the cause of death? Justify your answer with right physical explanation.

![Diagram of Execution Cavity](https://via.placeholder.com/150)

**Solution:**

The cause of death is "cooking" due to heating caused by the collision of the free charge carriers to the static atoms or ions. The free charges are forced to move within the conductor by the applied high voltage. The heat energy is radiated in all directions including into the cavity raising the temperature of the cavity. Electrocution can not take place inside the cavity since the electric field everywhere inside the cavity is zero which means the electric potential is constant. To cause electrocution the electric field must be different from zero inside the cavity.
Surface charge and the force on a conductor: The force per unit area on a conductor carrying a surface charge density $\sigma$

$$\vec{f} = \sigma \vec{E}_{\text{ave}}$$  \hspace{1cm} (2.262)

where

$$\vec{E}_{\text{ave}} = \frac{1}{2} \left( \vec{E}_{\text{above}} + \vec{E}_{\text{below}} \right)$$  \hspace{1cm} (2.263)

The electrostatic pressure on the surface, tending to draw the conductor into the field,

$$P = \frac{\varepsilon_0}{2} E^2$$  \hspace{1cm} (2.264)

Figure 2.37: Electrostatic pressure

Example 1.22 Two large metal plates (each of area $A$) are held a distance $d$ apart. Suppose we put a charge $Q$ on each plate; what is the electrostatic pressure on the plates?

Solution: the surface charge density on each of the plate is given by

$$\sigma = \frac{Q}{A}$$  \hspace{1cm} (2.265)

the electric field between the plate is zero and outside the plate is

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$  \hspace{1cm} (2.266)

then the electrostatic pressure will be

$$P = \frac{\varepsilon_0}{2} E^2 = \frac{\varepsilon_0}{2} \left( \frac{Q}{\varepsilon_0 A} \right)^2 = \frac{1}{2\varepsilon_0} \left( \frac{Q}{A} \right)^2$$  \hspace{1cm} (2.267)
Capacitors: two conductors that can store equal and opposite charges ($\pm Q$), with a potential difference between it. The magnitude of the potential difference between the two conductors, $V$, is proportional to the magnitude of the charge stored on the capacitor,

$$C = \frac{|Q|}{V},$$  \hspace{1cm} (2.268)

where the constant $C$ is the constant of proportionality known as the capacitance of the conductors. Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation length between the two conductors. In SI units $C$ is measured in farads (F).
Figure 2.41: A parallel plate capacitor with decreased separation distance, increased surface area, and a dielectric inserted.

**Example 1.23** Find the capacitance of a "parallel-plate capacitor" consisting of two metal surfaces of area $A$ held a distance $d$ apart.

**Solution:** The electric field in the region between the parallel plates is $E = \frac{\sigma}{\epsilon_0}$. The potential on the positive plate relative to the negative plate can be expressed as

$$V = -\int_d^0 \vec{E} \cdot d\vec{r} = -\frac{\sigma}{\epsilon_0} \int_d^0 dr = \frac{\sigma d}{\epsilon_0}$$ (2.269)

Then the capacitance

$$C = \frac{Q}{V} = \frac{\sigma A}{\sigma d/\epsilon_0} = \frac{\epsilon_0 A}{d}.$$ (2.270)

**Example 1.24** Find the capacitance of two concentric spherical metal shells, with radii $a$ and $b$.

**Solution:** The electric potential on the positively charged shell relative to the negatively charged shell is given by

$$V = -\int_b^a \vec{E} \cdot d\vec{r}.$$ (2.271)

If we assume the inner shell to be positively charged, using Gauss’s Law it can be shown that the electric field in the region between $a < r < b$ is

$$\vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \hat{r}.$$ (2.272)

Hence the electric potential

$$V = -\int_b^a \vec{E} \cdot d\vec{r} = -\frac{Q}{4\pi \epsilon_0} \int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi \epsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right).$$ (2.273)

Then the capacitance

$$C = \frac{Q}{V} = \frac{\frac{Q}{4\pi \epsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right)}{b-a} = \frac{4\pi \epsilon_0 ab}{b-a}.$$ (2.274)
Example 1.25 For two long coaxial metal cylindrical tubes of radius \( a \) and \( b \) (see Fig. 2.42)

(a) Find the capacitance per unit length of

(b) Show that the energy stored in a capacitor with a capacitance \( C \) and total charge of \( Q \) on one is given by

\[
W = \frac{1}{2} \frac{Q}{C} = \frac{1}{2} CV^2
\]

where \( V \) is the potential difference across the capacitor.

Figure 2.42: Cylindrical capacitor.

Solution:

(a) If the inner metallic cylinder carries a surface charge of \( \sigma \) using Gauss’s law we can easily show that the electric field in the region \( a < r < b \) to be

\[
\int_S \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \Rightarrow E2\pi rl = \frac{\sigma 2\pi al}{\varepsilon_0} \Rightarrow E = \frac{\sigma a}{\varepsilon_0 r} \Rightarrow \vec{E} = \frac{\sigma a}{\varepsilon_0 r} \hat{r}.
\]

The electric potential on the inner cylinder as measured relative to the outer shell is

\[
V = -\int_b^a \vec{E} \cdot d\vec{r} = -\frac{\sigma a}{\varepsilon_0 r} \int_b^a \frac{dr}{r} = \frac{\sigma a}{\varepsilon_0} \ln \left( \frac{b}{a} \right)
\]

Noting that the charge on the shell is \( Q = \sigma 2\pi al \), the capacitance

\[
C = \frac{Q}{V} = \frac{\sigma 2\pi al}{\varepsilon_0} \frac{\varepsilon_0}{\ln \left( \frac{b}{a} \right)} = \frac{2\pi \varepsilon_0 l}{\ln \left( \frac{b}{a} \right)}
\]

Therefore, the capacitance per unit length is found to be

\[
\frac{C}{l} = \frac{2\pi \varepsilon_0}{\ln \left( \frac{b}{a} \right)}
\]

(b) Energy of a Capacitor:
Chapter 3

Special techniques

In this chapter we will learn how we determine the electric potential and field in space where the charge is localized to a specific region. Specifically, we will focus how when charge is localized to surfaces separating regions. We recall that in electrostatics, the electric field is a conservative vector field that can be expressed in terms of the electrostatic potential,

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}).$$

(3.1)

In terms of the potential, we saw that Gauss’s law leads to Poisson’s equation

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0} \Rightarrow \nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}.$$  

(3.2)

In the region of space where there is no charge ($\rho(\vec{r}) = 0$), we also saw Poisson’s equation becomes Laplace’s equation

$$\nabla^2 V(\vec{r}) = 0.$$  

(3.3)

At a point on the boundary separating two regions, we have shown that the electric field is discontinuous by the surface charge density at that point but the potential is independent of the surface charge density and it is continuous,

$$\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r}) = \frac{\sigma(\vec{r})}{\varepsilon_0} \hat{n},$$

(3.4a)

$$V_1(\vec{r}) = V_2(\vec{r})$$

(3.4b)

where $V_1(\vec{r}) \left( \vec{E}_1(\vec{r}) \right)$ and $V_2(\vec{r}) \left( \vec{E}_2(\vec{r}) \right)$ are the electric potential (field) in regions 1 and 2, respectively, evaluated at a point on the surface separating the two regions, and $\hat{n}$ is a unit vector pointing from region 1 to region 2 (see Fig. 3.1). In terms of the of the electric potential, the electric field in the two regions
CHAPTER 3. SPECIAL TECHNIQUES

Figure 3.1: Electric field at the boundaries of two region. (Bottom is region 1 and top is region 2). At the interface of the two regions, there is a surface charge.

is given by

\[ \vec{E}_1 (\vec{r}) = \nabla V_1 (\vec{r}) \]
\[ \Rightarrow E_{1n} (\vec{r}) \hat{n} + E_{1t} (\vec{r}) \hat{t} = - \frac{\partial V_1 (\vec{r})}{\partial n} \hat{n} - \frac{\partial V_1 (\vec{r})}{\partial t} \hat{t} \]
\[ \vec{E}_2 (\vec{r}) = - \nabla V_2 (\vec{r}) \]
\[ \Rightarrow E_{2n} (\vec{r}) \hat{n} + E_{2t} (\vec{r}) \hat{t} = - \frac{\partial V_2 (\vec{r})}{\partial n} \hat{n} - \frac{\partial V_2 (\vec{r})}{\partial t} \hat{t} \]

where \( \hat{t} \) is the unit vector tangent to the surface separating the two regions. The boundary condition in Eq. (3.4a) can then be expressed as

\[ V_1 (\vec{r}) = V_2 (\vec{r}), \frac{\partial V_1 (\vec{r})}{\partial n} - \frac{\partial V_2 (\vec{r})}{\partial n} = \frac{\sigma (\vec{r})}{\epsilon_0}. \tag{3.6a} \]

In this chapter, we will focus how we determine the electric potential and the electric field by solving Laplace’s equation and imposing the boundary conditions in Eqs. (3.6a).

3.1 Properties of Laplace’s Equation

Let’s consider Laplace’s equation in Cartesian coordinates

\[ \nabla^2 V(x, y, z) = \frac{\partial^2 V(x, y, z)}{\partial x^2} + \frac{\partial^2 V(x, y, z)}{\partial y^2} + \frac{\partial^2 V(x, y, z)}{\partial z^2} = 0. \tag{3.7} \]

The solution to Laplace’s equations are called Harmonic function. For one dimensional case, we have
3.1. PROPERTIES OF LAPLACE’S EQUATION

\[
\frac{d^2 V(x)}{dx^2} = 0 \Rightarrow V(x) = mx + b,
\]  

(3.8)

the constants \(m\) and \(b\) are determined by the boundary conditions for the electric potential \(V(x)\) and the electric field \(\vec{E}(x) = -\frac{dV(x)}{dx}\). Suppose the boundaries in this case are \(x - a\) and \(x + a\), at these points where the potential is known, we have

\[
V(x - a) = m(x - a) + b, V(x + a) = m(x + a) + b
\]

\[
\Rightarrow \frac{V(x - a) + V(x + a)}{2} = mx + b = V(x)
\]  

(3.9)

Equation (3.9) shows that the solution to the Laplace equation (the potential at \(x\)) is just the average of the potential at the boundaries.

We now consider Laplace’s equation in two dimensions. Suppose there is a point on the \(x\)-\(y\) plane with coordinates \((x, y)\). In view of the result for one dimensional case, the solution to Laplace’s equation

\[
\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0
\]  

(3.10)

is the average of the potential over some kind of boundary. If this boundary is a circle of radius, \(R\), centered about \((x, y)\); one can write the average as

\[
V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\left(\vec{R}\right) dl,
\]  

(3.11)

when \(dl\) is an infinitesimal length on the circle and \(V\left(\vec{R}\right)\) is the potential on the circular boundary, which is known. Note that the function, \(V(x, y)\), defines a surface with no local maxima or minima inside the circle (see Fig.3). Then the
solution to Laplace’s equation for three dimensional case

\[ \nabla^2 V(x, y, z) = \frac{\partial^2 V(x, y, z)}{\partial x^2} + \frac{\partial^2 V(x, y, z)}{\partial y^2} + \frac{\partial^2 V(x, y, z)}{\partial z^2} = 0 \quad (3.12) \]

(the potential at the point \((x, y, z)\)) is just the average of the potential on the boundary. Suppose this boundary is a sphere with radius \(R\) centered about the point \((x, y, z)\), one can express

\[ V(x, y, z) = \frac{1}{4\pi R^2} \int_{\text{sphere}} V(\vec{R}) \, da. \quad (3.13) \]

where \(da\) is an infinitesimal area on the surface of the sphere and \(V(\vec{R})\) is again the potential on the boundary, which is a sphere in this case.

Next we verify this property of the solution to the Laplace’s equation using the potential for a point charge, \(q\), positioned at a point on the z-axis, \((0, 0, z)\). From what we studied in chapter 2, the potential at the origin \((0, 0, 0)\), is given by

\[ V(0, 0, 0) = \frac{1}{4\pi \epsilon_0} \frac{1}{z} \quad (3.14) \]

Consider the sphere with radius \(R\) centered at the origin shown in Fig. 3.2.

Figure 3.2: A point charge placed on the z-axis at \((0, 0, z)\). A spherical surface of radius, \(R\), centered at the origin.

Suppose on the surface of the sphere the potential is known and is given by

\[ V(\vec{R}) = \frac{1}{4\pi \epsilon_0} \frac{1}{\sqrt{z^2 + R^2 - 2zR \cos(\theta)}} \quad (3.15) \]

and there is no charge inside the sphere. That means the potential satisfies Laplace’s equation,

\[ \nabla^2 V(x, y, z) = 0, \quad (3.16) \]
3.1. PROPERTIES OF LAPLACE’S EQUATION

and at the origin \((0, 0, 0)\) can be expressed as

\[
V(0, 0, 0) = \frac{1}{4\pi R^2} \int_{\text{sphere}} V(R) \, da
= \frac{1}{16\pi^2 R^2 \epsilon_0} \int_{\text{sphere}} \frac{1}{\sqrt{z^2 + R^2 - 2zR \cos(\theta)}} \, da.
\]  

(3.17)

Using

\[da = R^2 \sin(\theta) \, d\theta d\varphi\]  

(3.18)

we may write

\[
V(0, 0, 0) = \frac{1}{16\pi^2 R^2 \epsilon_0} \int_0^\pi \int_0^{2\pi} \frac{R^2 \sin(\theta)}{\sqrt{z^2 + R^2 - 2zR \cos(\theta)}} \, d\theta d\varphi
= \frac{1}{8\pi \epsilon_0} \int_0^\pi \frac{\sin(\theta)}{\sqrt{z^2 + R^2 - 2zR \cos(\theta)}} \, d\theta
= \frac{1}{8\pi \epsilon_0} \left[ \frac{z + R - (z - R)}{zR} \right]_0^\pi \Rightarrow V(0, 0, 0) = \frac{1}{4\pi \epsilon_0} \frac{1}{z}
\]  

(3.19)

The result in Eq. (3.19) verifies that the solution to Laplace’s equation, \(V(x, y, z)\), in Eq. (3.13)

Green’s theorem, boundary conditions, and uniqueness theorem: consider a vector field \(\vec{A}\). From the divergence theorem this vector field satisfies the equation

\[
\int_{\text{Volume}} \left( \nabla \cdot \vec{A}(x, y, z) \right) \, d\tau = \oint_{\text{Surface}} \vec{A}(x, y, z) \cdot d\vec{a}.
\]

Suppose the vector field is expressible in terms of some well-behaved scalar functions \(U(x, y, z)\) and \(V(x, y, z)\) as

\[
\vec{A}(x, y, z) = U(x, y, z) \nabla V(x, y, z)
\]  

(3.20)

we have

\[
\nabla \cdot \vec{A}(x, y, z) = \nabla \cdot [U(x, y, z) \nabla V(x, y, z)] = (\nabla U(x, y, z)) \cdot (\nabla V(x, y, z)) + U(x, y, z) \nabla^2 V(x, y, z)
\]

\[
\Rightarrow \nabla \cdot \vec{A}(x, y, z) = (\nabla U(x, y, z)) \cdot (\nabla V(x, y, z)) + U(x, y, z) \nabla^2 V(x, y, z)
\]  

(3.21)

so that

\[
\int_{\text{Volume}} [(\nabla U) \cdot (\nabla V) + U \nabla^2 V] \, d\tau = \oint_{\text{Surface}} [U(x, y, z) \nabla V(x, y, z)] \cdot d\vec{a}.
\]  

(3.22)

In terms \(\hat{n}\) (the outward normal) and \(\hat{t}\) (the tangent) unit vectors to the surface, we may write

\[d\vec{a} = \hat{n} \, da\]  

(3.23)
and
\[ \nabla V(x, y, z) = \frac{\partial V(x, y, z)}{\partial n} \hat{n} + \frac{\partial V(x, y, z)}{\partial t} \hat{t}, \]  
so that one finds
\[ \int_{\text{Volume}} \left[ (\nabla U(x, y, z)) \cdot (\nabla V(x, y, z)) + U(x, y, z) \nabla^2 V(x, y, z) \right] \, d\tau 
= \oint_{\text{Surface}} U(x, y, z) \frac{\partial V(x, y, z)}{\partial n} \, da \quad (3.25) \]
Interchanging \( U \) and \( V \) in the above expression, we have
\[ \int_{\text{Volume}} \left[ (\nabla V(x, y, z)) \cdot (\nabla U(x, y, z)) + V(x, y, z) \nabla^2 U(x, y, z) \right] \, d\tau 
= \oint_{\text{Surface}} V(x, y, z) \frac{\partial U(x, y, z)}{\partial n} \, da \quad (3.26) \]
so that upon subtracting the two equations
\[ \int_{\text{Volume}} \left[ U(x, y, z) \nabla^2 V(x, y, z) - V(x, y, z) \nabla^2 U(x, y, z) \right] \, d\tau 
= \oint_{\text{Surface}} \left[ U(x, y, z) \frac{\partial V(x, y, z)}{\partial n} - V(x, y, z) \frac{\partial U(x, y, z)}{\partial n} \right] \, da, \quad (3.27) \]
This is called Green's Theorem.

**Dirichlet Boundary Condition:** the potential, \( V(x, y, z) = 0 \) on the boundary. Under this condition the Green’s theorem becomes
\[ \int_{\text{Volume}} \left[ U(x, y, z) \nabla^2 V(x, y, z) - V(x, y, z) \nabla^2 U(x, y, z) \right] \, d\tau 
= \oint_{\text{Surface}} V(x, y, z) \frac{\partial U(x, y, z)}{\partial n} \, da \quad (3.28) \]

**Neumann Boundary Condition:** the normal component of the vector field is zero \( (\frac{\partial V(x, y, z)}{\partial n} = 0) \) on the boundary and the Green’s theorem becomes
\[ \int_{\text{Volume}} \left[ U(x, y, z) \nabla^2 V(x, y, z) - V(x, y, z) \nabla^2 U(x, y, z) \right] \, d\tau 
= -\oint_{\text{Surface}} \left[ V(x, y, z) \frac{\partial U(x, y, z)}{\partial n} \right] \, da. \quad (3.29) \]

**The Uniqueness theorem:** The solution to Laplace’s equation in some volume is uniquely determined if \( V(x, y, z) \) is specified on the boundary surface.
3.1. PROPERTIES OF LAPLACE’S EQUATION

That means two solutions of Laplace’s equation that satisfy the same boundary conditions differ at most by an additive constant

\[ V_1(x, y, z) = V_2(x, y, z) + C, \quad (3.30) \]

where \( C \) is a constant independent of \( x, y, \) and \( z \).

Consider the closed region \( V \) exterior to the surface \( S_1, S_2, \ldots, S_N \) of the various conductors in the problem and bounded on the outside by the surface \( S \) that can be either a surface at infinity or a finite surface enclosing \( V \). Suppose the solution is not unique and there are two solutions \( V_1(x, y, z) \) and \( V_2(x, y, z) \), of Laplace’s equation in \( V \) with the same boundary conditions. These boundary conditions may be specified by assigning either \( V \) (Dirichlet condition) or the normal derivative \( \partial V / \partial n \) (Neumann condition) on the bounding surfaces.

\[ \nabla^2 V_1(x, y, z) = 0, \nabla^2 V_2(x, y, z) = 0 \quad (3.31) \]

Let’s consider the difference

\[ V_3(x, y, z) = V_1(x, y, z) - V_2(x, y, z), \quad (3.32) \]

It satisfies Laplace’s equation

\[ \nabla^2 V_3(x, y, z) = 0 \quad (3.33) \]

since

\[ \nabla^2 V_3(x, y, z) = \nabla^2 V_1(x, y, z) - \nabla^2 V_2(x, y, z) = 0. \quad (3.34) \]

Using Green’s theorem

\[
\int_{\text{Volume}} \left[ U(x, y, z) \nabla^2 V(x, y, z) - V(x, y, z) \nabla^2 U(x, y, z) \right] d\tau = \\
\int_{\text{Surface}} \left[ U(x, y, z) \frac{\partial V(x, y, z)}{\partial n} - V(x, y, z) \frac{\partial U(x, y, z)}{\partial n} \right] d\alpha, \quad (3.35)
\]

for

\[ U(x, y, z) = V(x, y, z) = V_3(x, y, z) \quad (3.36) \]

we find

\[
\int_{\text{Volume}} \left( |\nabla V_3(x, y, z)|^2 + V_3(x, y, z) \nabla^2 V_3(x, y, z) \right) d\tau = \\
\int_{\text{Surface}} V_3(x, y, z) \frac{\partial V_3(x, y, z)}{\partial n} d\alpha.
\]

Using the result

\[ \nabla^2 V_3(x, y, z) = 0 \quad (3.37) \]
one finds
\[ \int_{\text{Volume}} |\nabla V_3(x, y, z)|^2 d\tau = \int_{\text{Surface}} V_3(x, y, z) \left( \frac{\partial V_3(x, y, z)}{\partial n} \right) d\alpha. \] (3.38)

For Dirichlet Boundary Condition \( V_3(x, y, z) \) is known on the boundary if we take this boundary to infinity where usually the potential becomes zero \( (V_3(x, y, z) = 0 \text{ on the surface}) \), we find

\[ \int_{\text{Volume}} |\nabla V_3(x, y, z)|^2 d\tau = 0 \Rightarrow |\nabla V_3(x, y, z)| = 0 \Rightarrow \nabla V_3(x, y, z) = 0 \]
\[ \Rightarrow V_3(x, y, z) = C. \] (3.39)

This means the potential \( V_3(x, y, z) \) is a constant inside the volume and this constant must be zero since \( V_3(x, y, z) = 0 \) on the surface. Therefore

\[ V_3(x, y, z) = V_1(x, y, z) - V_2(x, y, z) = 0 \Rightarrow V_1(x, y, z) = V_2(x, y, z). \] (3.40)

For Neumann Boundary Condition the normal component of the electric field is zero on the boundary surface \( \left( \frac{\partial V_3}{\partial n} = 0 \right) \), and we also find

\[ \int_{\Omega} |\nabla V_3(x, y, z)|^2 d\tau = 0. \] (3.41)

that leads to

\[ \nabla V_3(x, y, z) = 0 \Rightarrow V_3(x, y, z) = V_1(x, y, z) - V_2(x, y, z) = C, \] (3.42)

where \( C \) is a constant.

### 3.2 The Method of Images

The method of images is founded on the fact that the solution to Laplace’s equation is Unique. It means there is only one solution to the Laplace’s equation.

**Example 3.1** A point charge \( q \) is held a distance \( d \) above an infinite grounded conducting plane.

(a) What is the potential in the region above the plane?

(b) Find the surface charge density and the total induced charge on the plane

(c) Find the force exerted by the conductor on the point charge.

(d) What is the energy.

**Solution:**
3.2. THE METHOD OF IMAGES

Figure 3.3: A positive point charge, \( q \), a distance \( d \) away from a grounded infinite conducting plate on the x-y plane.

(a) Electric Potential: We are interested to find the potential in the region \( Z > 0 \) which must satisfy the following boundary conditions

i. On the surface of the conducting plate the potential is known and it is zero (i.e. \( V(x, y, 0) = 0 \))

ii. Since all the free charges in the conductor are drawn towards the point charge, a point far away from the plate, we expect the potential to be zero (i.e. \( V(x, y, z) = 0 \) for \( r \to \infty \))

Consider two point charges as shown in the figure below. The potential at a point above the plane located at a distance \( r \) from the origin due to these point charges is

\[
V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 \frac{1}{r_+}} - \frac{q}{4\pi\varepsilon_0 \frac{1}{r_-}}
\]

\[
\Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}}
\]

From this potential we note that on the x-y plane (\( z = 0 \))

\[
V(x, y, 0) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + d^2}} - \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + d^2}} \Rightarrow V(x, y, 0) = 0
\]

(3.43)

and at a point far from the plane (\( \sqrt{x^2 + y^2 + z^2} >> d \)), \( V(x, y, z) = 0 \). Therefore, the potential

\[
V(x, y, z) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}}
\]

(3.44)
satisfies the boundary conditions described above. From the uniqueness of the solution to the Laplace equation we conclude that the potential in the region \( z > 0 \) due to a point charge placed a distance \( d \) above a grounded conducting plate is given by

\[
V(\mathbf{r}) = \frac{q}{4\pi \varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{4\pi \varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}} \tag{3.45}
\]

(b) **Induced surface charge:** We recall that if we have a surface charge \( \sigma \) on a conducting surface the normal component of the electric field is discontinuous and it is related to the surface charge density by

\[
\sigma = \varepsilon_0 E_\perp = -\varepsilon_0 \frac{\partial V}{\partial n} \tag{3.46}
\]

where \( E_\perp \) is the normal component of the electric field on the surface which is related to the electric potential by \( E_\perp = -\frac{\partial V}{\partial n} \) (which must be evaluated on the surface). Using the result obtained above for the electric potential we find

\[
\sigma = -\varepsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} = -\frac{q}{4\pi} \frac{\partial}{\partial z} \left[ \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}} \right] \tag{3.47}
\]
3.2. THE METHOD OF IMAGES

\[
\sigma = -\frac{q}{4\pi} \left[ \frac{z-d}{(x^2 + y^2 + (z-d)^2)^{3/2}} - \frac{z+d}{(x^2 + y^2 + (z+d)^2)^{3/2}} \right]_{z=0}
\]

\[
= -\frac{q}{4\pi} \left[ \frac{d}{(x^2 + y^2 + d^2)^{3/2}} + \frac{d}{(x^2 + y^2 + d^2)^{3/2}} \right]
\]

\[
\Rightarrow \sigma = -\frac{q}{2\pi} \left[ \frac{d}{(x^2 + y^2 + d^2)^{3/2}} \right]
\]

(3.48)

The induced surface charge is negative as expected and it is maximum at the center of the plate

\[
\sigma_{\text{max}} = -\frac{q}{2\pi} \left[ \frac{1}{d^2} \right]
\]

(3.49)

The total charge induced on the conducting plate is

\[
Q = \int \sigma \, da
\]

(3.50)

if we use polar coordinates instead of Cartesian coordinates we can get the total charge easily. In polar coordinates \((r, \theta)\), we have \(x = r \cos \theta\), \(y = r \sin \theta\), and \(da = r \, dr \, d\theta\) so that

\[
\sigma = -\frac{q}{2\pi} \left[ \frac{d}{(r^2 + d^2)^{3/2}} \right] = -\frac{q}{2\pi} \left[ \frac{d}{(r^2 + d^2)^{3/2}} \right]
\]

(3.51)

\[
Q = -\frac{q}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{d}{(r^2 + d^2)^{3/2}} r \, dr \, d\theta = -\frac{q2\pi d}{2\pi} \int_0^\infty \frac{r \, dr}{(r^2 + d^2)^{3/2}}
\]

\[
= -q d \int_0^\infty \frac{r \, dr}{(r^2 + d^2)^{3/2}} = -q d \int_0^\infty \frac{\, du}{u^2} = -2qd \int_d^\infty \frac{\, du}{u^2} \Rightarrow Q = -q.
\]

(3.52)

(c) Force: The charge \(q\) is attracted to the plate due to the induced surface charge. You can determine this force in two different ways. One is just to find the strength of the electric field at the position of the charge \(q\) due to the image charge. We recall the electric field (due to the image charge) is given by

\[
\vec{E}_I = -\frac{\partial V_I}{\partial z} \bigg|_{x=0,y=0,z=d}
\]

(3.53)

where \(V_I\) is the potential due to the image charge,

\[
V_I = -\frac{q}{4\pi \epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}.
\]

(3.54)

At the position of the charge \(q\) \((x, 0, y = 0, z = d)\), we then find

\[
\vec{E}_I = -\frac{\partial}{\partial z} \left[ -\frac{q}{4\pi \epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right]
\]

(3.55)
\[ \vec{E}_I = -\frac{q}{4\pi\epsilon_0} \left[ \frac{z + d}{(x^2 + y^2 + (z + d)^2)^{3/2}} \right]_{x=0,y=0,z=d} = -\frac{q}{4\pi\epsilon_0} \left[ \frac{2d}{(2d)^3} \right] \hat{z} = -\frac{q}{4\pi\epsilon_0} \frac{1}{(2d)^2} \hat{z} \Rightarrow \vec{F} = q\vec{E}_I = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{(2d)^2} \hat{z} \quad (3.56) \]

The second way is to use the force of attraction between two point charges separated by a distance \( 2d \).

(d) **Energy:** The energy is the work needed to bring the charge \( q \) from infinity to a distance \( d \) from the plate. This work is given by

\[ W = \int_{\infty}^{d} \vec{F} \cdot d\vec{r} \quad (3.57) \]

The electrical force on the charge \( q \) (if the charge is a distance \( z \) from the center of the plate), from the above result it can be written as

\[ F = \frac{q^2}{4\pi\epsilon_0} \frac{1}{(2z)^2} \quad (3.58) \]

Therefore

\[ W = \frac{q^2}{4\pi\epsilon_0} \int_{\infty}^{d} \frac{dz}{(2z)^2} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{4d} \Rightarrow W = -\frac{1}{2} \frac{q^2}{4\pi\epsilon_0} \frac{1}{2d} \quad (3.59) \]

This shows that the energy, unlike the force, is half of the energy due to two point charges separated by a distance \( 2d \).

**Example 3.2** A point charge \( q \) is situated a distance \( a \) from the center of a grounded conducting sphere of radius \( R \) (Fig. 3.12). Find the potential outside the sphere.

(a) What is the potential in the region outside the sphere?

(b) Find the force exerted by the conductor on the point charge.

**Solution:**

(a) **The potential:** Let the point charge \( q \) be on the \( z \) axis. Then we may replace the induced charge by an image charge \( q' \) at a distance \( b \) from inside the sphere on the positive \( z \) axis as shown in the figure. For now we don’t know the magnitude of the charge, the distance \( b \) and also we don’t know whether it is positive or negative. But we do know that the potential outside the sphere is due to the point charge \( q \) and its image charge \( q' \). If we are able to find \( q' \) and \( b \) then, it means we know the potential outside the sphere. The potential at a distance \( r \) from the center of the sphere is
3.2. THE METHOD OF IMAGES

![Figure 3.5](image)

Figure 3.5: A positive point charge, \( q \), a distance \( d \) away from the center of a grounded conducting sphere of radius \( R \).

given by

\[
V(\hat{r}) = \frac{q}{4\pi\varepsilon_0 \hat{r}} + \frac{q}{4\pi\varepsilon_0 r}
\]

\[
= \frac{1}{4\pi\varepsilon_0} \left[ \frac{q'}{\sqrt{r'^2 + b'^2 - 2r'b'\cos\theta}} + \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} \right]
\]  

(3.60)

Now apply the boundary conditions. The sphere is grounded and therefore the potential must vanish everywhere on the surface of the sphere. Which means at \( \hat{r} = \hat{R} \)

\[
V(\hat{R}) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q'}{\sqrt{R'^2 + b'^2 - 2R'b'\cos\theta}} + \frac{q}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} \right] = 0
\]

\[
\Rightarrow q'^2 (R'^2 + a^2 - 2Ra\cos\theta) = q^2 (R'^2 + b'^2 - 2R'b\cos\theta)
\]

\[
\Rightarrow q'^2 (R'^2 + a^2) - q^2 (R'^2 + b'^2) - (q^2a - q^2b)\cos\theta = 0
\]

From this follows that

\[
q'^2 (R'^2 + a^2) - q^2 (R'^2 + b'^2) = 0, q'^2 a - q^2 b = 0
\]  

(3.61)

Dividing the two equations

\[
\frac{q'^2a}{q'^2 (R'^2 + a^2)} = \frac{q^2b}{q^2 (R^2 + b^2)} \Rightarrow \frac{a}{R^2 + a^2} = \frac{b}{R^2 + b^2}
\]

\[
a (R'^2 + b'^2) = b (R^2 + a^2) \Rightarrow R^2 (a - b) - ba (a - b) = 0
\]

\[
(R^2 - ba) (a - b) \Rightarrow b = \frac{R^2}{a}
\]  

(3.62)
Figure 3.6: A positive point charge, $q$, a distance $a$ away from the center of a grounded conducting sphere of radius $R$. The induced negative surface charge represented by an image charge $-q$ at a distance $b$ from the center inside the sphere.

Now substituting this result into the equation

$$q^2a = q^2b \Rightarrow q' = \pm q\sqrt{\frac{b}{a}} \quad (3.63)$$

and noting that the induced charge must be opposite to the charge $q$, we may write

$$q' = -\frac{R}{a}q \quad (3.64)$$

Therefore the electric potential outside the sphere is given by

$$V(r) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{\frac{R}{a}q}{\sqrt{r^2 + \left(\frac{R}{a}\right)^2 - 2r\frac{R}{a}\cos\theta}} \right] \quad (3.65)$$

(b) Force: the electrical force on the point charge due to the induced charges
Example 3.3 A uniform line charge \( \lambda \) is placed on an infinite straight wire, a distance \( d \) above a grounded conducting plane. (Let’s say the wire runs parallel to the \( x \)-axis and directly above it, and the conducting plane is in the \( x-y \) plane.)

(a) Find the potential in the region above the plane.

(b) Find the charge density \( \sigma \) induced on the conducting plane.

Solution:

(a) Due to the presence of the line charge there will be an induced charge on the conducting plane. This induced charge can be imagined as if we have a line charge of unknown charge density \( \lambda' \) at unknown distance \( d' \) below the \( x-y \) plane. Then the potential in the region above the plane is the sum of the potential due to the line charge and its image charge. First let’s find the potential for a uniform line charge. To this end, we recall that for a line charge the electric field at a distance \( s \) from the mid point of the line is given by

\[
E(s) = \frac{\lambda}{2 \pi \varepsilon_0 s} \tag{3.67}
\]

then the potential is given by

\[
V(s) = - \int_{s_0}^{s} E(s) ds = - \frac{\lambda}{2 \pi \varepsilon_0} \int_{s_0}^{s} \frac{ds}{s} = - \frac{\lambda}{2 \pi \varepsilon_0} (\ln s - \ln s_0) = \frac{\lambda}{2 \pi \varepsilon_0} (\ln s_0 - \ln s) \Rightarrow V(s) = \frac{\lambda}{2 \pi \varepsilon_0} \ln \left( \frac{s_0}{s} \right) \tag{3.68}
\]

Note that we chose the reference point for the potential a point a distance \( s_0 \) away from the mid point of the line. We can not use infinity since the line charge extends up to infinity and the potential diverges at infinity. Now using the result above we may write the potential due to a line charge placed a distance \( d \) above the \( x-y \) plane and parallel to the \( x \)-axis,

\[
V_+(s_+) = \frac{\lambda}{2 \pi \varepsilon_0} \ln \left( \frac{s_0}{s_+} \right) \tag{3.69}
\]
where
\[ s_+ = \sqrt{(z - d)^2 + y^2}. \]  
(3.70)

Similarly for the image charge we have
\[ V_-(s_-) = \frac{\lambda'}{2\pi\epsilon_0} \ln \left( \frac{s_0}{s_-} \right) \]  
(3.71)

where
\[ s_- = \sqrt{(z + d')^2 + y^2}. \]  
(3.72)

\[ V(\vec{r}) = \frac{\lambda}{2\pi\epsilon_0} \ln (z - d) \]  
(3.73)

Therefore the electric potential above the conducting plane is given by
\[ V(\vec{r}) = \frac{\lambda}{2\pi\epsilon_0} \ln \left( \frac{s_0}{s_+} \right) + \frac{\lambda'}{2\pi\epsilon_0} \ln \left( \frac{s_0}{s_-} \right) \]  
(3.74)

Now applying the boundary condition \( V(\vec{r}) = 0 \) on the surface of the plane \( (z = 0) \) we have
\[ \frac{\lambda}{2\pi\epsilon_0} \ln \left( \frac{s_0}{\sqrt{d^2 + y^2}} \right) + \frac{\lambda'}{2\pi\epsilon_0} \ln \left( \frac{s_0}{\sqrt{d'^2 + y^2}} \right) = 0 \]  
(3.75)

so that one can easily see that \( d' = d \) and \( \lambda' = -\lambda \). Then the potential is
3.3 Separation of Variables

Cartesian Coordinates

Example 3.4 Two infinite grounded metal plates lie parallel to the $x-z$ plane, one at $y = 0$, the other at $y = a$ (see figure below). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

Solution: The boundary conditions are

$$V(x, y = 0) = 0, V(x, y = a) = 0$$
$$V(x = 0, y) = V_0(y)$$
$$V(x \to \infty, y) \to 0$$

Since there is no condition for the potential in the $z$ direction, we can use the two-dimensional Laplace’s equation involving $x$ and $y$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (3.78)$$

(b) The induced surface charge density on the plane ($Z = 0$) is given by

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} \bigg|_{z = 0} = -\varepsilon_0 \frac{\partial}{\partial z} \left[ \frac{\lambda}{4\pi \varepsilon_0} \ln \left( \frac{(z + d)^2 + y^2}{(z - d)^2 + y^2} \right) \right]_{z = 0}$$

$$= -\varepsilon_0 \frac{\partial}{\partial z} \left[ \frac{\lambda}{4\pi \varepsilon_0} \ln \left( \frac{(z + d)^2 + y^2}{(z - d)^2 + y^2} \right) \right]_{z = 0}$$

$$= -\frac{\lambda}{4\pi} \left[ \frac{2(z + d)}{(z + d)^2 + y^2} - \frac{2(z - d)}{(z - d)^2 + y^2} \right]_{z = 0}$$

$$= -\frac{\lambda}{4\pi} \left[ \frac{2d}{d^2 + y^2} + \frac{2d}{d^2 + y^2} \right] \Rightarrow \sigma = -\frac{\lambda d}{\pi (d^2 + y^2)} \quad (3.77)$$

(c) Exercise: Find the total charge induced on a strip of length $l$ parallel to the $y$-axis the conducting plane. You should get $Q_{ind} = -\lambda l$. 

3.3 Separation of variables
We use separation of variables. Which means we assume the potential can be expressed as a product of two independent functions $X(x)$ and $Y(y)$

$$V(x, y) = X(x)Y(y)$$  \hspace{1cm} (3.79)

substituting Eq. (3.79) into Eq. (3.78) we find

$$\frac{\partial^2 (X(x)Y(y))}{\partial x^2} + \frac{\partial^2 (X(x)Y(y))}{\partial y^2} = 0 \Rightarrow Y(y) \frac{d^2 X(x)}{dx^2} + X(x) \frac{d^2 Y(y)}{dy^2} = 0$$  \hspace{1cm} (3.80)

Dividing the above equation by $X(x)Y(y)$ we find

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = 0 \Rightarrow f(x) + g(y) = 0$$  \hspace{1cm} (3.81)

where $f(x) = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}$ and $g(y) = \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2}$. Since $f(x)$ and $g(y)$ are two independent functions, these functions must be constants with one of these constants is negative of the other. If we define this constant to be $k^2$ then

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k^2, \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k^2$$

$$\Rightarrow \frac{d^2 X(x)}{dx^2} - k^2 X(x) = 0, \frac{d^2 Y(y)}{dy^2} + k^2 Y(y) = 0$$  \hspace{1cm} (3.82)

The solutions for these differential equations are given by

$$X(x) = Ae^{kx} + Be^{-kx}, Y(y) = C \cos(ky) + D \sin(ky)$$

$$V(x, y) = (Ae^{kx} + Be^{-kx}) (C \cos(ky) + D \sin(ky))$$  \hspace{1cm} (3.83)
From the given boundary conditions we have
\[ V(x, y = 0) = 0 \Rightarrow C = 0, \quad V(x \to \infty, y) \to 0 \Rightarrow A = 0 \] (3.84)
so that the solution can be written as
\[ V(x, y) = De^{-kx} \sin(ky), \] (3.85)
where we absorbed the constant \( B \) into \( D \). Now using the boundary condition \( V(x, y = a) = 0 \), we find
\[ V(x, a) = De^{-kx} \sin(ka) = 0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}, \] (3.86)
where \( n = 1, 2, 3, \ldots \). Therefore, the general solution of the Laplace’s equation for the given problem can be written as
\[ V(x, y) = \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi}{a} x} \sin\left(\frac{n\pi}{a} y\right). \] (3.87)
To find the constant \( D_n \), we use the last boundary condition, \( V(x = 0, y) = V_0(y) \)
\[ V(0, y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a} y\right) = V_0(y) \] (3.88)
Now to find an explicit expression for \( D_n \) in terms of \( V_0(y) \) multiply both sides by \( \sin\left(\frac{m\pi}{a} y\right) \) and integrate with respect to \( y \)
\[ \sum_{n=1}^{\infty} D_n \int_0^a \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{n\pi}{a} y\right) dy = \int_0^a V_0(y) \sin\left(\frac{m\pi}{a} y\right) dy. \] (3.89)
If we use the relation
\[ \sin(\alpha) \sin(\beta) = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \] (3.90)
we can write the integral
\[
\int_0^a \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{n\pi}{a} y\right) dy = \frac{1}{2} \int_0^a \left\{ \cos\left(\frac{m\pi}{a} y - \frac{n\pi}{a} y\right) - \cos\left(\frac{m\pi}{a} y + \frac{n\pi}{a} y\right) \right\} dy
\]
\[
- \frac{1}{2} \int_0^a \cos\left[\frac{(m+n)\pi}{a} y\right] dy = \frac{1}{2} \int_0^a \frac{(m-n)\pi}{a} \sin\left[\frac{(m-n)\pi}{a} y\right] \bigg|_0^a - \frac{1}{2} \int_0^a \frac{(m+n)\pi}{a} \sin\left[\frac{(m+n)\pi}{a} y\right] \bigg|_0^a
\]
\[
= \frac{1}{2} \sin\left[\frac{(m-n)\pi}{a} y\right] \bigg|_0^a - \frac{1}{2} \sin\left[\frac{(m+n)\pi}{a} y\right] \bigg|_0^a \] (3.91)
The second term is always zero and the first term gives
\[
\frac{1}{2} \left. \sin \left( \frac{(m-n)\pi x}{a} \right) \right|_0^a = \begin{cases} 
0 & m \neq n \\
\frac{a}{2} & m = n
\end{cases}.
\] (3.92)

Therefore, we find
\[
\int_0^a \sin \left( \frac{m\pi y}{a} \right) \sin \left( \frac{n\pi y}{a} \right) dy = \begin{cases} 
0 & m \neq n \\
\frac{a}{2} & m = n
\end{cases}
\] (3.93)

Using this result in Eq. (3.89)
\[
a D_m = \int_0^a V_0(y) \sin \left( \frac{m\pi y}{a} \right) dy \Rightarrow D_m = \frac{2}{a} \int_0^a V_0(y) \sin \left( \frac{m\pi y}{a} \right) dy
\] (3.94)

The electric potential for the boundary conditions specified in this problem will then be
\[
V(x, y) = \sum_{n=1}^{\infty} D_n e^{-\frac{m\pi x}{a}} \sin \left( \frac{n\pi y}{a} \right),
\] (3.95)

where
\[
D_n = \frac{2}{a} \int_0^a V_0(y) \sin \left( \frac{n\pi y}{a} \right) dy.
\] (3.96)

As an example let’s consider the left end metal plate (the plate at \( x = 0 \)) is kept at a constant potential \( V_0 \). Which means \( V_0(y) = V_0 \) and the constant \( D_n \) is given by

\[
D_n = \frac{2}{a} \int_0^a V_0 \sin \left( \frac{n\pi y}{a} \right) dy = -\frac{2V_0}{a} \frac{\cos \left( \frac{n\pi y}{a} \right)}{\frac{n\pi}{a}} \bigg|_0^a = \frac{2V_0}{n\pi} \left[ 1 - \cos \frac{n\pi}{a} \right]
\] (3.97)

there follows that
\[
D_n = \begin{cases} 
0 & n \text{ even} \\
\frac{4V_0}{n\pi} & n \text{ odd}
\end{cases}
\] (3.98)

and the electrical potential becomes
\[
V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,2,3}^{\infty} e^{-\frac{m\pi x}{a}} \sin \left( \frac{n\pi y}{a} \right)
\] (3.99)

This equation can be summed up to give
\[
V(x, y) = \frac{2V_0}{\pi} \tan^{-1} \left[ \frac{\sin \left( \frac{\pi x}{a} \right)}{\sinh \left( \frac{\pi x}{a} \right)} \right]
\] (3.100)
3.3. **SEPARATION OF VARIABLES**

**Complete functions:** A set of functions $f_n(x)$ are said to be complete if any other function $g(x)$ can be expressed as a linear combination of these functions

$$g(x) = \sum_{n=1}^{\infty} C_n f_n(x)$$

(3.101)

**Orthogonal functions:** A set of functions $f_n(x)$ are said to be orthogonal if they satisfy the condition

$$\int_{0}^{\infty} f_n(x)f_{n'}(x)dx = 0 \text{ for } n \neq n'$$

(3.102)

**Example 3.5** An infinitely long rectangular metal pipe (sides $a$ and $b$) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y,z)$, as indicated in Fig.3.22. Find the potential inside the pipe.
Solution: The electric potential should satisfy the boundary conditions

\[ V(x, 0, z) = 0, V(x, a, z) = 0 \]
\[ V(x, y, 0) = 0, V(x, y, b) = 0, \]
\[ V(0, y, z) = V_0(y, z) \]
\[ V(x, y, z) \to 0 \text{ as } x \to \infty \] (3.103)

From these boundary conditions we note that the required electric potential satisfies three dimensional Laplace’s equation. Therefore we need to solve the three-dimensional Laplace’s equation

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \] (3.104)

Using separation of variables we can write

\[ V(x, y, z) = X(x)Y(y)Z(z) \] (3.105)

and substituting Eq. (3.105) into Eq. (3.104)

\[ \frac{\partial^2 (X(x)Y(y)Z(z))}{\partial x^2} + \frac{\partial^2 (X(x)Y(y)Z(z))}{\partial y^2} + \frac{\partial^2 (X(x)Y(y)Z(z))}{\partial z^2} = 0 \]

\[ Y(y)Z(z) \frac{d^2 X(x)}{dx^2} + X(x)Z(z) \frac{d^2 Y(y)}{dy^2} + X(x)Y(y) \frac{d^2 Z(z)}{dz^2} = 0 \] (3.107)

and dividing Eq. (3.107) by \( X(x)Y(y)Z(z) \), we find

\[ \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0 \] (3.108)

which can be written as

\[ f(x) + g(y) + h(z) = 0 \] (3.109)
where
\[ f(x) = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}, \quad g(y) = \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2}, \quad h(z) = \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2}. \]  
(3.110)

Since \( f(x), g(y), \) and \( h(z) \) are independent functions in order Eq. (3.109) to be true each of these functions must be a constant. We chose these constants to be expressed in the following form
\[ f(x) = 1 \quad X(x) \quad \frac{d^2 X(x)}{dx^2} = k^2 + l^2 \quad X(x) = 0 \quad (3.111) \]
\[ g(y) = 1 \quad Y(y) \quad \frac{d^2 Y(y)}{dy^2} = -k^2 \Rightarrow \frac{d^2 Y(y)}{dy^2} + k^2 Y(y) = 0 \quad (3.112) \]
\[ h(z) = 1 \quad Z(z) \quad \frac{d^2 Z(z)}{dz^2} = -l^2 \Rightarrow \frac{d^2 Z(z)}{dz^2} + l^2 Z(z) = 0 \quad (3.113) \]

The general solutions to the differential equations Eqs. (3.111-??) are given by
\[ X(x) = Ae^{-\sqrt{k^2+l^2}x} + Be^{\sqrt{k^2+l^2}x}, \quad Y(y) = C \sin(ky) + D \cos(ky), \]
\[ Z(z) = F \sin(lz) + G \cos(lz) \quad (3.114) \]

The electric potential can be expressed as
\[ V(x, y, z) = \left( Ae^{-\sqrt{k^2+l^2}x} + Be^{\sqrt{k^2+l^2}x} \right) (C \sin(ky) + D \cos(ky)) \]
\[ \quad (F \sin(lz) + G \cos(lz)) \quad (3.115) \]

Now we use the boundary conditions. The first condition \( V(x, y, z) \to 0 \) as \( x \to \infty \) requires \( B = 0 \). The second boundary conditions \( V(x, 0, z) = 0, V(x, y, 0) = 0 \) requires \( D = 0 \) and \( G = 0 \), respectively. The resulting expression for the electric potential can be written as
\[ V(x, y, z) = Ae^{-\sqrt{k^2+l^2}x}C \sin(ky)F \sin(lz) \quad (3.116) \]
or if we absorb the constants \( C \) and \( F \) in \( A \),
\[ V(x, y, z) = Ae^{-\sqrt{k^2+l^2}x} \sin(ky) \sin(lz). \quad (3.117) \]

Next applying the third boundary conditions \( V(x, y, b) = 0, V(x, a, z) = 0 \) we have
\[ V(x, y, b) = 0 \Rightarrow Ae^{-\sqrt{k^2+l^2}x} \sin(ky) \sin(bl) = 0 \]
\[ \Rightarrow \quad lb = m\pi \Rightarrow l = \frac{m\pi}{b}, \quad m = 1, 2, 3.. \quad (3.118) \]
\[ V(x, a, z) = 0 \Rightarrow Ae^{-\sqrt{k^2+l^2}x} \sin(ka) \sin(lz) = 0 \]
\[ \Rightarrow \quad ka = n\pi \Rightarrow k = \frac{n\pi}{a}, \quad n = 1, 2, 3.. \quad (3.119) \]
Using the results in Eq. (3.33) and (3.34), the electric potential can be expressed as
\[ V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} e^{-\sqrt{\frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}} \pi x} \sin \left( \frac{n \pi y}{a} \right) \sin \left( \frac{m \pi z}{b} \right). \]

To determine the constant \( A_{mn} \), we use the last boundary condition \( V(0, y, z) = V_0(y, z) \). Using Eq. (3.33) this boundary condition may be written as
\[ V(0, y, z) = V_0(y, z) \Rightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin \left( \frac{n \pi y}{a} \right) \sin \left( \frac{m \pi z}{b} \right) = V_0(y, z). \] (3.120)

Following a similar technique used in Example 3.4, we multiply Eq. (3.120) by \( \sin \left( \frac{n' \pi y}{a} \right) \sin \left( \frac{m' \pi z}{b} \right) \) and integrate with respect to \( y \) and \( z \) over the dimension of the box,
\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \int_0^a \sin \left( \frac{n' \pi y}{a} \right) \sin \left( \frac{n \pi y}{a} \right) \, dy \int_0^b \sin \left( \frac{m' \pi z}{b} \right) \sin \left( \frac{m \pi z}{b} \right) \, dz = \int_0^a \int_0^b V_0(y, z) \sin \left( \frac{n' \pi y}{a} \right) \sin \left( \frac{m' \pi z}{b} \right) \, dy \, dz \] (3.121)

Applying the relation in Eq. (3.93), we find
\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \frac{ab}{2} \delta_{nn'} \delta_{mm'} = \int_0^a \int_0^b V_0(y, z) \sin \left( \frac{n' \pi y}{a} \right) \sin \left( \frac{m' \pi z}{b} \right) \, dy \, dz \]

\[ A_{m'n'} \frac{ab}{4} = \int_0^a \int_0^b V_0(y, z) \sin \left( \frac{n' \pi y}{a} \right) \sin \left( \frac{m' \pi z}{b} \right) \, dy \, dz \]
\[ A_{m'n'} = \frac{4}{ab} \int_0^a \int_0^b V_0(y, z) \sin \left( \frac{n' \pi y}{a} \right) \sin \left( \frac{m' \pi z}{b} \right) \, dy \, dz. \] (3.122)

Therefore the final complete solution to the Laplace equation satisfying the specified boundary conditions is given by
\[ V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} e^{-\sqrt{\frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}} \pi x} \sin \left( \frac{n \pi y}{a} \right) \sin \left( \frac{m \pi z}{b} \right), \] (3.123)

where
\[ A_{mn} = \frac{4}{ab} \int_0^a \int_0^b V_0(y, z) \sin \left( \frac{n \pi y}{a} \right) \sin \left( \frac{m \pi z}{b} \right) \, dy \, dz \] (3.124)
As an example consider the end of the tube is a conductor kept at constant potential $V_0$ then we find from Eq. (3.124) that

$$A_{mn} = \frac{4V_0}{ab} \int_{0}^{a} \sin \left( \frac{n\pi y}{a} \right) dy \int_{0}^{b} \sin \left( \frac{m\pi z}{b} \right) dz$$

$$= \frac{4V_0}{ab} \left[ 1 - (1)^n \right] \left[ 1 - (1)^m \right] \Rightarrow A_{mn} = \begin{cases} \frac{0}{\pi nm} & \text{if } n \text{ or } m \text{ is even} \\ \frac{16V_0}{\pi nm} & \text{if both } n \text{ and } m \text{ are odd} \end{cases}$$

(3.125)

so that the electric potential is given by

$$V(x, y, z) = \frac{16V_0}{\pi^2} \sum_{n=\text{odd}}^{\infty} \sum_{m=\text{odd}}^{\infty} \frac{1}{nm} \frac{1}{\pi} \sin \left( \frac{n\pi y}{a} \right) \sin \left( \frac{m\pi z}{b} \right).$$

(3.126)

**Example 3.6** Two infinitely long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential $V_0$ as shown in the figure below (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

![Diagram of a rectangular pipe with grounded metal plates](image)

**Sol:** To find the potential inside the rectangular pipe we need to solve the Laplace’s equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

(3.127)

under the following boundary conditions

$$V(x, 0) = 0, V(x, a) = 0, V(-b, y) = V_0, V(b, y) = V_0$$

(3.128)
Using separation of variables

\[ \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k^2, \quad \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k^2 \]

\[ \frac{d^2 X(x)}{dx^2} - k^2 X(x) = 0, \quad \frac{d^2 Y(y)}{dy^2} + k^2 Y(y) = 0 \quad (3.129) \]

whose solutions are given by

\[ X(x) = Ae^{kx} + Be^{-kx}, \quad Y(y) = C \cos(ky) + D \sin(ky). \quad (3.130) \]

Then the electric potential satisfying the Laplace equation is given by

\[ V(x, y) = (Ae^{kx} + Be^{-kx}) (C \cos(ky) + D \sin(ky)) \quad (3.131) \]

Now using the boundary condition \( V(x, 0) = 0 \), we have

\[ (Ae^{kx} + Be^{-kx}) (C \cos(ky) + D \sin(ky)) = 0 \Rightarrow C = 0. \quad (3.132) \]

Which leads to the simplified expression

\[ V(x, y) = (Ae^{kx} + Be^{-kx}) \sin(ky), \quad (3.133) \]

where we have absorbed the constant \( D \) into \( A \) and \( B \). Using the second boundary condition \( V(x, a) = 0 \) in Eq. (3.133) we get

\[ V(x, y) = \left( Ae^{kx} + Be^{-kx} \right) \sin(ka) = 0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a} \quad (3.134) \]

where \( n = 1, 2, 3... \) Using the result in Eq. (3.134), we may rewrite Eq. (3.133) as

\[ V(x, y) = \left( Ae^{\frac{n\pi}{a} x} + Be^{-\frac{n\pi}{a} x} \right) \sin\left( \frac{n\pi}{a} y \right) \quad (3.135) \]

Now using the remaining boundary conditions \( V(-b, y) = V_0 \) and \( V(b, y) = V_0 \) in Eq. (3.135), we find

\[ V(-b, y) = \left( Ae^{-\frac{n\pi}{a} b} + Be^{\frac{n\pi}{a} b} \right) \sin\left( \frac{n\pi}{a} y \right) = V_0, \]

\[ V(b, y) = \left( Ae^{\frac{n\pi}{a} b} + Be^{-\frac{n\pi}{a} b} \right) \sin\left( \frac{n\pi}{a} y \right) = V_0 \quad (3.136) \]

There follows that

\[ Ae^{-\frac{n\pi}{a} b} + Be^{\frac{n\pi}{a} b} = Ae^{\frac{n\pi}{a} b} + Be^{-\frac{n\pi}{a} b} \Rightarrow A = B \quad (3.137) \]

which leads to

\[ V(x, y) = A \left( e^{\frac{n\pi}{a} x} + e^{-\frac{n\pi}{a} x} \right) \sin\left( \frac{n\pi}{a} y \right) = A \cosh\left( \frac{n\pi}{a} x \right) \sin\left( \frac{n\pi}{a} y \right). \quad (3.138) \]
Then the general solutions for the electric potential can be expressed as

\[ V(x, y) = \sum_{n=1}^{\infty} A_n \cosh \left( \frac{n\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \quad (3.139) \]

Using the boundary condition \( V(b, y) = V_0 \)

\[ V_0 = A_n \cosh \left( \frac{n\pi b}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \Rightarrow A_n \cosh \left( \frac{n\pi b}{a} \right) \sin \left( \frac{n\pi y}{a} \right) = V_0 \quad (3.140) \]

and following a similar procedure we used in the previous example, we may write

\[ A_n \cosh \left( \frac{n\pi b}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \sin \left( \frac{m\pi y}{a} \right) = V_0 \sin \left( \frac{m\pi y}{a} \right) \]

\[ A_n \int_0^a \sin \left( \frac{m\pi y}{a} \right) \sin \left( \frac{n\pi y}{a} \right) dy = V_0 \int_0^a \sin \left( \frac{m\pi y}{a} \right) \cosh \left( \frac{n\pi b}{a} \right) dy \quad (3.141) \]

Applying the result in Eq. (3.93) for the integral on the left side, we find

\[ A_n \frac{a}{2} = \frac{V_0}{\cosh \left( \frac{n\pi b}{a} \right)} \int_0^a \sin \left( \frac{m\pi y}{a} \right) dy \Rightarrow A_n = \frac{2V_0}{a \cosh \left( \frac{n\pi b}{a} \right)} \int_0^a \sin \left( \frac{m\pi y}{a} \right) dy \]

\[ \Rightarrow A_n = \frac{2V_0}{n\pi \cosh \left( \frac{n\pi b}{a} \right)} [1 - (-1)^n] \quad (3.142) \]

Therefore the electric potential is given by

\[ V(x, y) = \sum_{n=odd}^{\infty} \frac{4V_0}{n\pi \cosh \left( \frac{n\pi y}{a} \right)} \sin \left( \frac{n\pi y}{a} \right) \]

\[ V(x, y) = \frac{4V_0}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n} \cosh \left( \frac{n\pi b}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \cosh \left( \frac{n\pi b}{a} \right) \quad (3.143) \]

**Spherical coordinates**

For round objects Laplace’s equation in spherical coordinates are more appropriate than Cartesian coordinates. In Spherical coordinates a point in space is described by three coordinates \((r, \theta, \varphi)\) as shown in the figure below.

Laplace’s equation in spherical coordinates can be written as

\[ \nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 V}{\partial \varphi^2} = 0 \quad (3.144) \]

If there is azimuthal symmetry (i.e. if the potential is independent of the azimuthal angle \(\varphi\)) we have

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial V}{\partial \theta} \right) = 0 \quad (3.145) \]
Using separation of variables

\[ V(r, \theta) = R(r)\Theta(\theta) \]  

we get

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial [R(r)\Theta(\theta)]}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial [R(r)\Theta(\theta)]}{\partial \theta} \right) = 0 \]

\[ \Rightarrow \frac{\Theta(\theta)}{r^2} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{R(r)}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) = 0. \]  

(3.147)

Multiplying this equation by \( \frac{r^2}{R(r)\Theta(\theta)} \)

\[ \frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{1}{\Theta(\theta) \sin (\theta)} \frac{d}{d\theta} \left( \sin (\theta) \frac{d\Theta(\theta)}{d\theta} \right) = 0 \]

\[ f(r) + g(\theta) = 0, \]  

(3.148)

where

\[ f(r) = \frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right), \quad g(\theta) = \frac{1}{\Theta(\theta) \sin (\theta)} \frac{d}{d\theta} \left( \sin (\theta) \frac{d\Theta(\theta)}{d\theta} \right). \]  

(3.149)

Since \( f(r) \) and \( g(\theta) \) are independent functions, we must have

\[ f(r) = \text{Constant}, \quad g(\theta) = -\text{Constant}. \]  

(3.150)

so that \( f(r) + g(\theta) = 0 \). We chose this constant to be \( l(l + 1) \) and we express Eq. (3.149) as

\[ \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) - l(l + 1)R(r) = 0, \]  

(3.151)

\[ \frac{d}{d\theta} \left( \sin (\theta) \frac{d\Theta(\theta)}{d\theta} \right) + l(l + 1)\sin (\theta) \Theta(\theta) = 0. \]  

(3.152)
The solution of Eq. (3.151) is given by
\[ R(r) = Ar^l + B r^{l+1} \] (3.153)
and the solutions of Eq. (3.152) are the Legendre polynomials
\[ \Theta(\theta) = P_l(\cos(\theta)). \] (3.154)

\( P_l(x) \) is generated by the Rodrigue’s formula
\[ P_l(x) = \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l \] (3.155)

A few Legendre polynomials are listed below
\[ P_0(x) = \frac{1}{2^0 0!} \left( \frac{d}{dx} \right)^0 (x^2 - 1)^0 = 1, \] (3.156a)
\[ P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx} (x^2 - 1)^1 = x \] (3.156b)
\[ P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2 - 1)^2 = \frac{3x^2 - 1}{2}, \] (3.157)
\[ P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2 - 1)^3 = \frac{5x^3 - 3x}{2} \] (3.158)
\[ P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4} (x^2 - 1)^4 = \frac{35x^4 - 30x^2 + 3}{8} \] (3.159)
\[ P_5(x) = \frac{1}{2^55!} \frac{d^5}{dx^5} \left(x^2 - 1\right)^5 = \left(63x^5 - 70x^3 + 15x\right)/8 \] (3.160)

The Legendre polynomials form an orthogonal set of functions in the interval \([0 < \theta < \pi]\).

\[ \int_0^\pi P_m(\cos(\theta))P_l(\cos(\theta))\sin(\theta)d\theta = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases} \] (3.161)

Introducing the transformation of variable defined by \(x = \cos(\theta)\) \(\Rightarrow\) \(dx = -\sin(\theta)d\theta\), \(\theta = 0 \Rightarrow x = 1, \theta = \pi \Rightarrow x = -1\)

\[ \int_0^1 P_m(x)P_l(x)\sin(\theta)d\theta = -\int_1^{-1} P_m(x)P_l(x)dx = \int_{-1}^{1} P_m(x)P_l(x)dx \]

\(\Rightarrow \int_{-1}^{1} P_m(x)P_l(x)dx = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases} . \) (3.163)

**N. B.: For more information about how the solutions to differential equations are derived using power series substitution method please refer to PHYS 3160 lecture note**

The complete solution for the potential can then be written as

\[ V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)). \] (3.164)

**Example 3.7** The potential is specified on the surface of a hollow sphere of radius \(R\). It is given by \(V_0(\theta) = k\sin^2(\theta/2)\). Find the potential inside the sphere.

**Solution:** The solution is given by

\[ V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)). \] (3.165)

We are interested in the electric potential inside the sphere and the potential must be defined everywhere this region including the center of the sphere. Since at the center \((r = 0)\), the term \(\frac{B_l}{r^{l+1}}\) diverges, we must have \(B_l = 0\) for all \(l\) and Eq. (3.165) reduces to

\[ V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)). \] (3.166)

On the surface of the sphere \(r = R\) the potential is \(V_0(\theta)\) then

\[ V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta)). \] (3.167)
Multiplying both sides by $P_m(\cos(\theta))\sin(\theta)$ and integrating over $\theta$,

$$
\int_0^\pi V_0(\theta)P_m(\cos(\theta))\sin(\theta)d\theta = \sum_{l=0}^\infty A_l R_l \int_0^\pi P_m(\cos(\theta))P_l(\cos(\theta))\sin(\theta)d\theta
$$

(3.168)

Applying the orthogonality condition for Legendre polynomials in Eq. (3.161), we have

$$
\int_0^\pi P_m(\cos(\theta))P_l(\cos(\theta))\sin(\theta)d\theta = \begin{cases} 
0 & l \neq m \\
\frac{2}{2l+1} & l = m
\end{cases} = \frac{2}{2l+1}\delta_{lm}
$$

(3.169)

so that Eq.(3.168) becomes

$$
\int_0^\pi V_0(\theta)P_m(\cos(\theta))\sin(\theta)d\theta = \sum_{l=0}^\infty A_l R_l \frac{2}{2l+1}\delta_{lm}
$$

(3.170)

$$
\Rightarrow A_m R_m \frac{2}{2m+1} = \int_0^\pi V_0(\theta)P_m(\cos(\theta))\sin(\theta)d\theta
$$

(3.171)

and this leads to

$$
A_m = \frac{2m+1}{2R_m} \int_0^\pi V_0(\theta)P_m(\cos(\theta))\sin(\theta)d\theta.
$$

(3.172)

For the potential $V_0(\theta) = k\sin^2(\theta/2)$,

$$
A_m = \frac{2m+1}{2R_m} \int_0^\pi k\sin^2(\theta/2)P_m(\cos(\theta))\sin(\theta)d\theta.
$$

(3.173)

In order to perform this integral it is more convenient if we express $\sin^2(\theta/2)$ as a linear combination of the Legendre polynomials. We note that

$$
\cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \left(1 - \sin^2\left(\frac{\theta}{2}\right)\right) - \sin^2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\left(\frac{\theta}{2}\right) \Rightarrow \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2} \left(1 - \cos(\theta)\right).
$$

(3.174)

From the Legendre polynomials listed above, we have

$$
1 = P_0(\cos(\theta)), \cos(\theta) = P_1(\cos(\theta))
$$

(3.175)

so that

$$
\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2} \left(1 - \cos(\theta)\right) = \frac{1}{2} \left(P_0(\cos(\theta)) - P_1(\cos(\theta))\right)
$$

(3.176)
Now substituting Eq. (3.176) into Eq. (3.173)
\[ A_m = \frac{2m + 1}{2R^m} k \int_0^\pi \frac{1}{2} (P_0(\cos(\theta)) - P_1(\cos(\theta))) P_m(\cos(\theta)) \sin(\theta)d\theta \]
\[ = \frac{2m + 1}{4R^m} k \left\{ \int_0^\pi P_0(\cos(\theta)) P_m(\cos(\theta)) \sin(\theta)d\theta - \int_0^\pi P_1(\cos(\theta)) P_m(\cos(\theta)) \sin(\theta)d\theta \right\} 
\]
\[ (3.177) \]

Using the orthogonality of the Legendre polynomials stated in Eq. (3.169), we find
\[ A_m = k \frac{2m + 1}{4R^m} \left( 2\delta_{0m} - \frac{2}{3} \delta_{1m} \right) \Rightarrow A_m = \begin{cases} \frac{k}{2R} & m = 0 \\ -\frac{k}{2R} & m = 1 \\ 0 & m > 1 \end{cases} 
\]
\[ (3.178) \]

Using the result in Eq. (3.178) the electric potential inside the sphere (Eq. (3.166)) can then be expressed as
\[ V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) = \frac{k}{2} \left( 1 - \frac{r}{R} P_1(\cos(\theta)) \right) = \frac{k}{2} \left( 1 - \frac{r \cos(\theta)}{R} \right). 
\]
\[ (3.179) \]

**Example 3.8** An uncharged metal sphere of radius \( R \) is placed in an otherwise uniform electric field \( \vec{E} = E_0 \hat{z} \). The field will push positive charge to the "northern" surface of the sphere, leaving negative charge on the "southern" surface (see Figure below). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere and the induced surface charge on the surface of the sphere.

**Solution:** Since the charge distribution inside and outside the sphere is zero, the electric potential satisfies the Laplace’s equation in spherical coordinates.
\[ \nabla^2 V(r, \theta) = 0 \]
\[ (3.180) \]

The solution of which is given by
\[ V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)). 
\]
\[ (3.181) \]

In terms of the electric potential the electric field is given by
\[ \vec{E} = -\nabla V(r, \theta) = -\frac{\partial V(r, \theta)}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V(r, \theta)}{\partial \theta} \hat{\theta} \Rightarrow \vec{E} = E_r \hat{r} + E_\theta \hat{\theta}, 
\]
\[ (3.182) \]
where

\[ E_r = -\frac{\partial V(r, \theta)}{\partial r}, \quad E_\theta = -\frac{1}{r} \frac{\partial V(r, \theta)}{\partial \theta} \quad (3.183) \]

For the radial and tangential component of the electric field, one can then write

\[ E_r = -\frac{\partial}{\partial r} \left[ \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)) \right] \]

\[ = -\sum_{l=0}^{\infty} \left( lA_l r^{l-1} - \frac{(l + 1)B_l}{r^{l+2}} \right) P_l(\cos(\theta)) \quad (3.184) \]

and

\[ E_\theta = -\frac{1}{r} \frac{\partial}{\partial \theta} \left[ \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)) \right] \]

\[ = -\sum_{l=0}^{\infty} \left( A_l r^{l-1} + \frac{B_l}{r^{l+2}} \right) \frac{d}{d \theta} P_l(\cos(\theta)) \quad (3.185) \]

**Boundary condition:**

(a) The sphere is a conductor and the electric field is zero inside \((E_{i\perp} = E_{i||} = 0)\). This means on the surface of the sphere \(r = R\), we must have

\[ [E_{\alpha\perp} - E_{i\perp}]_{r=R} = \frac{\sigma(R, \theta)}{\epsilon_0}, [E_{i||}]_{r=R} = [E_{i\perp}]_{r=R} \]

\[ \Rightarrow [E_r]_{r=R} = \frac{\sigma(R, \theta)}{\epsilon_0}, [E_\theta]_{r=R} = 0. \quad (3.186) \]
Also since the sphere is not grounded, the total charge on the surface of the sphere must be zero. This means

\[ Q = \int_0^{2\pi} \int_0^\infty \sigma(R, \theta) R^2 \sin(\theta) \, d\theta \, d\varphi = 0. \tag{3.187} \]

(b) The effect of the induced charge on the electric field is negligible far away from the sphere. As a result as we go far away from the sphere, we must expect the electric field to be a constant. This means \( r \to \infty \Rightarrow \vec{E}(r, \theta) = E_0 \hat{z} \) and this can be expressed, in terms of the unit vectors in spherical coordinates \( \hat{r} \) and \( \hat{\theta} \), as

\[ \vec{E} = E_0 \hat{z} = E_0 \cos(\theta) \hat{r} - E_0 \sin(\theta) \hat{\theta} \tag{3.188} \]

Now using Eq. (3.188) and the radial component in Eq. (3.188), for \( r \to \infty \)

\[ - \lim_{r \to \infty} \sum_{l=0}^{\infty} \left( A_l r^{l-1} - \frac{(l+1)B_l}{r^{l+2}} \right) P_l(\cos(\theta)) = E_0 \cos \theta \]

\[ \Rightarrow - \lim_{r \to \infty} \sum_{l=0}^{\infty} \left( A_l r^{l-1} \right) P_l(\cos(\theta)) = E_0 \cos \theta = E_0 P_1(\cos \theta) \]

\[ \lim_{r \to \infty} \left[ A_1 P_1(\cos(\theta)) + \sum_{l=2}^{\infty} \left( A_l r^{l-1} \right) P_l(\cos(\theta)) \right] = -E_0 P_1(\cos \theta) \tag{3.189} \]

The left side must be independent of \( r \) for the above equation to be true. This happens if

\[ A_l = \begin{cases} -E_0 & l = 1 \\ 0 & l \neq 1 \end{cases} \tag{3.190} \]

Therefore, using the result in Eq. (3.190), we may write the electric potential as

\[ V(r, \theta) = A_1 r P_1(\cos(\theta)) + \sum_{l=0}^{\infty} \left( \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)) \]

\[ \Rightarrow V(r, \theta) = -E_0 r \cos(\theta) + \sum_{l=0}^{\infty} \left( \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)) \tag{3.191} \]

Due to the external electric field we expect positive charges pushed up to the top surface while equal magnitude of negative charges pulled down to bottom surface of the sphere. This tells us that the equatorial plane on the surface of the sphere must experience a zero potential. We also know that a conductor surface is an equipotential surface and the potential must
be a constant. Therefore this leads to the conclusion that \( V(r, \theta) = 0 \) for \( r = R \). Hence

\[
V(R, \theta) = -E_0 R \cos(\theta) + \sum_{l=0}^{\infty} \left( \frac{B_l}{R^{l+1}} \right) P_l(\cos(\theta)) = 0
\] (3.192)

which results in

\[
B_l = \begin{cases} 
E_0 R^3 & l = 1 \\
0 & l \neq 1
\end{cases}.
\] (3.193)

Then the electric potential outside the sphere can then be written as

\[
V(r, \theta) = -E_0 r \cos(\theta) + \frac{E_0 R^3}{r^2} \cos(\theta)
\]

\[
\Rightarrow V(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos(\theta).
\] (3.194)

To find the induced surface charge we use

\[
E_{\perp \text{out}} - E_{\perp \text{in}} = \frac{\sigma}{\varepsilon_0}
\] (3.195)

where \( E_{\perp \text{out}} \) and \( E_{\perp \text{in}} \) are the normal component of the electric field outside and inside the sphere, respectively. Since the sphere is a conductor, the electric field is zero inside. Hence

\[
E_{\perp \text{out}} = \frac{\sigma}{\varepsilon_0} \Rightarrow -\frac{\partial V(r, \theta)}{\partial r} \bigg|_{r=R} = \frac{\sigma}{\varepsilon_0}
\]

\[
\Rightarrow \sigma = -\varepsilon_0 \frac{\partial}{\partial r} \left[ -E_0 \left( r - \frac{R^3}{r^2} \right) \cos(\theta) \right] \bigg|_{r=R}
\]

\[
\Rightarrow \sigma = \varepsilon_0 \left[ E_0 \left(1 + \frac{2R^3}{r^3} \right) \cos(\theta) \right] \bigg|_{r=R} \Rightarrow \sigma = 3\varepsilon_0 E_0 \cos(\theta)
\] (3.196)

**Example 3.9** A charge density \( \sigma_0(\theta) = k \cos(\theta) \) is glued over the surface of a spherical shell of radius \( R \). Find the resulting potential inside and outside the sphere.

**Solution:** From the general solution for the Laplace’s equation in spherical coordinates

\[
V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)).
\] (3.197)

the electric potential inside the shell can be expressed as

\[
V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))
\] (3.198)
since \( \frac{B_l}{r^l} \) diverges for \( r = 0 \), we must set \( B_l = 0 \) for \( l \). However, for the potential outside the shell since \( A_l r^l \) diverges as \( r \to \infty \), \( A_l = 0 \) for all \( l \).

Then the potential outside the sphere is given by

\[
V_{\text{out}}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta)). \tag{3.199}
\]

Using the boundary condition for the continuity of the electric potential

\[
V_{\text{in}}(r = R, \theta) = V_{\text{out}}(r = R, \theta) \tag{3.200}
\]

we find

\[
\sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta)) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos(\theta)) \Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow B_l = A_l R^{2l+1} \tag{3.201}
\]

and the potential outside the shell can be rewritten as

\[
V_{\text{out}}(r, \theta) = \sum_{l=0}^{\infty} \frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos(\theta)). \tag{3.202}
\]

From the second boundary condition (the normal component of the electric field is discontinuous by an amount proportional to the surface charge density, we have

\[
E_{\perp \text{out}}(R, \theta) - E_{\perp \text{in}}(R, \theta) = \frac{\sigma(\theta)}{\epsilon_0}, \tag{3.203}
\]

where \( E_{\perp \text{out}} \) and \( E_{\perp \text{in}} \) are the normal component of the electric fields outside and inside the sphere, respectively. The normal component of the electric fields are the radial components and therefore we may write the surface charge density in terms of the electric potentials as

\[
- \left[ \frac{\partial V_{\text{out}}(r, \theta)}{\partial r} - \frac{\partial V_{\text{in}}(r, \theta)}{\partial r} \right]_{r=R} = \frac{\sigma(\theta)}{\epsilon_0}. \tag{3.204}
\]

We now substitute the expressions for the electric potentials and for the surface charge density

\[
\left[ - \frac{\partial}{\partial r} \left( \sum_{l=0}^{\infty} \frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos(\theta)) \right) + \frac{\partial}{\partial r} \left( \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) \right) \right]_{r=R} = \frac{k \cos(\theta)}{\epsilon_0} \tag{3.205}
\]

\[
\sum_{l=0}^{\infty} \left[ A_l (l + 1) \frac{R^{2l+1}}{R^{l+2}} + A_l l R^{l-1} \right] P_l(\cos(\theta)) = \frac{k \cos(\theta)}{\epsilon_0} = \frac{k P_1(\cos(\theta))}{\epsilon_0} \tag{3.206}
\]
Multiplying both sides by $P_m(\cos(\theta)) \sin(\theta) \, d\theta$ and integrating with respect to $\theta$, we have

$$\sum_{l=0}^{\infty} \left[ \frac{(l+1) R^{2l+1}}{R^{l+2}} + lR^{l-1} \right] A_l \int_{0}^{\pi} P_l(\cos(\theta)) P_m(\cos(\theta)) \sin(\theta) \, d\theta$$

$$= \frac{k}{\epsilon_0} \int_{0}^{\pi} P_1(\cos(\theta)) P_m(\cos(\theta)) \sin(\theta) \, d\theta.$$  

Applying the orthogonality condition for the Legendre polynomials

$$\int_{0}^{\pi} P_m(\cos(\theta)) P_l(\cos(\theta)) \sin(\theta) \, d\theta = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases} = \delta_{lm} \frac{2}{2l+1}$$

we find

$$\sum_{l=0}^{\infty} \left[ \frac{(l+1) R^{2l+1}}{R^{l+2}} + lR^{l-1} \right] 2A_l \delta_{lm} = \frac{2k}{3\epsilon_0} \delta_{lm}$$

$$\Rightarrow \left[ \frac{(m+1) R^{2m+1}}{R^{m+2}} + mR^{m-1} \right] 2A_m = \frac{2k}{3\epsilon_0} \delta_{1m}$$

$$\Rightarrow 2A_m R^{m-1} = \frac{2k}{3\epsilon_0} \delta_{1m} \Rightarrow A_m = \frac{k}{3\epsilon_0 R^{m-1}} \delta_{1m}. \quad (3.209)$$

Substituting this result into Eqs. (3.198) and (3.202), the potentials inside and outside the shell are given by

$$V(r, \theta) = \begin{cases} \frac{kR^3}{3\epsilon_0} \frac{1}{r} \cos \theta, & r \geq R \\ \frac{k}{3\epsilon_0} \int \frac{1}{r} \cos \theta, & r \leq R \end{cases} \quad (3.210)$$

**Example 3.10** A spherical shell of radius $R$ carries a uniform surface charge $\sigma_0$ on the northern hemisphere and a uniform surface charge $-\sigma_0$ on the southern hemisphere. Find the potential inside and outside the sphere, calculating the coefficients explicitly up to $A_6$ and $B_6$

**Solution:** There is no charge both inside and outside the sphere. That means the potential satisfies Laplace’s equation in the region $r < R$ and $r > R$. Then from the general solution for the Laplace’s equation

$$V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)).$$  

the electric potential inside the shell ($r < R$) can be expressed as

$$V_{in}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$  

(3.212)
since \( \frac{B_l}{r^{l+1}} \) diverges for \( r = 0 \), we must set \( B_l = 0 \) for all \( l \). However, for the potential outside the shell since \( A_l r^l \) diverges as \( r \to \infty \), \( A_l = 0 \) for all \( l \). Then the potential outside the sphere is given by

\[
V_{\text{out}}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta)). \tag{3.213}
\]

Using the boundary condition for the continuity of the electric potential

\[
V_{\text{in}}(r = R, \theta) = V_{\text{out}}(r = R, \theta) \tag{3.214}
\]

we find

\[
\sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta)) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos(\theta)) \Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow B_l = A_l R^{2l+1} \tag{3.215}
\]

and the potential outside the shell can be rewritten as

\[
V_{\text{out}}(r, \theta) = \sum_{l=0}^{\infty} \frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos(\theta)). \tag{3.216}
\]

From the second boundary condition (the normal component of the electric field is discontinuous by an amount proportional to the surface charge density, \( \sigma(\theta) \), we have

\[
E_{\perp \text{out}}(R, \theta) - E_{\perp \text{in}}(R, \theta) = \frac{\sigma(\theta)}{\varepsilon_0}, \tag{3.217}
\]

or in terms of the electric potentials

\[
- \left[ \frac{\partial V_{\text{out}}(r, \theta)}{\partial r} - \frac{\partial V_{\text{in}}(r, \theta)}{\partial r} \right]_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0}. \tag{3.218}
\]

We now substitute the expressions for the electric potentials and for the surface charge density

\[
\left[ \frac{\partial}{\partial r} \left( \sum_{l=0}^{\infty} A_l R^{2l+1} P_l(\cos(\theta)) \right) + \frac{\partial}{\partial r} \left( \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) \right) \right]_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0} \tag{3.219}
\]

\[
\sum_{l=0}^{\infty} \frac{(l+1) R^{2l+1}}{R^{l+2}} + l R^{l-1} A_l P_l(\cos(\theta)) = \frac{\sigma(\theta)}{\varepsilon_0}.
\]
Multiplying both sides by $P_m(\cos (\theta)) \sin (\theta) \, d\theta$ and integrating with respect to $\theta$, we have

$$
\sum_{l=0}^{\infty} \left[ \frac{(l+1) R^{2l+1}}{R^{l+2}} + l R^{l-1} \right] A_l \int_0^\pi P_l(\cos (\theta)) P_m(\cos (\theta)) \sin (\theta) \, d\theta
= \frac{1}{\epsilon_0} \int_0^\pi \sigma (\theta) P_m(\cos (\theta)) \sin (\theta) \, d\theta.
$$

(3.220)

Applying the orthogonality condition for the Legendre polynomials

$$
\int_0^\pi P_m(\cos (\theta)) P_l(\cos (\theta)) \sin (\theta) \, d\theta = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m = \delta_{lm} \\ \frac{2}{2l+1} & l = m \end{cases}
$$

(3.222)

we find

$$
\sum_{l=0}^{\infty} \left[ \frac{(l+1) R^{2l+1}}{R^{l+2}} + l R^{l-1} \right] A_l \frac{2 A_{lm}}{2l+1} = \frac{1}{\epsilon_0} \int_0^\pi \sigma (\theta) P_m(\cos (\theta)) \sin (\theta) \, d\theta
$$

$$
\Rightarrow \left[ \frac{(m+1) R^{2m+1}}{R^{m+2}} + m R^{m-1} \right] A_m \frac{2 A_m}{2m+1} = \frac{1}{\epsilon_0} \int_0^\pi \sigma (\theta) P_m(\cos (\theta)) \sin (\theta) \, d\theta
$$

$$
\Rightarrow 2 A_m R^{m-1} = \frac{1}{\epsilon_0} \int_0^\pi \sigma (\theta) P_m(\cos (\theta)) \sin (\theta) \, d\theta
$$

$$
\Rightarrow A_m = \frac{1}{2 \epsilon_0 R^{m-1}} \int_0^\pi \sigma (\theta) P_m(\cos (\theta)) \sin (\theta) \, d\theta.
$$

(3.223)

We recall that the spherical shell carries a uniform surface charge $\sigma_0$ on the northern hemisphere and a uniform surface charge $-\sigma_0$ on the southern hemisphere,

$$
\sigma (\theta) = \begin{cases} \sigma_0 & 0 \leq \theta \leq \pi/2 \\ -\sigma_0 & \pi/2 \leq \theta \leq \pi \end{cases}.
$$

(3.224)
Substituting this into the expression for $A_m$, we may write

$$A_m = \frac{\sigma_0}{2\epsilon_0 R^{m-1}} \left[ \int_0^{\pi/2} P_m(\cos(\theta)) \sin(\theta) d\theta - \int_{\pi/2}^{\pi} P_m(\cos(\theta)) \sin(\theta) d\theta \right].$$

(3.225)

Introducing the transformation defined by

$$\theta = \theta' - \pi$$

(3.226)

we may write the second integral as

$$I_2 = \int_{-\pi/2}^{0} P_m(\cos(\theta')) (\sin(\theta')) d\theta' = -\int_{0}^{-\pi/2} P_m(-\cos(\theta')) (\sin(\theta')) d\theta'$$

(3.227)

Noting that sin is an odd function and cos is an even function we can put the above equation in the form

$$I_2 = \int_{0}^{\pi/2} P_m(-\cos(\theta')) (\sin(\theta')) d\theta' = \int_{0}^{\pi/2} P_m(-\cos(\theta')) (\sin(\theta')) d\theta'$$

(3.228)

Substituting this into the expression for $A_m$, we find

$$A_m = \frac{\sigma_0}{2\epsilon_0 R^{m-1}} \left[ \int_0^{\pi/2} P_m(\cos(\theta)) \sin(\theta) d\theta - \int_{0}^{\pi/2} P_m(-\cos(\theta')) \sin(\theta') d\theta' \right].$$

(3.229)

Recalling that $P_m(x)$ is generated by the Rodrigue’s formula

$$P_m(x) = \frac{1}{2^m m!} \left( \frac{d}{dx} \right)^m (x^2 - 1)^m$$

(3.231)

we have

$$P_m(-x) = \begin{cases} -P_m(x) & m = \text{odd} \\ P_m(x) & m = \text{even} \end{cases}$$

(3.232)
which leads to

\[ P_m(\cos(\theta)) - P_m(-\cos(\theta)) = \begin{cases} 2P_m(x) & m = \text{odd} \\ 0 & m = \text{even} \end{cases} \quad (3.233) \]

Therefore

\[ A_m = \frac{\sigma_0}{2\epsilon_0 R^{m-1}} \int_0^{\pi/2} [P_m(\cos(\theta)) - P_m(-\cos(\theta)) \sin(\theta)] \, d\theta \quad (3.234) \]

becomes

\[ A_m = \frac{\sigma_0}{\epsilon_0 R^{m-1}} \int_0^{\pi/2} P_m(\cos(\theta)) \sin(\theta) \, d\theta \quad \text{For odd } m \quad (3.235) \]

\[ A_m = 0, \quad \text{For even } m \quad (3.236) \]

We can then write the the electric potential inside the sphere as

\[ V_{in}(r, \theta) = \sum_{l=0}^{\infty} A_{2l+1} r^{2l+1} P_{2l+1}(\cos(\theta)) \quad (3.237) \]

and outside the sphere becomes

\[ V_{out}(r, \theta) = \sum_{l=0}^{\infty} A_{2l+1} \frac{R^{4l+3}}{r^{2l+2}} P_{2l+1}(\cos(\theta)), \quad (3.238) \]

where

\[ A_{2l+1} = \frac{\sigma_0}{\epsilon_0 R^{2l+1}} \int_0^{\pi/2} P_{2l+1}(\cos(\theta)) \sin(\theta) \, d\theta \quad (3.239) \]

Using Mathematica, we may evaluate the integral in the above expression.
so that
\[ V_{in}(r, \theta) = \frac{\sigma_0 r}{2\epsilon_0} \left[ P_1(\cos \theta) - \frac{1}{4} \left( \frac{r}{R} \right)^2 P_3(\cos \theta) + \frac{1}{8} \left( \frac{r}{R} \right)^4 P_5(\cos \theta) + \ldots \right], r \leq R, \]

and
\[ V_{out}(r, \theta) = \frac{\sigma_0 R^3}{2\epsilon_0 r^2} \left[ P_1(\cos \theta) - \frac{1}{4} \left( \frac{R}{r} \right)^2 P_3(\cos \theta) + \frac{1}{8} \left( \frac{R}{r} \right)^4 P_5(\cos \theta) + \ldots \right], r \geq R. \]

### 3.4 Multipole Expansion

Multipole expansion is useful to find approximate Potential and the electric field at large distances for localized charge distribution.

**Example 3.11** A (Physical) electric dipole consists of two equal and opposite charges (±q) separated by a distance d. In the previous chapter we have seen the electric field vector and lines for a dipole are very different from a monopole (see Fig. 3.7) close to the dipole. We are interested in the behavior of the field of a dipole at a point far away from the dipole. We use the potential at this distance to find the electric field.
Find the approximate potential and electric field of a dipole at points far from the dipole.

**Sol:** To this end we put the dipole along the z-axis centered about the origin as shown in Fig. 3.8 The electric potential due to the two charges a distance $r$ from the mid point of the charges is given by

$$V(r) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{r_+} - \frac{1}{r_-} \right] = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{\sqrt{r^2 + \frac{d^2}{2}}} - \frac{1}{\sqrt{r^2 + \frac{d^2}{2}}} \right]$$

(3.242)
Using cosine law we have

\[ \vec{r}^+ = \sqrt{r^2 + \frac{d^2}{4} - 2rd \cos (\theta)} = r \sqrt{1 + \left( \frac{d}{2r} \right)^2 - \frac{d}{r} \cos (\theta)} \] (3.243)

\[ \vec{r}^- = \sqrt{r^2 + \frac{d^2}{4} - rd \cos(180 - \theta)} = \sqrt{r^2 + \frac{d^2}{4} + rd \cos(\theta)} \]

\[ = r \sqrt{1 + \left( \frac{d}{2r} \right)^2 + \frac{d}{r} \cos (\theta)} \] (3.244)

so that

\[ V(r) = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{\vec{r}^+} - \frac{1}{\vec{r}^-} \right] = \frac{q}{4\pi \varepsilon_0 r} \left[ \frac{1}{\sqrt{1 + \left( \frac{d}{2r} \right)^2 - \frac{d}{r} \cos (\theta)}} - \left( 1 + \left( \frac{d}{2r} \right)^2 + \frac{d}{r} \cos (\theta) \right)^{-1/2} \right] \] (3.245)

For \( r >> d \) we can make the approximation using Binomial expansion

\[ \left( 1 + \left( \frac{d}{2r} \right)^2 - \frac{d}{r} \cos (\theta) \right)^{-1/2} \simeq 1 - \frac{1}{2} \left( \frac{d}{2r} \right)^2 + \frac{d}{r} \cos (\theta) \]

\[ \left( 1 + \left( \frac{d}{2r} \right)^2 + \frac{d}{r} \cos (\theta) \right)^{-1/2} \simeq 1 - \frac{1}{2} \left( \frac{d}{2r} \right)^2 - \frac{d}{r} \cos (\theta) . \] (3.246)

Then the approximate electric potential becomes

\[ V(r) = \frac{q}{4\pi \varepsilon_0 r} \left[ 1 - \frac{1}{2} \left( \frac{d}{2r} \right)^2 + \frac{d}{2r} \cos (\theta) - 1 + \frac{1}{2} \left( \frac{d}{2r} \right)^2 + \frac{d}{2r} \cos (\theta) \right] \]

\[ = \frac{q}{4\pi \varepsilon_0 r^2} d \cos (\theta) = \frac{qd}{4\pi \varepsilon_0 r^2} \cos (\theta) \Rightarrow V(r) = \frac{qd}{4\pi \varepsilon_0 r^2} \cos (\theta) \] (3.247)

This is the electric potential for a dipole. It is proportional to \( 1/r^2 \) unlike a monopole which is proportional to \( 1/r \).

The electric field of a dipole: Let’s define a vector \( \vec{d} \) that points from the negative to the positive charge,

\[ \vec{d} = d\hat{z}, \] (3.248)
and a unit vector, \( \hat{r} \), along \( \vec{r} \). Then using the relation for the dot product of two vectors one can write,

\[
\cos (\theta) = \frac{\hat{r} \cdot \vec{d}}{d},
\]

and the electric potential for a dipole can be put in the form,

\[
V_{\text{dip}}(r, \theta) = \frac{\hat{r} \cdot q \vec{d}}{4\pi \epsilon_0 r^2} = \frac{\hat{r} \cdot \vec{p}}{4\pi \epsilon_0 r^2},
\]

where

\[
\vec{p} = q d \hat{z}
\]

is known as the dipole moment. A dipole moment is a vector quantity directed from the negative to the positive charge making the dipole. The electric field

\[
\vec{E} = -\nabla V_{\text{dip}}(r, \theta)
\]

in spherical coordinates can be written as

\[
\vec{E} = E_r \hat{r} + E_\theta \hat{\theta} + E_\varphi \hat{\varphi}
\]

where

\[
\hat{r} = \sin (\theta) \cos (\varphi) \hat{x} + \sin (\theta) \sin (\varphi) \hat{y} + \cos (\theta) \hat{z},
\]

\[
\hat{\theta} = \cos (\theta) \cos (\varphi) \hat{x} + \cos (\theta) \sin (\varphi) \hat{y} - \sin (\theta) \hat{z},
\]

\[
\hat{\varphi} = -\sin (\varphi) \hat{x} + \cos (\varphi) \hat{y}.
\]

and

\[
\frac{\partial \hat{r}}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ \sin (\theta) \cos (\varphi) \hat{x} + \sin (\theta) \sin (\varphi) \hat{y} + \cos (\theta) \hat{z} \right]
\]

\[
= \cos (\theta) \cos (\varphi) \hat{x} + \cos (\theta) \sin (\varphi) \hat{y} - \sin (\theta) \hat{z} = \hat{\theta},
\]

\[
\frac{\partial \hat{r}}{\partial \varphi} = \frac{\partial}{\partial \varphi} \left[ \sin (\theta) \cos (\varphi) \hat{x} + \sin (\theta) \sin (\varphi) \hat{y} + \cos (\theta) \hat{z} \right]
\]

\[
= -\sin (\varphi) \hat{x} + \sin (\theta) \cos (\varphi) \hat{y} = \sin (\theta) \hat{\varphi}
\]

Noting that

\[
E_r = -\frac{\partial V_{\text{dip}}(r, \theta)}{\partial r} = -\frac{\partial}{\partial r} \left( \frac{\hat{r} \cdot \vec{p}}{4\pi \epsilon_0 r^2} \right) = \frac{2\hat{r} \cdot \vec{p}}{4\pi \epsilon_0 r^3},
\]

\[
E_\theta = -\frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{\hat{r} \cdot \vec{p}}{4\pi \epsilon_0 r^2} \right] = -\frac{\vec{p} \cdot \partial \hat{r}}{4\pi \epsilon_0 r^3} = -\frac{\vec{p} \cdot \hat{\theta}}{4\pi \epsilon_0 r^3},
\]

and

\[
E_\varphi = -\frac{1}{r \sin (\theta)} \frac{\partial V_{\text{dip}}(r, \theta)}{\partial \varphi} = -\frac{1}{r \sin (\theta)} \frac{\partial}{\partial \varphi} \left( \frac{\hat{r} \cdot \vec{p}}{4\pi \epsilon_0 r^2} \right)
\]

\[
= -\frac{1}{r \sin (\theta)} \frac{\vec{p} \cdot \partial \hat{r}}{4\pi \epsilon_0 r^2} = -\frac{\vec{p} \cdot \hat{\varphi}}{4\pi \epsilon_0 r^3} = -\frac{\vec{p} \cdot \hat{\varphi}}{4\pi \epsilon_0 r^3}
\]
The electric field can be expressed as

\[ \vec{E} = \left( \frac{2\hat{r} \cdot \vec{p}}{4\pi \varepsilon_0 r^3} \right) \hat{r} - \left( \frac{\vec{p} \cdot \hat{\theta}}{4\pi \varepsilon_0 r^3} \right) \hat{\theta} - \left( \frac{\vec{p} \cdot \hat{\varphi}}{4\pi \varepsilon_0 r^3} \right) \hat{\varphi} \]

Suppose the dipole moment has all its three components in spherical coordinates. Then one can write

\[ \vec{p} = p_r \hat{r} + p_\theta \hat{\theta} + p_\varphi \hat{\varphi} = (\vec{p} \cdot \hat{r}) \hat{r} + (\vec{p} \cdot \hat{\theta}) \hat{\theta} + (\vec{p} \cdot \hat{\varphi}) \hat{\varphi} \]

\[ \Rightarrow (\vec{p} \cdot \hat{\theta}) \hat{\theta} + (\vec{p} \cdot \hat{\varphi}) \hat{\varphi} = \vec{p} - (\vec{p} \cdot \hat{r}) \hat{r} \]

and the electric field becomes

\[ \vec{E} = \left( \frac{2\hat{r} \cdot \vec{p}}{4\pi \varepsilon_0 r^3} \right) \hat{r} - \frac{1}{4\pi \varepsilon_0 r^3} [\vec{p} - (\vec{p} \cdot \hat{r}) \hat{r}] = \frac{1}{4\pi \varepsilon_0 r^3} [3 (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}] \]  

Equation (3.262) represent the electric field of a dipole.

**Example 3.12** Develop a systematic expansion of the electric potential for arbitrary volume charge distribution \( \rho(r) \) outside the charge distribution.

**Solution:** The electric potential at point \( p \) due to the volume charge distribution is given by

\[ V(r) = \frac{1}{4\pi \varepsilon_0} \int_{\text{vol}} \frac{\rho(r')}{|r - r'|} d\tau' \]  

Using the cosine law we may write

\[ |r - r'| = \sqrt{r^2 + r'^2 - 2rr' \cos \theta'} = r \left( 1 + \frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \theta' \right)^{1/2} \]  

\[ (3.264) \]
if we define a new variable

\[ \epsilon = \frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \theta' = \frac{r'}{r} \left( \frac{r'}{r} - 2 \cos \theta' \right) \]  
(3.265)

for points outside the charge distribution (\( \epsilon < 1 \)) using Binomial expansion we can write

\[ \left( 1 + \frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \theta' \right)^{1/2} = (1 + \epsilon)^{-1/2} \]
\[ \Rightarrow (1 + \epsilon)^{-1/2} = 1 - \frac{1}{2} \epsilon + \frac{3}{8} \epsilon^2 - \frac{5}{16} \epsilon^3 + \ldots \]  
(3.266)

then

\[ \frac{1}{|\mathbf{r}' - \mathbf{r}|^2} = \frac{1}{r} \left( 1 + \frac{r'^2}{r^2} - 2 \frac{r'}{r} \cos \theta' \right)^{1/2} = \frac{1}{r} (1 + \epsilon)^{-1/2} \]
\[ = \frac{1}{r} \left[ 1 - \frac{1}{2} \epsilon + \frac{3}{8} \epsilon^2 - \frac{5}{16} \epsilon^3 + \ldots \right] \]  
(3.267)

\[ \frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r} \left[ 1 - \frac{1}{2} \frac{r'}{r} \left( \frac{r'}{r} - 2 \cos \theta' \right) + \frac{3}{8} \frac{r'^2}{r^2} \left( \frac{r'}{r} - 2 \cos \theta' \right)^2 \right. \]
\[ - \frac{5}{16} \frac{r'^3}{r^3} \left( \frac{r'}{r} - 2 \cos \theta' \right)^3 + \ldots \]  
(3.268)

The above expression can be rewritten as

\[ \frac{1}{|\mathbf{r}' - \mathbf{r}|^2} = \frac{1}{r} \left[ 1 + \frac{r'}{r} \cos \theta' + \frac{r'^2}{r^2} \left( 3 \cos \theta' - 1 \right) + \frac{r'^3}{r^3} \left( 5 \cos^3 \theta' - 3 \right) + \ldots \right] \]  
(3.269)

Using the Legendre polynomials, we can write

\[ \frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r} \left[ P_0(\cos \theta') + \frac{r'}{r} P_1(\cos \theta') + \frac{r'^2}{r^2} P_2(\cos \theta') + \frac{r'^3}{r^3} P_3(\cos \theta') + \ldots \right] \]  
(3.270)

\[ \Rightarrow \frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r} \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \theta') = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \theta'). \]  
(3.271)

Therefore, the electric potential is expressible as

\[ V(r) = \frac{1}{4\pi \epsilon_0} \int_{\text{vol}} \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \theta') \rho(\mathbf{r'}) d\mathbf{r'} \]
\[ V(r) = \frac{1}{4\pi \epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int_{\text{vol}} r'^l P_l(\cos \theta') \rho(\mathbf{r'}) d\mathbf{r'} \]  
(3.272)
or

\[
V(r) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{1}{r} \int_{\text{vol}} \rho(r')dr' + \frac{1}{r^2} \int_{\text{vol}} r' P_1(\cos\theta')\rho(r')dr' + \frac{1}{r^3} \int_{\text{vol}} r'^2 P_2(\cos\theta')\rho(r')dr' + \ldots \right\}. 
\]

This is the multipole expansion. It shows that the electric potential for

(i) A monopole: \( V \sim 1/r \)

(ii) A dipole: \( V \sim 1/r^2 \)

(iii) A quadrupole: \( V \sim 1/r^3 \)

\[\text{Figure 3.9: The electric field and lines of a quadrupole.}\]

**Example 3.13** A sphere of radius \( R \), centered at the origin, carries charge density

\[
\rho(r, \theta) = k \frac{R}{r^2} (R - 2r) \sin \theta
\]

where \( k \) is a constant, and \( r, \theta \) are the usual spherical coordinates. Find the approximate potential for points on the \( z \) axis, far from the sphere

**Solution** The electric potential for points on the \( z \) axis can be expressed using the multipole expansion in Eq. (3.273). For the first terms (the monopole
3.4. MULTIPOLE EXPANSION

\[ \int_{\text{vol}} \rho(\vec{r}) d\tau = \int \int \int k \frac{R}{r^2} (R - 2r) \sin \theta r^2 dr \sin \theta d\theta d\varphi \]

\[ = k2\pi R \int \int \left( \frac{R}{r^2} - \frac{2}{r} \right) r^2 dr \sin^2(\theta) d\theta = k2\pi R \int \frac{R}{r^2} \left( \frac{R}{r^2} - \frac{2}{r} \right) r^2 dr \]

\[ = k\pi^2 R \left( \int Rdr - 2 \int r dr \right) = k\pi^2 R (R^2 - R^2) = \int_{\text{vol}} \rho(\vec{r}) d\tau' = 0 \]

(3.275)

This means the monopole contribution is zero. Next let’s consider the dipole term

\[ \int_{\text{vol}} rP_1(\cos \theta) \rho(\vec{r}) d\tau = \int \int \int k \frac{R}{r^2} (R - 2r) \sin \theta \cos \theta r^3 dr \sin \theta d\theta d\varphi \]

\[ = k2\pi R \int \int \left( \frac{R}{r^2} - \frac{2}{r} \right) r^3 dr \sin^2(\theta) \cos \theta d\theta \]

(3.276)

Since integration with respect to \( \theta \) gives

\[ \int_0^\pi \sin^2(\theta) \cos \theta d\theta = 0 \]

(3.277)

the dipole contribution is also zero

\[ \int_{\text{vol}} rP_1(\cos \theta) \rho(\vec{r}) d\tau = 0 \]

(3.278)

Let’s consider the next term the quadrupole term which can be expressed as

\[ \int_{\text{vol}} r^2P_2(\cos \theta) \rho(\vec{r}) d\tau = \int \int \int k \frac{R}{r^2} (R - 2r) \sin \theta r^2 P_2(\cos \theta) r^2 dr \sin \theta d\theta d\varphi \]

\[ = kR \int_0^R (R - 2r) r^2 dr \int_0^\pi P_2(\cos \theta) \sin^2 \theta d\theta \int_0^{2\pi} d\varphi \]

\[ = kR \left[ \frac{R^4}{3} - \frac{2R^4}{4} \right] \int_0^\pi P_2(\cos \theta) \sin^2 \theta d\theta (2\pi) \]

(3.279)

Using the result
Integration with respect to $r$, $\theta$, and $\varphi$ leads to

$$
\int_{\text{vol}} r^2 P_2(\cos \theta) \rho(\mathbf{r}) d\tau = k \frac{R^5 \pi^2}{48}
$$

(3.280)

Therefore the potential for points far from the sphere is given by

$$
V(r) \approx \frac{1}{4\pi \epsilon_0} \left( \frac{k\pi^2 R^5}{48 z^3} \right)
$$

(3.281)

The monopole and dipole terms

The monopole term

$$
V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \frac{Q}{r}
$$

(3.282)

represents the exact potential for a point charge placed at the origin.

The dipole term

$$
V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \frac{1}{r^2} \int_{\text{vol}} r' \cos \theta' \rho(\mathbf{r}') d\tau'
$$

(3.283)

Recalling that $\theta'$ is the angle between $\mathbf{r}$ and $\mathbf{r}'$ and using the dot product of these two vectors, we have

$$
\mathbf{r} \cdot \mathbf{r}' = rr' \cos \theta' \Rightarrow \cos \theta' = \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'}
$$

(3.284)

so that

$$
V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \frac{1}{r^2} \int_{\text{vol}} \mathbf{r} \cdot \mathbf{r}' \rho(\mathbf{r}') d\tau' = \frac{1}{4\pi \epsilon_0} \frac{1}{r^3} \int_{\text{vol}} \mathbf{r} \cdot \mathbf{r}' \rho(\mathbf{r}') d\tau'
$$

$$
= \frac{1}{4\pi \epsilon_0} \int_{\text{vol}} \mathbf{r} \cdot \rho(\mathbf{r}') d\tau' \Rightarrow V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \frac{\mathbf{r} \cdot \mathbf{p}}{r^3} = \frac{1}{4\pi \epsilon_0} \frac{\mathbf{r} \cdot \mathbf{p}'}{r^2}
$$

(3.285)

where

$$
\mathbf{p} = \int_{\text{vol}} \mathbf{r} \rho(\mathbf{r}') d\tau'
$$

(3.286)

is known as the dipole moment. It is determined by the geometry of the charge distribution (size, shape, and density). For a collection of point charges, the dipole moment is given by

$$
\mathbf{p} = \sum_{i=1}^{n} \mathbf{r}_i q_i
$$

(3.287)
for a "physical" dipole (equal and opposite charges, $\pm q$)

$$\vec{p} = q (\vec{r}_+ - \vec{r}_-) = q\vec{d}$$ (3.288)

$d$ is the vector from the negative to the positive charge

*Dipole moments are vectors and should be added like vectors.* For example the net dipole moment for the following charge configuration (quadrupole) is zero.

*Origin of Coordinates in Multipole expansion.* Moving the origin does not change the monopole, $Q$ since the total charge is independent of the origin. However, it does change the dipole term except when the total charge is zero.
Example 3.14 A "pure" dipole $p$ is situated at the origin, pointing in the $z$-direction.

(a) What is the force on a point charge $q$ at $(a, 0, 0)$ (Cartesian coordinates)?

(b) What is the force on $q$ at $(0, 0, a)$?

(c) How much work does it take to move $q$ from $(a, 0, 0)$ to $(0, 0, a)$

Solution:

(a) To find the force we need to get the electric field due to the dipole at the position of the charge. This electric field, from the result above, is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3} (3 (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}) \tag{3.289}$$

and for a point charge located at $(a, 0, 0)$, we have

$$\hat{r} = \hat{x}, r = a. \tag{3.290}$$

The dipole is located at the origin and pointing along the $z$-direction and is given by

$$\vec{p} = p\hat{z} \tag{3.291}$$

Therefore, the electric field becomes

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 a^3} (3 (p\hat{z} \cdot \hat{x}) \hat{x} - p\hat{z}) \Rightarrow \vec{E} = -\frac{p}{4\pi\varepsilon_0 a^3} \hat{z} \tag{3.292}$$

and the force on a point charge $q$

$$\vec{F} = q\vec{E} = -\frac{pq}{4\pi\varepsilon_0 a^3} \hat{z}. \tag{3.293}$$
(b) For a charge located at $(0,0,a)$, we have

$$\hat{r} = \hat{z}, r = a$$  \hspace{1cm} (3.294)

and the electric field

$$E = \frac{1}{4\pi\varepsilon_0 a^3} \left[ 3 (p\hat{z} \cdot \hat{z}) \hat{z} - p\hat{z} \right] \Rightarrow E = \frac{2p}{4\pi\varepsilon_0 a^3} \hat{z} = \frac{p}{2\pi\varepsilon_0 a^3} \hat{z}. \hspace{1cm} (3.295)$$

The work done can be obtained using

$$W = q \left( V(0,0,a) - V(a,0,0) \right). \hspace{1cm} (3.296)$$

We recall the electric potential of a dipole is

$$V_{dp}(r,\theta) = \frac{\hat{r} \cdot \vec{p}}{4\pi\varepsilon_0 r^2}. \hspace{1cm} (3.297)$$

Using this expression the electric potential at

$$(0,0,a) \Rightarrow \hat{r} = \hat{z}, r = a$$  \hspace{1cm} (3.298)

is

$$V(0,0,a) = \frac{\hat{z} \cdot p\hat{z}}{4\pi\varepsilon_0 a^2} = \frac{p}{4\pi\varepsilon_0 a^2} \hspace{1cm} (3.299)$$

and

$$(a,0,0) \Rightarrow \hat{r} = \hat{x}, r = a$$  \hspace{1cm} (3.300)

$$V(a,0,0) = \frac{\hat{x} \cdot p\hat{z}}{4\pi\varepsilon_0 a^2} = 0. \hspace{1cm} (3.301)$$

Then, the work done becomes

$$W = q \left( V(0,0,a) - V(a,0,0) \right) \Rightarrow W = \frac{pq}{4\pi\varepsilon_0 a^2}. \hspace{1cm} (3.302)$$
Chapter 4

Electric Fields in Matter

In our study so far we considered the medium to be either free space or a conductor where we have free charges. In this chapter we are interested in a medium where we have bound positive and negative charges. We will study the behavior of the electric field and the effect of the electric field on the medium.

4.1 Polarization of an atom

Induced dipole moment for an atom

\[ \vec{p} = \alpha \vec{E} \tag{4.1} \]

\( \alpha \) is called atomic polarizability.

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<th>Li</th>
<th>Be</th>
<th>C</th>
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<td>5.6</td>
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<td>K</td>
<td>Cs</td>
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<td>43.4</td>
<td>59.6</td>
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</tbody>
</table>

**The Primitive Model:** In the primitive model the electron with charge \(-q\) is considered to be uniformly distributed in a sphere with a radius \(a\) centered about the positively charge nucleus with charge \(q\). The radius is also considered to be the Bohr radius. We want to determine the atomic polarizability of such an atom.

As shown in Fig. 4.1 the presence of the external electric field causes the positive charge to be displaced to the right by a distance of \(d\). The induced electric field at a distance \(d\) from the center of a uniformly charges sphere of radius \(a\)

\[ \vec{E}_{\text{in}} = -\frac{qd}{4\pi\epsilon_0 a^3} \hat{d} = -\frac{p}{4\pi\epsilon_0 a^3} \hat{d} = -\frac{\vec{p}}{4\pi\epsilon_0 a^3} \tag{4.2} \]

Noting that the induced electric field is directed opposite to the applied electric field, \(\vec{E}_{\text{in}} = -\vec{E}\), we have

\[ \vec{E} = \frac{\vec{p}}{4\pi\epsilon_0 a^3} \tag{4.3} \]
for a neutral atom the induced electric dipole can then be expressed, in terms of the applied electric field, as

$$\vec{p} = \alpha \vec{E} \quad (4.4)$$

where the atomic polarizability $\alpha$, in the primitive model is given by

$$\alpha = 4\pi \varepsilon_0 a^3 = 3\varepsilon_0 V \quad (4.5)$$

Note that we used $V = \frac{4}{3} \pi a^3$ for the volume of the Bohr sphere.

The Quantum Model: According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a} \quad (4.6)$$

where $q$ is the charge of the electron and $a$ is the Bohr radius. To find the polarizability of an atom in quantum model, we first need to find the electric field. We first use Gauss’s law

$$\int_S \vec{E}_e \cdot d\vec{a} = \frac{Q}{\varepsilon_0} \Rightarrow E_4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E_e = \frac{Q}{4\pi \varepsilon_0 r^2} \quad (4.7)$$

The total charge enclosed

$$Q = \int_V \rho(r) d\tau = \frac{q}{\pi a^3} \int_0^r \int_0^{2\pi} \int_0^\pi e^{-2r/a} r^2 d\theta d\phi d\tau = \frac{q}{\pi a^3} 4\pi \int_0^r e^{-2r/a} r^2 dr$$

$$\Rightarrow Q = \frac{4q}{a^3} \int_0^r e^{-2r/a} r^2 dr. \quad (4.8)$$
If we replace $2/a$ by $x$, we can write the integral in the above expression as

$$
\int_0^r e^{-2r/a} r^2 dr = \int_0^r e^{-xr} r^2 dr = \int_0^r \frac{\partial^2}{\partial x^2} \left( e^{-xr} \right) dr = \frac{\partial^2}{\partial x^2} \left[ \int_0^r e^{-xr} dr \right]
$$

$$
= \frac{\partial^2}{\partial x^2} \left[ \frac{1}{x} - \frac{e^{-xr}}{x} \right] = \frac{\partial}{\partial x} \left( -\frac{1}{x^2} + \frac{1}{x^2} e^{-xr} + \frac{re^{-xr}}{x} \right)
$$

$$
= \frac{2}{x^3} - \frac{2}{x^3} e^{-xr} - \frac{r}{x^2} e^{-xr} - \frac{re^{-xr}}{x} = \frac{2}{x^3} \left[ 1 - e^{-xr} \left( 1 + r + \frac{r^2 x^2}{2} \right) \right]
$$

$$
\int_0^r e^{-2r/a} r^2 dr = \frac{2}{x^3} \left[ 1 - e^{-xr} \left( 1 + r + \frac{r^2 x^2}{2} \right) \right]. \tag{4.10}
$$

Now substituting back $2/a$ for $x$, we find

$$
\int_0^r e^{-2r/a} r^2 dr = \frac{a^3}{4} \left[ 1 - e^{-2r/a} \left( 1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right]. \tag{4.10}
$$

Then the total charge in Eq. (??) becomes

$$
Q = \frac{4q}{a^3} \left[ 1 - e^{-2r/a} \left( 1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right] \Rightarrow Q = q \left[ 1 - e^{-2r/a} \left( 1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right] \tag{4.11}
$$

Therefore, the electric field becomes

$$
\vec{E}_e = \frac{q}{4\pi \varepsilon_0 r^2} \left[ 1 - e^{-2r/a} \left( 1 + \frac{2r}{a} + \frac{2r^2}{a^2} \right) \right] \hat{r} \tag{4.12}
$$

When the proton shifted from the point $r = 0$ to the point $r = d$ due to the external electric field, $\vec{E}_e$, the electric field due to the electron could at this new position of the nucleus balances the eternal electric field, $E_e = E$. Therefore, the magnitude of the external field that induces the dipole moment is

$$
E = \frac{q}{4\pi \varepsilon_0 d^2} \left[ 1 - e^{-2d/a} \left( 1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right) \right] \tag{4.13}
$$
Expressing $e^{-2d/a}$ in power series, we have

$$
e^{-2d/a} = 1 - \frac{2d}{1!} \frac{1}{a} + \frac{1}{2!} \left( \frac{2d}{a} \right)^2 - \frac{1}{3!} \left( \frac{2d}{a} \right)^3 \ldots$$

$$= 1 - \frac{2d}{a} + 2 \left( \frac{d}{a} \right)^2 - \frac{4}{3} \left( \frac{d}{a} \right)^3 \ldots \quad (4.14)$$

so that

$$1 - e^{-2d/a} \left( 1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right)$$

$$= 1 - \left( 1 - \frac{2d}{a} + 2 \left( \frac{d}{a} \right)^2 - \frac{4}{3} \left( \frac{d}{a} \right)^3 \ldots \right) \left( 1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right) \quad (4.15)$$

$$= 1 - \left[ 1 + \frac{2d}{a} + \frac{2d^2}{a^2} - \frac{2d}{a} - \frac{4d^2}{a^2} - \frac{4d^3}{a^3} + \frac{2d^2}{a^2} + \frac{4d^3}{a^3} + \frac{4d^4}{a^4} - \frac{4d^7}{3a^5} + \ldots \right] \quad (4.16)$$

where we have kept the terms only up to the third order. This can be simplified to give

$$1 - e^{-2d/a} \left( 1 + \frac{2d}{a} + \frac{2d^2}{a^2} \right) = \frac{4d^3}{3a^3} + \text{higher order terms} \quad (4.17)$$

Therefore, the external electric field that induces the polarization

$$\vec{E}' = \frac{q}{4\pi\varepsilon_0 d^2} \left[ \frac{4}{3} \frac{d^3}{a^3} \right] \hat{d} = \frac{qd}{3\pi\varepsilon_0 a^3} \hat{d} = \frac{\vec{\rho}}{3\pi\varepsilon_0 a^3}$$

$$\Rightarrow \vec{\rho} = \left( 3\pi\varepsilon_0 a^3 \right) \vec{E}' \quad (4.18)$$

The atomic polarizability will then be

$$\alpha = 3\pi\varepsilon_0 a^3 = 3\pi\varepsilon_0 \frac{3}{4\pi} V \Rightarrow \alpha = \frac{9}{4} V \varepsilon_0. \quad (4.19)$$

### 4.2 Polarization of molecules and electric torque

For a molecule in general the induced dipole moment

$$p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

$$p_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

$$p_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z \quad (4.20)$$

**Alignment of polar molecules**
4.2. POLARIZATION OF MOLECULES AND ELECTRIC TORQUE

No permanent dipole moment polar molecule: e.g. CO$_2$

Built-in permanent dipole moment polar molecule: e.g. H$_2$O (water)

Such molecules with a permanent electric dipole moment, such as water molecule, experiences an electrical torque. The net torque ($\vec{N}$) about the center of the dipole is

$$\vec{N} = \vec{N}_- + \vec{N}_+ = \vec{r}_- \times \vec{F}_- + \vec{r}_+ \times \vec{F}_+$$

Noting that the moment arms for the positive and negative charges about the center of the dipole are

$$\vec{r}_- = -\frac{d}{2} \hat{d}, \quad \vec{r}_+ = \frac{d}{2} \hat{d}$$

and for a uniform electric field the forces on these charges

$$\vec{F}_- = -q\vec{E}, \quad \vec{F}_+ = q\vec{E}$$

we find for the net torque on the dipole to be

$$\vec{N} = \left( -\frac{d}{2} \hat{d} \right) \times \left( -q\vec{E} \right) + \left( \frac{d}{2} \hat{d} \right) \times \left( q\vec{E} \right)$$

$$\Rightarrow \quad \vec{N} = qdd \times \vec{E} = \vec{p} \times \vec{E}.$$  \hspace{1cm} (4.24)

For a nonuniform electric field the force on a dipole can be expressed as

$$\vec{F} = q\vec{E} (x + \Delta x, y + \Delta y, z + \Delta z) - qq\vec{E} (x, y, z) = q\Delta \vec{E} (x, y, z)$$  \hspace{1cm} (4.25)
CHAPTER 4. ELECTRIC FIELDS IN MATTER

Noting that

\[ \Delta \vec{E}(x, y, z) = \frac{\partial \vec{E}(x, y, z)}{\partial x} \Delta x + \frac{\partial \vec{E}(x, y, z)}{\partial y} \Delta y + \frac{\partial \vec{E}(x, y, z)}{\partial z} \Delta z \]  

(4.26)

one can write

\[ \vec{F} = q \Delta \vec{E}(x, y, z) = \frac{\partial \vec{E}(x, y, z)}{\partial x} q \Delta x + \frac{\partial \vec{E}(x, y, z)}{\partial y} q \Delta y + \frac{\partial \vec{E}(x, y, z)}{\partial z} q \Delta z. \]  

(4.27)

Using the dipole moment

\[ \vec{p} = p_x \hat{x} + p_y \hat{y} + p_z \hat{z} = (q \Delta x) \hat{x} + (q \Delta y) \hat{y} + (q \Delta z) \hat{z}. \]  

(4.28)

we have

\[ \vec{F} = \left[ p_x \frac{\partial \vec{E}(x, y, z)}{\partial x} + p_y \frac{\partial \vec{E}(x, y, z)}{\partial y} + p_z \frac{\partial \vec{E}(x, y, z)}{\partial z} \right] \vec{E}(x, y, z). \]  

(4.29)

so that the force can be expressed as

\[ \vec{F} = (\vec{p}, \nabla) \vec{E} \]  

(4.31)
4.2. POLARIZATION OF MOLECULES AND ELECTRIC TORQUE

For a perfect dipole of infinitesimal length the torque about the center of the dipole is given by Eq. (4.2) even if the field is not uniform. For any other axis

\[ \vec{N} = \vec{p} \times \vec{E} + \vec{r} \times \vec{F} \]  

(4.32)

**Example 4.1** A dipole \( \vec{p} \) is situated a distance \( z \) above an infinite grounded conducting plane (See figure below). The dipole makes an angle \( \theta \) with the perpendicular to the plane. Find the torque on \( \vec{p} \) if the dipole is free to rotate, in what orientation will it come to rest?

_N.B. Alternative (probably easier) approach is to use cylindrical coordinates.... I use this in class_

**Solution:** We use our knowledge of image methods to find the electric field due to the grounded conducting plane at the position of the dipole. In spherical coordinates we may write the dipole moment for the figure shown as

\[ \vec{p} = \vec{p} \hat{r} \]  

(4.33)

Then recalling that the electric field of a dipole is given by

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0 r^3} (3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}) . \]  

(4.34)

Now let the origin of the coordinate system be at position of the image charge. Then we may write for the dipole moment of the image dipole as

\[ \vec{p} = p \left[ \cos (2\pi - \theta) \hat{r} + \sin (2\pi - \theta) \hat{\theta} \right] = p \left[ \cos \theta \hat{r} - \sin \theta \hat{\theta} \right] \]

\[ \Rightarrow \vec{p} \cdot \hat{r} = p \cos \theta \]  

(4.35)

so that the electric field can be expressed as

\[ \vec{E} = \frac{p}{4\pi\varepsilon_0 r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] . \]  

(4.36)
Thus the electric field at the position of the dipole due to its image charge can be written as

\[ E_i = \frac{p}{4\pi \varepsilon_0 (2z)^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right]. \quad (4.37) \]

Noting that the dipole moment in terms of the radial and angular directions is given by

\[ \vec{p} = p \left( \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \quad (4.38) \]

we have for the torque

\[ \vec{N} = \vec{p} \times \vec{E} = p \left( \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \times \frac{p}{4\pi \varepsilon_0 (2z)^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \]

\[ = \frac{p^2}{32 \pi \varepsilon_0 z^3} \left( \cos \theta \sin \theta \hat{\varphi} - 2 \sin \theta \cos \theta \hat{\varphi} \right) \]

\[ = -\frac{p^2}{32 \pi \varepsilon_0 z^3} \cos \theta \sin \theta \hat{\varphi} \quad (4.39) \]

Using the relation \( \sin (2\theta) = 2 \sin \theta \cos \theta \), we may write

\[ \vec{N} = -\frac{p^2 \sin (2\theta)}{64 \pi \varepsilon_0 z^3} \hat{\varphi} \quad (4.40) \]

The torque is out of the page. If \( 0 < \theta < \pi/2 \), we have \( \sin (2\theta) \) positive then the torque is going to be in the negative direction which means the dipole rotates in the counterclockwise direction. If \( \pi/2 < \theta < \pi \) we have \( \sin (2\theta) \) negative and the torque is going to be in the positive \( \hat{\varphi} \) which meant the dipole rotate in the clockwise direction.

**Polarization** (\( \vec{P} \)): dipole moments per unit volume

### 4.3 The field of a polarized object

**Bound charges**: Consider a polarized object with a polarization \( \vec{P} \). We can subdivide this object into infinitely small volumes \( d\tau \) with dipole moment \( \vec{p} \). We recall the potential of a dipole like the one shown in Figure below is given by

\[ V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \frac{(\vec{r} - \vec{r}') \cdot \vec{p}}{|\vec{r} - \vec{r}'|^3} \quad (4.41) \]

This is the electric potential due to a dipole with a dipole moment \( \vec{p} \) shown in the figure above. If this dipole moment is due to the dipole in the infinitesimal volume \( d\tau' \), then using the polarization \( \vec{P} \) of the material we can express it as

\[ \vec{p} = \vec{P} d\tau'. \quad (4.42) \]
4.3. THE FIELD OF A POLARIZED OBJECT

We recall that the electric potential of a dipole is given by

\[ V_{dip}(\vec{r}) = \frac{\vec{r} \cdot \vec{p}}{4\pi\epsilon_0 r^2} \]  

(4.43)

so that the total electric potential at the position \( \vec{r} \) due to the polarized material can be written as

\[ V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{(\vec{r} - \vec{r}') \cdot \vec{P} d\tau'}{|\vec{r} - \vec{r}'|^3}. \]  

(4.44)

Using the relation

\[ \nabla' \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right) = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \]  

(4.45)

we have

\[ V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{vol} \left[ \nabla' \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right) \right] \cdot \vec{P} d\tau'. \]  

(4.46)

Noting that

\[ \nabla \cdot [f \vec{g}] = f \nabla \cdot \vec{g} + \vec{g} \cdot \nabla f \Rightarrow \vec{g} \cdot \nabla f = \nabla \cdot \left[ f \vec{g} \right] - f \nabla \cdot \vec{g} \]  

(4.47)

we can write

\[ \left[ \nabla' \left( \frac{1}{|\vec{r} - \vec{r}'|^3} \right) \right] \cdot \vec{p} = \nabla' \cdot \left( \frac{\vec{P}}{|\vec{r} - \vec{r}'|^3} \right) - \frac{1}{|\vec{r} - \vec{r}'|} \nabla' \cdot \vec{p} \]  

(4.48)
Then the potential becomes

\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \left[ \nabla' \cdot \left( \frac{\vec{P}}{|\vec{r}' - \vec{r}|} \right) - \frac{1}{|\vec{r}' - \vec{r}|} \nabla' \cdot \vec{P} \right] d\tau' \]

\[ = \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \nabla' \cdot \left( \frac{\vec{P}}{|\vec{r}' - \vec{r}|} \right) d\tau' - \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \frac{1}{|\vec{r}' - \vec{r}|} \nabla' \cdot \vec{P} \right] d\tau' \]  (4.49)

Using the divergence theorem we have

\[ \int_{\text{vol}} \nabla' \cdot \left( \frac{\vec{P}}{|\vec{r}' - \vec{r}|} \right) d\tau' = \oint_{\text{sur}} \left( \frac{\vec{P}}{|\vec{r}' - \vec{r}|} \right) \cdot d\vec{a}' \]  (4.50)

so that

\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_{\text{sur}} \vec{P} \cdot d\vec{a}' - \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \frac{\nabla' \cdot \vec{P}}{|\vec{r}' - \vec{r}|} d\tau' \]  (4.51)

or using \( d\vec{a}' = \hat{n}da' \), where \( \hat{n} \) is the unit vector normal to the area \( da' \),

\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_{\text{sur}} \vec{P} \cdot \hat{n} \cdot da' - \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \nabla' \cdot \vec{P} d\tau' \]  (4.52)

Comparing \( \vec{P} \cdot \hat{n} \) and \( \nabla' \cdot \vec{P} \) with surface charge and volume change densities we studied

\[ \sigma = \varepsilon_0 \vec{E} \cdot \hat{n}, \text{ and } \rho = \varepsilon_0 \nabla \cdot \vec{E}, \]  (4.53)

for polarized material we have a bound surface and volume change densities given by

\[ \sigma_b(\vec{r}') = \vec{P} \cdot \hat{n}, \text{ and } \rho(\vec{r}') = -\nabla' \cdot \vec{P} \]  (4.54)

Therefore, the electric potential due to a polarized material is given by

\[ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint_{\text{sur}} \frac{\sigma_b(\vec{r}')}{|\vec{r}' - \vec{r}|} \cdot da' - \frac{1}{4\pi\varepsilon_0} \int_{\text{vol}} \frac{\rho(\vec{r}')}{|\vec{r}' - \vec{r}|} d\tau' \]  (4.55)

**Example 4.2** A sphere of radius \( R \) carries a polarization

\[ \vec{P}(r) = k\vec{r} \]  (4.56)

where \( k \) is a constant and \( \vec{r} \) is the vector from the center.

(a) Calculate the bound charge densities \( \sigma_b \) and \( \rho_b \). What can you say about the total bound charges?

(b) Find the electric potential.
(c) Find the field inside and outside the sphere.

**Solution:**

(a) The bound surface and volume charges densities are given by

\[ \sigma_b(\vec{r}) = \vec{P} \cdot \hat{n}, \quad \rho_b(\vec{r}) = -\vec{\nabla} \cdot \vec{P} \quad (4.57) \]

for a spherical surface, we have \( \hat{n} = \hat{r} \) and the surface charged density is found to be

\[ \sigma_b(\vec{r}) = \vec{P} \cdot \hat{n}\big|_R = k \vec{r} \cdot \hat{r}\big|_R = kR. \quad (4.58) \]

For the volume charge density, \( \rho_b(\vec{r}) \), we find

\[ \rho_b(\vec{r}) = -\vec{\nabla} \cdot \vec{P} = -k \vec{\nabla} \cdot \vec{r} = -k \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) = -3k \quad (4.59) \]

(b) The potential in terms of the bound surface and volume charges is expressible as

\[ V(\vec{r}) = V_S(\vec{r}) + V_V(\vec{r}) \quad (4.60) \]

where

\[ V_S(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int_{\text{sur}} \frac{\sigma_b(\vec{r}')}{|\vec{r} - \vec{r}'|} d\sigma' \]

is the potential due to the surface bound charges and

\[ V_V(\vec{r}) = \frac{1}{4\pi \epsilon_0} \int_{\text{vol}} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \]

is the potential due to the volume bound charges. We now proceed to find these potentials. To these end, we note that

\[ |\vec{r} - \vec{r}'| = \sqrt{\vec{r}^2 + \vec{r}'^2 - 2\vec{r}\vec{r}' \cos \theta'} \quad (4.61) \]

On the surface of the sphere, \( \vec{r}' = R \),

\[ d\sigma' = R^2 \sin \theta' d\theta' d\varphi' \quad (4.62) \]

and

\[ |\vec{r} - \vec{R}| = \sqrt{\vec{r}^2 + R^2 - 2R \vec{r} \cos \theta} \quad (4.63) \]

Then the potential due to the bound surface charges, \( \sigma_b(\vec{r}) = kR \) becomes

\[ V_S(\vec{r}) = \frac{kR^3}{4\pi \epsilon_0} \frac{2\pi}{0} \int_0^{2\pi} \frac{\sin \theta' d\theta' d\varphi'}{\sqrt{\vec{r}^2 + R^2 - 2R \vec{r} \cos \theta'}} \]

\[ = \frac{kR^3}{2\epsilon_0} \frac{\pi}{0} \frac{\sin \theta' d\theta'}{\sqrt{\vec{r}^2 + R^2 - 2R \vec{r} \cos \theta'}} \quad (4.64) \]
We can evaluate the integral as
\[
\int_{0}^{\pi} \frac{\sin (\theta') d\theta'}{\sqrt{r^2 + R^2 - 2rr\cos (\theta')}} = \int_{-1}^{1} \frac{du}{\sqrt{r^2 + R^2 - 2rru}}
\]
\[
= - \sqrt{r^2 + R^2 - 2rru} \Bigg|_{-1}^{1} = \frac{\sqrt{r^2 + R^2 + 2rru} - \sqrt{r^2 + R^2 + 2rru}}{rru} = \frac{\sqrt{(r + R)^2} - \sqrt{(r - R)^2}}{rru},
\]
\[
\Rightarrow \int_{0}^{\pi} \frac{\sin (\theta') d\theta'}{\sqrt{r^2 + R^2 - 2rr\cos (\theta')}} = \frac{1}{rru} (r + R - |r - R|), \quad (4.65)
\]
where we used the transformation of variable defined by, \( u = \cos (\theta') \). The potential due to the surface charges can then be written as
\[
V_S(\vec{r}) = \frac{kR^2}{2\epsilon_0} (r + R - |r - R|). \quad (4.66)
\]
For a point inside \((r < R)\) and outside \((r > R)\) the sphere, we have
\[
|r - R| = \begin{cases} 
-(r - R) & r < R \\
-(r - R) & r > R
\end{cases} \quad (4.67)
\]
so that the potential becomes
\[
V_S(\vec{r}) = \begin{cases} 
\frac{kR^2}{\epsilon_0} & r < R \\
\frac{kR^3}{\epsilon_0 r} & r > R
\end{cases} \quad (4.68)
\]
Using the relations
\[
|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos (\theta')}, \quad dr' = r'^2 dr' \sin (\theta') d\theta' d\varphi', \quad (4.69)
\]
and the bound charge density determined in part (a), \( \rho_b(\vec{r}') = -3k \), one can write
\[
V_V(\vec{r}) = - \frac{1}{4\pi\epsilon_0} \int V(\vec{r}) \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \sin (\theta') d\theta' d\varphi' = - \frac{3k}{4\pi\epsilon_0} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{r'^2 dr' \sin (\theta') d\theta' d\varphi'}{\sqrt{r^2 + r'^2 - 2rr'\cos (\theta')}}
\]
\[
\Rightarrow V_V(\vec{r}) = \frac{3k}{2\epsilon_0} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{r'^2 dr' \sin (\theta') d\theta'}{\sqrt{r^2 + r'^2 - 2rr'\cos (\theta')}} \quad (4.70)
\]
Noting that this expression can be put in the form
\[
V_V(\vec{r}) = - \frac{3k}{2\epsilon_0} \int_{0}^{\pi} \frac{\sin (\theta') d\theta'}{\sqrt{r^2 + r'^2 - 2rr'\cos (\theta')}} \int_{0}^{2\pi} d\varphi'. \quad (4.71)
\]
applying the results in Eq. (4.65), one finds

\[ V_V(\bar{r}) = -\frac{3k}{2\epsilon_0} \int_0^R r'^2 \left[ \frac{1}{rr'} (r + r' - |r - r'|) \right] dr' \]

\[ = -\frac{3k}{2\epsilon_0} \int_0^R r' (r + r' - |r - r'|) dr' \quad (4.72) \]

For a point outside the sphere, \( r > R \), we have \( r > r' \), and in view of the relation in Eq. (4.67), Eq. (4.72) becomes

\[ V_V(\bar{r}) = -\frac{3k}{2\epsilon_0} \int_0^R r' dr' [r + r' - (r - r')] = -\frac{3k}{\epsilon_0} \int_0^R r'^2 dr = -\frac{kR^3}{\epsilon_0 r} r > R. \quad (4.73) \]

On the other hand inside the sphere \((r < R)\), we need to split the integral in Eq. (4.72) into two parts \(0 < r' < r\) and \(r < r' < R\),

\[ V_V(\bar{r}) = -\frac{3k}{2\epsilon_0} \left\{ \int_0^r r' (r + r' - |r - r'|) dr' + \int_r^R (r + r' - |r - r'|) dr' \right\}. \quad (4.74) \]

Noting that for the in the first integral \( r' < r \) and \( r < r' \), in view of Eq. (4.67), Eq. (4.74) becomes

\[ V_V(\bar{r}) = -\frac{3k}{2\epsilon_0} \left\{ \int_0^r r' dr' + 2r \int_r^R r' dr' \right\} = -\frac{3k}{2\epsilon_0} \left[ \frac{2}{3} r^3 + r (R^2 - r^2) \right] \]

\[ = -\frac{3k}{2\epsilon_0} \left[ \frac{2}{3} r^3 + r (R^2 - r^2) \right] \]

\[ \Rightarrow V_V(\bar{r}) = -\frac{3k}{2\epsilon_0} \left( R^2 - \frac{1}{3} r^2 \right). \quad (4.75) \]

Therefore the potential due to the volume charge is found to be

\[ V_V(\bar{r}) = \begin{cases} -\frac{k}{\epsilon_0} \left( 3R^2 - r^2 \right) & r < R \\ -\frac{kR^2}{\epsilon_0} & r > R \end{cases}. \]

Then using the result for the potential for the surface bound charges, total potential inside the sphere \((r < R)\) becomes

\[ V(\bar{r}) = V_S(\bar{r}) + V_V(\bar{r}) = \frac{kR^2}{\epsilon_0} - \frac{k}{2\epsilon_0} (3R^2 - r^2) = -\frac{k}{2\epsilon_0} (R^2 - r^2) \quad (4.76) \]
and outside the sphere \((r > R)\)

\[
V(\vec{r}) = \frac{kR^3}{\varepsilon_0 r} - \frac{kR^3}{\varepsilon_0 r} = 0 \quad (4.77)
\]

(c) The electric field can then be obtained using the potential in part \(b\). In the region inside the sphere using Eq. (4.76), we find

\[
E(\vec{r}) = -\nabla V(\vec{r}) = -\hat{r} \frac{\partial}{\partial r} \left[ -\frac{k}{2\varepsilon_0} \left( R^2 - r^2 \right) \right] = -\frac{kr}{\varepsilon_0} \hat{r} = -\frac{k}{\varepsilon_0} \hat{r} \quad (4.78)
\]

and outside the sphere the electric field is zero since the electric potential is zero. Do you know why the electric field outside is zero? Is there a simple physical justification that can provide and find the electric field outside without calculating it?

4.4 Physical interpretation of bound charges

Consider a cylinder with area \(A = \pi R^2\) and length \(L\) placed in a uniform electric field parallel to the axis of the cylinder. Suppose this cylinder has \(N\) dipoles each with a dipole moment \(\vec{p}_i\). At room temperature these dipoles have random direction and the polarization of the cylinder

\[
\vec{P} = \frac{1}{V} \sum_{i=1}^{N} \vec{p}_i = \frac{1}{AL} \sum_{i=1}^{N} \vec{p}_i = 0.
\]

When this cylinder is placed in a uniform electric field parallel to the axis of the cylinder all the dipoles will be directed parallel to the electric field as shown in Fig.4.2 For an electric field directed parallel to the axis of the cylinder (z-axis), the polarization will then be

\[
\vec{P} = \frac{N}{AL} \vec{p} = \frac{N}{AL} \vec{p} = \frac{N}{AL} p\hat{z} = P\hat{z} \quad (4.79)
\]

where

\[
P = \frac{Np}{AL}.
\]

Now if we make a slice of the cylinder with length \(d\), and cross sectional area \(\Delta a\) perpendicular to the axis of the cylinder, then one can write the total dipole moment as

\[
\Delta p = P \Delta V = P (\Delta a) \, d \quad (4.80)
\]

but this dipole moment is a result of a negative bound charge \(-\Delta q\) and positive bound charge \(\Delta q\) residing on the two surface of the sliced cylinder of length \(d\). Hence we have

\[
\Delta p = (\Delta q) \, d \quad (4.81)
\]
4.4. PHYSICAL INTERPRETATION OF BOUND CHARGES

Figure 4.2: A dielectric placed in a uniform electric field.

therefore

\[(\Delta q) d = Pd (\Delta a) \Rightarrow \Delta q = P \Delta a \Rightarrow \sigma_b = \frac{\Delta q}{\Delta a} = P. \quad (4.82)\]

For a slice with an oblique cut as shown in the figure below

the area \(\Delta a_{\text{end}} = \Delta a \cos \theta\) and we have

\[
\Delta p = P \Delta V \Rightarrow \Delta q d = P \Delta a \cos (\theta) d \Rightarrow \Delta q = P \Delta a \cos \theta
\]

\[
\Rightarrow \sigma_b = \frac{\Delta q}{\Delta a} = P \cos \theta \Rightarrow \sigma_b = \vec{P} \cdot \hat{n} \quad (4.83)
\]

If the polarization is not uniform there will be a negative bound charge inside the dielectric equal in magnitude to the bound charge on the surface of the dielectric (see figure below)

If we denote this inside bound charge distribution by volume charge density \(\rho(r)\). Then the total bound charge inside the dielectric is given by

\[
Q_b^v = \int_{\text{vol}} \rho(r) d\tau. \quad (4.84)
\]

This charge is the negative of the total charge on the surface, \(Q_b^s\), which is given by

\[
Q_b^s = \oint \vec{P} \cdot \hat{n} da = \oint \vec{P} \cdot \vec{a} \quad (4.85)
\]
Therefore
\[ Q_v^b = Q_s^b \Rightarrow \int_{vol} \rho(r) d\tau = -\oint P \cdot a \] (4.86)

Using the divergence theorem we find
\[ \int_{vol} \rho(r) d\tau = -\oint P \cdot a \Rightarrow \int_{vol} \rho(r) d\tau = -\int \nabla \cdot \vec{P} d\tau \]
\[ = \rho(r) = -\nabla \cdot \vec{P} \] (4.87)

4.5 The electric Displacement

Gauss’s Law in the presence of Dielectrics: We recall from chapter 2 the differential form of Gauss’s law in a free space with some localized free charge density, \( \rho_f(\vec{r}) \), is given by
\[ \epsilon_0 \nabla \cdot \vec{E} = \rho_f(\vec{r}), \] (4.88)
where \( \rho_f(\vec{r}) \) is the free charge density. We have seen that the electric field outside a polarized material is the result of the bound volume change density.
\( \rho_b(\vec{r}) \) and surface charge density \( \sigma_b(\vec{r}) \) given by
\[
\rho_b(\vec{r}) = -\nabla \cdot \vec{P}, \quad \sigma_b(\vec{r}) = \vec{P} \cdot \hat{n}.
\] (4.89)
respectively. Suppose now we replace the free space with a dielectric material with polarization \( \vec{P} \). Gauss’s law, inside the volume of the polarizable medium becomes
\[
\epsilon_0 \nabla \cdot \vec{E} = \rho_{\text{total}}(\vec{r}) = \rho_f(\vec{r}) + \rho_b(\vec{r}).
\] (4.90)
Using \( \rho_b(\vec{r}) = -\nabla \cdot \vec{P} \), one can write
\[
\epsilon_0 \nabla \cdot \vec{E} = \rho_f(\vec{r}) - \nabla \cdot \vec{P} \Rightarrow \nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f(\vec{r})
\]
\[
\Rightarrow \nabla \cdot \vec{D} = \rho_f(\vec{r}),
\] (4.91)
where
\[
\vec{D} = \epsilon_0 \vec{E} + \vec{P}
\] (4.92)
is known as the electric displacement. We can then write the integral form of Gauss’s Law as
\[
\oint S \vec{D} \cdot \hat{a} = \int_V \rho_f(\vec{r})d\tau = \text{Total enclosed free charge.}
\] (4.93)

**Example 4.3** A long straight wire, carry uniform line charge \( \lambda \) is surrounded by rubber insulation out to a radius \( a \) (see Fig. below). Find the electric displacement.

![Diagram of a long straight wire surrounded by rubber insulation](image)

**Solution:** To find the electric displacement we use the integral form of Gauss’s Law in a dielectric medium,
\[
\oint_S \vec{D} \cdot \hat{a} = \int_V \rho_f(\vec{r})d\tau = Q_{\text{free}} \text{ (Total enclosed free charge).}
\] (4.94)
Considering a cylindrical Gaussian surface of length \( L \) and radius \( s \), as shown in the figure above, we may write
\[
\oint_S \vec{D} \cdot \hat{a} = \lambda L \Rightarrow D2\pi s L = \lambda L \Rightarrow D = \frac{\lambda}{2\pi s} \Rightarrow \vec{D} = \frac{\lambda}{2\pi s} \hat{s}
\] (4.95)
Example 4.4 A thick spherical shell (inner radius $a$, outer radius $b$) is made of dielectric material with a "frozen-in" polarization.

$$\vec{P}(r) = \frac{k}{r}\hat{r}, \quad (4.96)$$

where $k$ is a constant and $r$ is the distance from the center (see Fig. below). (There is no free charge in the problem). Find the electric field in all three regions by two different methods.

(a) Locate all the bound charge, and use Gauss’s law

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \quad (4.97)$$

to calculate the field it produces.

(b) Use the electric displacement given by

$$\oint_S \vec{D} \cdot d\vec{a} = \int_V \rho_f (\vec{r}) d\tau = \text{Total enclosed free charge} \quad (4.98)$$

to find $\vec{D}$, and then get $\vec{E}$ from Eq. (4.92)

Solution:

(a) From Gauss’s Law

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \quad (4.99)$$

Here the enclosed charge is due to the volume and/or surface charges if there are any in the region we want to find the electric field. For $r < a$, 

...
since we have an empty space there is no bound charges, \( Q_{\text{enc}} = 0 \Rightarrow \vec{E} = 0 \). In the region, \( a < r < b \), the volume bound charge density is given by

\[
\rho_b(\vec{r}) = -\nabla \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 P \right) = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{k}{r} \right) = -\frac{1}{r^2} (2k - k)
\]

\[
\Rightarrow \rho_b(\vec{r}) = -\frac{k}{r^2}.
\]

(4.100)

The surface charge densities

\[
\sigma_b(\vec{r}) = \vec{P} \cdot \hat{n} = \begin{cases} -\frac{k}{r^2} \frac{\vec{r}}{r} & r = a \\ \frac{k}{r^2} \frac{\vec{r}}{r} & r = b \end{cases}
\]

(4.101)

Then the total charge enclosed in the spherical shell of volume of inner radius \( a \) and outer radius \( r \) is given by

\[
Q_{\text{enc}} = Q_{\text{surface}} + Q_{\text{vol}} = -\frac{k}{a} 4\pi a^2 + \int_{V_{\text{vol}}} -\frac{k}{r^2} d\tau = -\frac{4\pi k a}{a} + 4\pi \int_{a}^{r} \frac{k}{r^2} r^2 dr = -4\pi k (a + r - a)
\]

\[
\Rightarrow Q_{\text{enc}} = -4\pi kr.
\]

(4.102)

Therefore, the electric field in the region \( a < r < b \) becomes

\[
\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{Q_{\text{enc}}}{r^2} \frac{\vec{r}}{r} = \frac{1}{4\pi \epsilon_0} \left( -\frac{4\pi k r}{r^2} \right) \Rightarrow \vec{E} = -\frac{k}{\epsilon_0 r^2}.
\]

(4.103)

In the region, \( r > b \), the total charge enclosed by the Gaussian spherical surface is given by

\[
Q_{\text{enc}} = Q_{\text{surface}} + Q_{\text{vol}} = -\frac{4\pi k a}{a} + 4\pi \int_{a}^{b} \left( -\frac{k}{r^2} \right) r^2 dr = 0
\]

\[
\Rightarrow Q_{\text{enc}} = 0
\]

(4.104)

The electric field is zero (\( \vec{E} = 0 \) for \( r > b \))

(b) Now using Eq. (4.93)

\[
\oint_S \vec{D} \cdot d\vec{a} = \int_V \rho_f(\vec{r}) d\tau = \text{Total enclosed free charge}
\]

(4.105)

but the total free charge is zero and therefore the electric displacement is zero everywhere. This leads to

\[
\vec{D} = 0 \Rightarrow \epsilon_0 \vec{E} + \vec{P} = 0 \Rightarrow \vec{E} = -\frac{1}{\epsilon_0} \vec{P}
\]
To find the electric field in the three regions what we need to do find the polarization in these regions.

For $r < a$, $\vec{P} = 0 \Rightarrow \vec{E} = 0$

For $a < r < b$, $\vec{P} = \frac{k}{r} \vec{r} \Rightarrow \vec{E} = -\frac{k}{\epsilon_0 r} \vec{r}$

For $r > b$, $\vec{P} = 0 \Rightarrow \vec{E} = 0$

### 4.6 Boundary conditions in a polarizable media

In this section, we basically revisit what we discussed in Section 2.7 except that here we are interested in how the electric potential and field changes at the boundaries of two regions filled with two different polarizable media. To this end, we consider a polarizable medium where there is some free volume charge density, $\rho_f(\vec{r})$ and bound volume charge density, $\rho_b(\vec{r})$. Gauss’s law becomes

$$\epsilon_0 \nabla \cdot \vec{E} = \rho_f(\vec{r}) + \rho_b(\vec{r}) \Rightarrow \nabla \cdot \left( \epsilon_0 \vec{E} + \vec{P} \right) = \rho_f(\vec{r}), \quad (4.106)$$

where we used

$$\rho_b(\vec{r}) = -\nabla \cdot \vec{P}. \quad (4.107)$$

We have defined the electric displacement vector, $\vec{D}$,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (4.108)$$

so that Gauss’s law can be put in the form

$$\nabla \cdot \vec{D} = \rho_f(\vec{r}) \Rightarrow \oint_S \vec{D} \cdot d\vec{a} = \int_V \rho_f(\vec{r}) d\tau = Q_{\text{free}}, \quad (4.109)$$

Then following the same procedure as we did in a free space, at the boundaries of two dielectric media (medium 1 and medium 2) as shown in the figure below
4.6. BOUNDARY CONDITIONS IN A POLARIZABLE MEDIA

\[ \int_S \vec{D} \cdot d\vec{a} = Q_f = \int_{\Delta A} \sigma_f (\vec{r}) \, da \]

\[ \Rightarrow \int_{\text{Below}} \vec{D}_1 (\vec{r}) \cdot d\vec{a}_1 + \int_{\text{Above}} \vec{D}_2 (\vec{r}) \cdot d\vec{a}_2 + \int_{\text{side}} \vec{D} (\vec{r}) \cdot d\vec{a} = \int_{\Delta A} \sigma_f (\vec{r}) \, da \]

(4.110)

We are interested in the behavior of the electric displacement at the boundary. So we consider the case where the thickness of the pillbox becomes zero. In this limit, the flux contribution from the side is zero

\[ \int_{\text{side}} \vec{D} (\vec{r}) \cdot d\vec{a} = 0. \]

(4.111)

Moreover, in this limit, from Fig.4.6, we note that

\[ d\vec{a}_1 = \hat{n} da, \quad d\vec{a}_2 = \hat{n} da. \]

(4.112)

Now substituting Eqs. (4.111) and (4.112), into Eq. (4.110), we have

\[ \int_{\Delta A} \left( D_2^+ (\vec{r}) \hat{n} + D_2^{||} (\vec{r}) \hat{i} \right) \cdot \hat{n} da + \int_{\Delta A} \left( D_1^+ (\vec{r}) \hat{n} + D_1^{||} (\vec{r}) \hat{i} \right) \cdot (-\hat{n} da) \]

\[ = \int_{\Delta A} \sigma_f (\vec{r}) \, da \Rightarrow \int_{\Delta A} D_2^+ (\vec{r}) \, da - \int_{\Delta A} D_1^+ (\vec{r}) \, da = \frac{1}{\varepsilon_0} \int_{\Delta A} \sigma_f (\vec{r}) \, da. \]

\[ \Rightarrow \int_{\Delta A} \left( D_2^+ (\vec{r}) da - D_1^+ (\vec{r}) \right) da = \int_{\Delta A} \sigma_f (\vec{r}) \, da. \]

(4.113)

There follows that

\[ D_2^+ (\vec{r}) - D_1^+ (\vec{r}) = \sigma_f (\vec{r}). \]

(4.114)

The normal component of the electric displacement is discontinuous at the boundary of the dielectric media if there is a free charge at the interface of the two media.

Using the irrotational nature of the electric field, in electrostatic

\[ \nabla \times \vec{E} = 0 \Rightarrow \int_{\text{surface}} \left( \nabla \times \vec{E} \right) \cdot d\vec{a} = \int_{\text{closed curve}} \vec{E} \cdot d\vec{l} = 0, \]

(4.115)

we have shown that

\[ E_2^{||} (\vec{r}) = E_1^{||} (\vec{r}). \]

(4.116)

For a polarizable medium, in terms of the displacement and the polarization, we have

\[ \vec{E} = \frac{1}{\varepsilon_0} \left( \vec{D} - \vec{P} \right). \]

(4.117)

one can then easily write

\[ \vec{D}_2^{||} (r) - \vec{P}_2^{||} (r) = \vec{D}_1^{||} (r) - \vec{P}_1^{||} (r), \]

(4.118)
at the boundary of the two dielectric media. For the electric potential one can also easily show that
\[ V_2(\vec{r}) = V_1(\vec{r}), \]  
(4.119)
at the boundary.

### 4.7 Linear Dielectrics

**Susceptibility, Permittivity, Dielectric constant:** We know that the polarization of a dielectric medium depends on the applied electric field,
\[ \vec{P} = \vec{P}(\vec{E}) = P_x(\vec{E}) \hat{x} + P_y(\vec{E}) \hat{y} + P_z(\vec{E}) \hat{z}. \]  
(4.120)

Let’s expand this function \( \vec{P}(\vec{E}) \) in Taylor series about \( \vec{E} = 0 \). We recall that the Taylor series expansion for a function, \( f(\vec{E}) \), that depends on a vector variable, \( \vec{E} \), is given by
\[ f(\vec{E}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \vec{E} \cdot \nabla_{E'} \right)^n f(\vec{E'}) \bigg|_{E'=0}, \]  
(4.121)
where the notation \( \nabla_{E'} \) mean, for example in Cartesian coordinates,
\[ \nabla_{E'} = \hat{x} \frac{\partial}{\partial E_x} + \hat{y} \frac{\partial}{\partial E_y} + \hat{z} \frac{\partial}{\partial E_z} \]
so that
\[ P_x(\vec{E}) = P_x(\vec{E} = 0) + \left[ \left( \vec{E} \cdot \nabla_{E'} \right) P_x(\vec{E'}) \bigg|_{E'=0} \right] + \cdots, \]  
(4.122)
with a similar expression for the \( y \) and \( z \) components. Note that in the absence of the electric field the polarization is zero for initially unpolarized media. Hence
\[ P_x(\vec{E}) = \left[ \left( \vec{E} \cdot \nabla_{E'} \right) P_x(\vec{E'}) \bigg|_{E'=0} \right] + \left[ \left( \vec{E} \cdot \nabla_{E'} \right) \left( \nabla_{E'} P_x(\vec{E'}) \bigg|_{E'=0} \right) \right] + \cdots \]  
(4.123)
If we drop the second and higher order terms in the above expression, we may write for the combined \( x \), \( y \), and \( z \) components
\[ \vec{P}(\vec{E}) = \left[ \left( \vec{E} \cdot \nabla_{E'} \right) P_x(\vec{E'}) \bigg|_{E'=0} \right] \hat{x} + \left[ \left( \vec{E} \cdot \nabla_{E'} \right) P_y(\vec{E'}) \bigg|_{E'=0} \right] \hat{y} \]
\[ + \left[ \left( \vec{E} \cdot \nabla_{E'} \right) P_z(\vec{E'}) \bigg|_{E'=0} \right] \hat{z} \]
\[ \Rightarrow \vec{P}(\vec{E}) = \vec{E} \cdot \left\{ \left[ \nabla_{E'} P_x(\vec{E'}) \bigg|_{E'=0} \right] \hat{x} \right. \]
\[ + \left[ \nabla_{E'} P_y(\vec{E'}) \bigg|_{E'=0} \right] \hat{y} + \left[ \nabla_{E'} P_z(\vec{E'}) \bigg|_{E'=0} \right] \hat{z} \} \]  
(4.124)
4.7. LINEAR DIELECTRICS

We note that

\[ \mathbf{h} = \mathbf{E} \mathbf{P} \]

\[ \mathbf{E} = \mathbf{E}_0 \]

\[ \mathbf{P} = \mathbf{P}_x \hat{x} + \mathbf{P}_y \hat{y} + \mathbf{P}_z \hat{z} = 0 \]

\[ \mathbf{E}_0 = \mathbf{E}_0 \]

\[ \mathbf{P}_x = \mathbf{P}_x \hat{x} + \mathbf{P}_y \hat{y} + \mathbf{P}_z \hat{z} = 0 \]

\[ \mathbf{E}_0 = \mathbf{E}_0 \]

\[ \mathbf{P}_y = \mathbf{P}_y \hat{x} + \mathbf{P}_y \hat{y} + \mathbf{P}_z \hat{z} = 0 \]

\[ \mathbf{E}_0 = \mathbf{E}_0 \]

\[ \mathbf{P}_z = \mathbf{P}_z \hat{x} + \mathbf{P}_y \hat{y} + \mathbf{P}_z \hat{z} = 0 \]

\[ \mathbf{E}_0 = \mathbf{E}_0 \]

where \( \alpha_{ij} \) for \( i, j = x, y, \) and \( z \) are constants that depend on the electrical property of the medium. Then the polarization can be put in the form

\[ \mathbf{P} = \mathbf{P}_x \hat{x} + \mathbf{P}_y \hat{y} + \mathbf{P}_z \hat{z} = 0 \]

\[ \mathbf{E} = \mathbf{E}_0 \]

so that using

\[ \mathbf{E} = \mathbf{E}_0 \hat{x} + \mathbf{E}_0 \hat{y} + \mathbf{E}_0 \hat{z} \]

one finds

\[ \mathbf{P} = \mathbf{P}_x \hat{x} + \mathbf{P}_y \hat{y} + \mathbf{P}_z \hat{z} = 0 \]

For an isotropic linear medium (i.e. the polarization does not depend on direction), we have

\[ \alpha_{xx} = \alpha_{yy} = \alpha_{yz} = \alpha \]

one can then write the polarization as

\[ \mathbf{P} = \alpha (\mathbf{E}_x \hat{x} + \mathbf{E}_y \hat{y} + \mathbf{E}_z \hat{z}) = \alpha \mathbf{E} \]

Upon defining the polarizability of the medium, \( \alpha \), in terms of the electrical permittivity of a free space, \( \varepsilon_0 \), as

\[ \alpha = \varepsilon_0 \chi_e \]

the polarization of a linear and isotropic medium can be expressed as

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \]

where \( \chi_e \) is known as dielectric susceptibility of the medium. We included \( \varepsilon_0 \) in order to make the dielectric susceptibility dimensionless. Linear and isotropic
dielectrics are those dielectrics in which the polarization is directly proportional to the electric field and satisfies Eq. (4.132). Using this expression one can write the electric displacement vector for a linear and isotropic dielectric as

$$D = \varepsilon_0 E + P \Rightarrow D = \varepsilon_0 E + \varepsilon_0 \chi_e \bar{E} \Rightarrow D = \varepsilon_0 (1 + \chi_e) \bar{E} = \varepsilon \bar{E}$$ \hspace{1cm} (4.133)

where

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$ \hspace{1cm} (4.134)

is the electrical permittivity of the dielectric (linear and isotropic). When \(\chi_e = 0\), we find \(\varepsilon = \varepsilon_0\) which is the electrical permittivity of a free space. The ratio of the electrical permittivity of the medium to that of a vacuum is known as the dielectric constant of the medium, \(\varepsilon_r\)

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$$ \hspace{1cm} (4.135)

**Example 4.5** A metal sphere of radius \(a\) carries a charge \(Q\) (Fig. below). It is surrounded, out to radius \(b\), by linear dielectric material of permittivity \(\varepsilon\). Find the potential at the center (relative to infinity).

![Diagram of a metal sphere with a dielectric medium surrounding it](image)

**Solution:** We recall the electric potential is given by

$$V(0) = -\int_\infty^0 \vec{E} \cdot d\vec{r}$$ \hspace{1cm} (4.136)

In the absence of the dielectric, the electric field is given by

$$\vec{E}_0 = \begin{cases} 0 & r < a \\ \frac{Q}{4\pi\varepsilon_0 r^2} & a < r < b \\ \frac{Q}{4\pi\varepsilon r^2} & r > b \end{cases}$$ \hspace{1cm} (4.137)
When the sphere is surrounded by a linear dielectric material in the region 
\(a < r < b\) the electric field would be different while in the other regions 
remains the same. This is because in a linear dielectric medium

\[
\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi) \vec{E} = \varepsilon \vec{E}.
\]  
(4.138)

Recalling that

\[
\oint \vec{D}.d\vec{a} = \int_V \rho_f(\vec{r})d\tau = \text{Total enclosed free charge}
\]  
(4.139)

we find

\[
\vec{D} = \frac{Q}{4\pi r^2} \hat{r} \Rightarrow \vec{E} = \frac{Q}{4\pi \varepsilon r^2} \hat{r}
\]  
(4.140)

Therefore in the presence of the dielectric field, we have

\[
\vec{E}(\vec{r}) = \begin{cases} 
0, & r < a \\
\frac{Q}{4\pi \varepsilon r^2} \hat{r}, & a < r < b \\
\frac{Q}{4\varepsilon_0 r^2} \hat{r}, & r > b
\end{cases}
\]  
(4.141)

so that the electric potential becomes

\[
V(0) = -\int_a^b \vec{E} \cdot d\vec{r} - \int_b^\infty \vec{E} \cdot d\vec{r} - \int_a^b \vec{E} \cdot d\vec{r} = -\int_b^\infty \frac{Q}{4\pi \varepsilon_0 r^2} dr - \int_b^a \frac{Q}{4\pi \varepsilon r^2} dr
\]

\[
= \frac{Q}{4\pi \varepsilon_0 b} + \frac{Q}{4\pi \varepsilon} \left(\frac{b - a}{ab}\right)
\]

\[
\Rightarrow V(0) = \frac{Q}{4\pi} \left[\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon} \left(\frac{b - a}{ab}\right)\right]
\]  
(4.142)

**Example 4.6** A certain coaxial cable consists of a copper wire, radius \(a\), surrounded by a concentric copper tube of inner radius \(c\) (see Fig. below). The space between is partially filled (from \(b\) out to \(c\)) with material tube of dielectric constant \(\varepsilon_r\), as shown. Find the capacitance per unit length of this cable.

**Solution:** If we assume a surface charge density of \(+\sigma\) on the inner copper wire and \(-\sigma\) on the outer concentric copper tube then the electric field in the region \(a < s < c\), in the absence of the dielectric is given by

\[
\vec{E}_0 = \frac{\sigma s}{\varepsilon_0 \hat{s}}.
\]  
(4.143)

In the presence of the dielectric (a linear dielectric), the electric field becomes

\[
\vec{E} = \begin{cases} 
\frac{\sigma s}{\varepsilon_0 \hat{s}} & a < r < b \\
\frac{\sigma s}{\varepsilon_0 \varepsilon_r \hat{s}} & b < r < c
\end{cases}
\]  
(4.144)
Then the potential difference between the inner copper wire and outer copper tube becomes

\[
V_{ac} = V(a) - V(c) = -\int_{a}^{c} \vec{E} \cdot d\vec{r} = -\int_{a}^{b} \vec{E} \cdot d\vec{s} = \int_{b}^{c} \vec{E} \cdot d\vec{s} =
\]

\[
-\int_{a}^{b} \frac{\sigma a}{\epsilon_0} ds - \int_{b}^{c} \frac{\sigma a}{\epsilon_r} ds
\]

\[
\Rightarrow V_{ac} = \frac{\sigma a}{\epsilon_0} \ln \frac{a}{b} + \frac{\sigma a}{\epsilon_0 \epsilon_r} \ln \frac{b}{c} = \frac{\sigma a}{\epsilon_0} \left( \ln \frac{a}{b} + \frac{1}{\epsilon_r} \ln \frac{b}{c} \right) \quad (4.145)
\]

Since \( c > b > a \), the above expression is negative

\[
V_{ac} = -\frac{\sigma a}{\epsilon_0} \left( \ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right) \quad (4.146)
\]

Therefore the capacitance per unit length

\[
\frac{C}{l} = \frac{Q}{l|V_{ac}|} = \frac{\sigma 2\pi al}{l \frac{\sigma a}{\epsilon_0} \left( \ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b} \right)} \Rightarrow \frac{C'}{l} = \frac{2\pi \epsilon_0}{\ln \frac{b}{a} + \frac{1}{\epsilon_r} \ln \frac{c}{b}} \quad (4.147)
\]

Does the capacity per unit length has increased or decreased compared to that of complete vacuum?

### 4.8 Boundary value problems with a linear dielectric

We recall for linear dielectric

\[
\vec{P} = \epsilon_0 \chi_c \vec{E} \quad \text{and} \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_c \vec{E} = \epsilon \vec{E} \Rightarrow \vec{E} = \frac{1}{\epsilon} \vec{D} \quad (4.148)
\]
4.8. **BOUNDARY VALUE PROBLEMS WITH A LINEAR DIELECTRIC**

where

$$\epsilon = \epsilon_0 \left(1 + \chi_e\right).$$  \hspace{1cm} (4.149)

We also know that the bound charge density is given by

$$\rho_b = -\nabla \cdot \vec{B} = -\nabla \cdot \left(\epsilon_0 \chi_e \vec{E}\right) \Rightarrow \rho_b = -\epsilon_0 \chi_e \nabla \cdot \vec{E}$$

$$\Rightarrow \rho_b = -\frac{\epsilon_0 \chi_e}{\epsilon} \left(\nabla \cdot \vec{D}\right).$$  \hspace{1cm} (4.150)

From Gauss’s Law for a dielectric, we have

$$\epsilon_0 \nabla \cdot \vec{E} = \rho_f + \rho_b \Rightarrow \nabla \cdot \left(\epsilon_0 \vec{E} + \vec{B}\right) = \rho_f \Rightarrow \nabla \cdot \vec{D} = \rho_f$$  \hspace{1cm} (4.151)

so that for linear and isotropic dielectrics

$$\rho_b = -\frac{\epsilon_0 \chi_e}{\epsilon} \left(\nabla \cdot \vec{D}\right) = -\frac{\epsilon_0 \chi_e}{\epsilon_0 \left(1 + \chi_e\right)} \rho_f$$  \hspace{1cm} (4.152)

We recall generally for any dielectric media (linear or none linear) we have shown that at the boundary of two different dielectric media, we have

$$D_2^\perp (\vec{r}) - D_1^\perp (\vec{r}) = \sigma_f (\vec{r}),$$  \hspace{1cm} (4.153)

$$\vec{D}_2^\parallel (r) - \vec{D}_1^\parallel (r) = \vec{D}_1^\parallel (r) - \vec{D}_1^\parallel (r),$$  \hspace{1cm} (4.154)

and

$$V_2 (\vec{r}) = V_1 (\vec{r}).$$  \hspace{1cm} (4.155)

Using the relation for linear and isotropic media

$$\vec{D} = \epsilon \vec{E}, \vec{P} = \epsilon_0 \chi_e \vec{E},$$  \hspace{1cm} (4.156)

at the boundary, for the normal component, we find

$$\epsilon_2 E_2^\perp (\vec{r}) - \epsilon_1 E_1^\perp (\vec{r}) = \sigma_f (\vec{r})$$

or in terms of the potential

$$\epsilon_2 \frac{\partial V_2 (\vec{r})}{\partial n} - \epsilon_1 \frac{\partial V_1 (\vec{r})}{\partial n} = -\sigma_f (\vec{r}).$$  \hspace{1cm} (4.157)

For the tangential component

$$\epsilon_2 E_2^\parallel (\vec{r}) - \epsilon_0 \chi_e E_2^\parallel (\vec{r}) = \epsilon_1 E_1^\parallel (\vec{r}) - \epsilon_0 \chi_e E_1^\parallel (\vec{r})$$

$$\Rightarrow \left(\epsilon_2 - \epsilon_0 \chi_e\right) E_2^\parallel (\vec{r}) = \left(\epsilon_1 - \epsilon_0 \chi_e\right) E_1^\parallel (\vec{r}),$$  \hspace{1cm} (4.158)

and the potential

$$V_2 (\vec{r}) = V_1 (\vec{r}),$$  \hspace{1cm} (4.159)

at the boundary of two linear and isotropic dielectric media.
Example 4.7 A sphere of homogeneous linear dielectric material of permittivity $\varepsilon$ and radius $R$ is placed in an otherwise uniform electric field $E_0$ (See Fig. below). Find the electric field inside the sphere.

Solution: There is no free charge both inside and outside the sphere and Laplace’s equation is valid in these regions.

$$\nabla^2 V = 0$$  \hspace{1cm} (4.160)

We recall the solution of this equation in spherical coordinate system is given by

$$V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta).$$ \hspace{1cm} (4.161)

The boundary conditions:

(a) 

$$\lim_{r \to \infty} E(r, \theta) = \lim_{r \to \infty} \left[ \frac{\partial V_2(r, \theta)}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V_2(r, \theta)}{\partial \theta} \hat{\theta} \right] = E_0 \cos(\theta) \hat{r} - E_0 \sin(\theta) \hat{\theta}$$ \hspace{1cm} (4.162)

(b) 

$$V_2(R, \theta) = V_1(R, \theta)$$ \hspace{1cm} (4.163)

(c) 

$$\left[ \frac{\varepsilon_2}{\partial r} \frac{\partial V_2(r, \theta)}{\partial r} - \frac{\varepsilon_1}{\partial r} \frac{\partial V_1(r, \theta)}{\partial r} \right]_{r=R} = 0, \text{ (since the is no free charge )}$$ \hspace{1cm} (4.164)

Inside the sphere the electric potential is given by

$$V_1(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta),$$ \hspace{1cm} (4.165)
4.8. BOUNDARY VALUE PROBLEMS WITH A LINEAR DIELECTRIC

since \( \frac{B_l}{r^l} \) diverges at \( r = 0 \), we must have \( B_l = 0 \) for all \( l \). Outside the sphere

\[
V_2(r, \theta) = \sum_{l=0}^{\infty} \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta). \tag{4.166}
\]

Applying the boundary condition in (a), one can write

\[
- \lim_{r \to \infty} \left[ \frac{\partial V_2(r, \theta)}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V_2(r, \theta)}{\partial \theta} \hat{\theta} \right] = - \lim_{r \to \infty} \sum_{l=0}^{\infty} \frac{1}{r} \hat{r} P_l(\cos \theta) \frac{d}{dr} \left( C_l r^l + \frac{D_l}{r^{l+1}} \right)
\]

\[
= - \lim_{r \to \infty} \sum_{l=0}^{\infty} \frac{1}{r} \hat{r} \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) \frac{d}{d\theta} P_l(\cos \theta) = E_0 \cos (\theta) \hat{r} - E_0 \sin (\theta) \hat{\theta}. \tag{4.167}
\]

Considering the radial part, we have

\[
\lim_{r \to \infty} \sum_{l=0}^{\infty} P_l(\cos \theta) \left( C_l r^{l-1} - (l+1) \frac{D_l}{r^{l+1}} \right) = - E_0 \cos (\theta)
\]

\[
\Rightarrow \sum_{l=0}^{\infty} P_l(\cos \theta) C_l r^{l-1} = - E_0 P_1 (\cos (\theta)) \tag{4.169}
\]

that leads to \( C_l = 0 \) for \( l > 1 \). On the other hand considering the angular part, we have

\[
\lim_{r \to \infty} \sum_{l=0}^{\infty} \frac{1}{r} \hat{r} \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) \frac{d}{d\theta} P_l(\cos \theta) = E_0 \sin (\theta)
\]

\[
\Rightarrow \sum_{l=0}^{\infty} C_l r^{l-1} \frac{d}{d\theta} P_l(\cos \theta) = E_0 \sin (\theta) = - E_0 \frac{d}{d\theta} P_1 (\cos (\theta))
\]

\[
\Rightarrow \frac{d}{d\theta} \sum_{l=0}^{\infty} C_l r^{l-1} P_l(\cos \theta) = \frac{d}{d\theta} \left[ - E_0 P_1 (\cos (\theta)) \right]
\]

\[
\Rightarrow \sum_{l=0}^{\infty} C_l r^{l-1} P_l(\cos \theta) = \left[ - E_0 P_1 (\cos (\theta)) \right] \tag{4.170}
\]

so that using the result above (i.e. \( C_l = 0 \) for \( l > 1 \)), one finds

\[
C_0 + C_1 P_1(\cos \theta) = \left[ - E_0 P_1 (\cos (\theta)) \right] \Rightarrow C_0 = 0, C_1 = - E_0. \tag{4.171}
\]
Thus the potential outside the sphere becomes

\[ V_2(r, \theta) = -E_0 r P_1(\cos \theta) + \sum_{l=0}^{\infty} \left( \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta) \]

\[ = \frac{D_0}{r} + \left( \frac{D_1}{r^2} - E_0 r \right) P_1(\cos \theta) + \sum_{l=2}^{\infty} \left( \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta). \quad (4.172) \]

Using the boundary condition in (b), we have

\[ \sum_{l=1}^{\infty} A_l R^l P_l(\cos \theta) = \frac{D_0}{R} + \left( \frac{D_1}{R^2} - E_0 R \right) P_1(\cos \theta) \]

\[ + \sum_{l=2}^{\infty} \left( \frac{D_l}{R^{l+1}} \right) P_l(\cos \theta) \Rightarrow \frac{D_0}{R} - A_0 + \left( \frac{D_1}{R^2} - E_0 R - A_1 R \right) P_1(\cos \theta) \]

\[ + \sum_{l=2}^{\infty} \left( \frac{D_l}{R^{l+1}} - A_l R^l \right) P_l(\cos \theta) = 0 \quad (4.173) \]

There follows that

\[ \frac{D_0}{R} - A_0 = 0, -E_0 R + \frac{D_1}{R^2} - A_1 R = 0, \frac{D_l}{R^{l+1}} - A_l R^l \]

\[ \Rightarrow A_0 = \frac{D_0}{R}, D_1 = (A_1 + E_0) R^3, D_l = A_l R^{2l+1} \text{ for } l > 1. \quad (4.174) \]

Using this result we may rewrite the potential outside as

\[ V_2(r, \theta) = A_0 + \left[ \frac{(A_1 + E_0) R^3}{r^2} - E_0 r \right] P_1(\cos \theta) \]

\[ + \sum_{l=2}^{\infty} \left( \frac{A_l R^{2l+1}}{r^{l+1}} \right) P_l(\cos \theta). \quad (4.175) \]

and the inside potential

\[ V_1(r, \theta) = A_0 + A_1 r P_1(\cos \theta) + \sum_{l=2}^{\infty} A_l r^l P_l(\cos \theta), \quad (4.176) \]

Using the last boundary condition

\[ \left[ \epsilon_2 \frac{\partial V_2(r, \theta)}{\partial r} - \epsilon_1 \frac{\partial V_1(r, \theta)}{\partial r} \right]_{r=R} = 0, \quad (4.177) \]
and $\epsilon_2 = \epsilon_0$, $\epsilon_1 = \epsilon$, we find

$$
\epsilon_0 \frac{\partial}{\partial r} \left[ \frac{(A_1 + E_0) R^3 - E_0 r^3}{r^2} P_1(\cos \theta) + \sum_{l=2}^{\infty} A_l R^{l+1} \frac{r^l}{r^l+1} P_l(\cos \theta) \right]_{r=R} - \epsilon \frac{\partial}{\partial r} \left[ A_1 r P_1(\cos \theta) + \sum_{l=2}^{\infty} A_l r^l P_l(\cos \theta) \right]_{r=R} = 0
$$

$$
\Rightarrow \left[ \epsilon_0 \left( -2 \frac{(A_1 + E_0) R^3}{r^3} - E_0 \right) P_1(\cos \theta) - \sum_{l=2}^{\infty} \epsilon_0 \left( \frac{A_l R^{l+1}}{r^{l+2}} \right) (l+1) P_l(\cos \theta),
-\epsilon A_1 P_1(\cos \theta) - \sum_{l=2}^{\infty} \epsilon_l A_l r^{l-1} P_l(\cos \theta) \right]_{r=R} = 0
$$

(4.178)

$$
\Rightarrow -\epsilon_0 (3E_0 + 2A_1) P_1(\cos \theta) - \sum_{l=2}^{\infty} \epsilon_0 (A_l R^{l-1}) (l+1) P_l(\cos \theta) = 0
$$

$$
\Rightarrow \left[ -\epsilon_0 (3E_0 + 2A_1) - \epsilon A_1 \right] P_1(\cos \theta) + \sum_{l=2}^{\infty} (A_l R^{l-1}) \left[ \epsilon_0 (l+1) - \epsilon l \right] P_l(\cos \theta) = 0.
$$

(4.179)

There follows that

$$
\Rightarrow -\epsilon_0 (3E_0 + 2A_1) - \epsilon A_1 = 0 \Rightarrow A_1 = -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0, A_l = 0 \text{ for } l > 1
$$

(4.180)

Therefore the potentials inside and outside the sphere becomes

$$
V_1(r, \theta) = A_0 - \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 r \cos (\theta),
$$

(4.181)

and

$$
V_2(r, \theta) = A_0 - E_0 r \cos (\theta) + \frac{\left( -E_0 + \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 \right) R^3}{r^2} \cos (\theta)
$$

$$
\Rightarrow V_2(r, \theta) = A_0 - E_0 r \cos (\theta) + E_0 \left( \frac{\epsilon_0 - \epsilon}{2\epsilon_0 + \epsilon} \right) \frac{R^3}{r^2} \cos \theta
$$

(4.182)
The electric field inside

\[
\vec{E} = -\frac{\partial V_1(r, \theta)}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V_2(r, \theta)}{\partial \theta} \hat{\theta} = \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} \left( E_0 \cos (\theta) \hat{r} - E_0 \sin (\theta) \hat{\theta} \right) = \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} \vec{E}_0. \quad (4.183)
\]

Note that when \( \epsilon = \epsilon_0 \), the electric field becomes \( \vec{E} = \vec{E}_0 \) as expected.

Example 4.8 A point charge finds itself at a height \( d \) above an infinite half-space of dielectric material. The charge has magnitude \( q \), the dielectric is a linear dielectric material of susceptibility \( \chi_e \), and there are no unpaired charges in the volume of the dielectric or on its surface (i.e. no free charges). The Cartesian coordinates \( x \) and \( y \) are in the plane of the dielectric interface, while \( z \) is directed perpendicular to the interface and into the free space region. Thus, the charge is at \( z = d \). The field in the free space region can be taken as the superposition of a particular solution due to the point charge and a homogeneous solution due to a charge \( q_b \) at \( z = -d \) below the interface. The field in the dielectric can be taken as that of a charge \( q_a \) at \( z = d \).

(a) Show that the potential is given by

\[
V(x, y, z) = \begin{cases} \frac{1}{4\pi\epsilon_0} \left( \frac{q}{r_x} + \frac{q_b}{r_z} \right) & z > 0 \\ \frac{1}{4\pi\epsilon_0} \frac{q_a}{r_x} & z < 0 \end{cases}
\]

where

\[
q_a = \frac{2\epsilon_0}{\epsilon + \epsilon_0} q, \quad q_b = -\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} q \quad (4.185)
\]

and

\[
r_\pm = \sqrt{x^2 + y^2 + (z \mp d)^2}, \quad \epsilon = \epsilon_0 (1 + \chi_e) \quad (4.186)
\]

(b) Show that the charge is attracted to the dielectric with the force

\[
\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qq_b}{(2d)^2} = -\frac{1}{16\pi\epsilon_0} \frac{\chi_e}{\chi_e + 2} \frac{q^2}{d^2} \hat{z}. \quad (4.187)
\]

Solution:

(a) We can write the potential outside the dielectric material as

\[
V_{out}(r, z) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{\sqrt{r^2 + (z - d)^2}} + \frac{q_b}{\sqrt{r^2 + (z + d)^2}} \right] \quad (4.188)
\]

and inside the dielectric

\[
V_{in}(r, z) = \frac{1}{4\pi\epsilon_0} \frac{q_a}{\sqrt{r^2 + (z - d)^2}}. \quad (4.189)
\]
Then applying the first boundary condition

\[ V_{\text{out}}(r, z = d) = V_{\text{in}}(r, z = d), \quad (4.190) \]

we find

\[ q = q_a - q_b. \quad (4.191) \]

Using the second boundary condition

\[ \frac{\epsilon_{\text{out}}}{\epsilon_{\text{in}}} \left. \frac{\partial V_{\text{out}}}{\partial z} \right|_{z=0} - \left. \frac{\partial V_{\text{in}}}{\partial z} \right|_{z=0} = -\sigma_f = 0 \quad (4.192) \]

we have

\[
\epsilon_0 \frac{\partial}{\partial z} \left[ \frac{1}{4\pi \epsilon_0} \frac{1}{\sqrt{r^2 + (z - d)^2}} \right] z=0 - \epsilon_0 \frac{\partial}{\partial z} \left[ \frac{q_a}{\sqrt{r^2 + (z - d)^2}} \right] z=0 - (q + q_b) - \epsilon q_a = 0 \Rightarrow q = \frac{\epsilon}{\epsilon_0} q_a + q_b \quad (4.193)
\]

Now combining the equations

\[ q = \frac{\epsilon}{\epsilon_0} q_a + q_b, \quad q = q_a - q_b, \quad (4.195) \]
we find

\[ q_a = \frac{2\epsilon_0}{\epsilon + \epsilon_0} q \]
\[ q_b = -\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} q \]  

(4.196)

Thus the potentials inside and outside are given by

\[ V(r, z) = \begin{cases} 
\frac{1}{4\pi\epsilon_0} \left( \frac{q_a}{r^2} + \frac{q_b}{r} \right) & z > 0 \\
\frac{1}{4\pi\epsilon_0} q_a r & z < 0 
\end{cases} \]  

(4.197)

The field in the dielectric can be taken as that of a charge \( q_a \) at \( z = d \).

\( (b) \) First let’s find the bound surface charge density on the surface of the dielectric at \( z = 0 \).

\[ \sigma_b(r, z = 0) = \vec{P} \cdot \hat{n} \bigg|_{z=0} \]  

(4.198)

we recall the polarization for a linear dielectric is given by

\[ \vec{P} = \epsilon_0 \chi_e \vec{E}_{in}, \]  

(4.199)

where \( \vec{E}_{in} \) is the electric field inside the dielectric. Then the bound charge density, using \( \hat{n} = \hat{z} \) and \( \vec{E}_{in} = E_{in,\hat{x}} \hat{x} + E_{in,\hat{y}} \hat{y} + E_{in,\hat{z}} \hat{z}, \) becomes

\[ \sigma_b(r, z = 0) = \vec{P} \cdot \hat{n} \bigg|_{z=0} = \epsilon_0 \chi_e E_{in,\hat{z}} \bigg|_{z=0} \]  

(4.200)

Using the electric potential, \( V(r, z) \), inside the dielectric \( (z < 0) \), the \( z \)-component of the electric field is given by

\[ E_{in,\hat{z}} = -\frac{\partial}{\partial z} \left[ \frac{1}{4\pi\epsilon_0} \frac{q_a}{\sqrt{r^2 + (z-d)^2}} \right] \]
\[ = \frac{q_a}{4\pi\epsilon_0} \frac{z-d}{(r^2 + (z-d)^2)^{3/2}} \]
\[ \Rightarrow \epsilon_0 \chi_e E_{in,\hat{z}} \bigg|_{z=0} = -\frac{q_a}{4\pi\epsilon_0} \frac{d}{(r^2 + d^2)^{3/2}} \]  

(4.201)

Then the surface charge density is \( \sigma_b \) becomes

\[ \sigma_b(r, z = 0) = -\frac{\chi_e q_a}{4\pi} \frac{d}{(r^2 + d^2)^{3/2}} = -\frac{\chi_e}{2\pi} \frac{\epsilon_0}{\epsilon + \epsilon_0} \frac{qd}{(r^2 + d^2)^{3/2}} \]
\[ \Rightarrow \sigma_b(r, z = 0) = \frac{\chi_e}{2\pi} \frac{qd}{(2 + \chi_e)} \frac{1}{(r^2 + d^2)^{3/2}} \]

Therefore, the total bound charge on the dielectric is given by

\[ q_b = \frac{\chi_e q d}{2\pi} \frac{1}{(2 + \chi_e)} \int_0^\infty \frac{qd}{(r^2 + d^2)^{3/2}} 2\pi r dr \]
\[ \Rightarrow q_b = \frac{\chi_e q d}{2 + \chi_e} \left( -\frac{1}{\sqrt{r^2 + d^2}} \right)_0^\infty = \frac{\chi_e q}{\chi_e + 2} \]  

(4.202)
From what we studied in image method, this charge can be considered as an image of a point charge located at \((0, 0, -d)\). Then the force on the point charge can be expressed as

\[
\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q q_b}{(2d)^2} = -\frac{1}{16\pi\varepsilon_0} \frac{1}{\varepsilon_e} \frac{q^2}{d^2} \hat{z}.
\] (4.203)

### 4.9 Energy in dielectric system

Consider a dielectric fixed at a given position. The work done needed \(\Delta W\) to add a free charge \(\Delta \rho_f d\tau\) onto the dielectric if the dielectric is at a potential \(V\), can be expressed as

\[
\Delta W = \int \Delta \rho_f V d\tau.
\] (4.204)

We recall that the free charge density is related to the electric displacement vector \(\vec{D}\)

\[
\rho_f = \nabla \cdot \vec{D} \Rightarrow \Delta \rho_f = \nabla \cdot (\Delta \vec{D})
\] (4.205)

and the work done can be expressed as

\[
\Delta W = \int V \left[ \nabla \cdot (\Delta \vec{D}) \right] d\tau.
\] (4.206)

Using the relation

\[
\nabla \cdot [V (\Delta \vec{D})] = V \nabla \cdot [(\Delta \vec{D})] + (\Delta \vec{D}) \cdot \nabla V
\]

\[
= \nabla \cdot V \left[ (\Delta \vec{D}) \right] = \nabla \cdot V \left[ (\Delta \vec{D}) \right] - (\Delta \vec{D}) \cdot \nabla V
\]

\[
\Rightarrow V \nabla \cdot \left[ (\Delta \vec{D}) \right] = \nabla \cdot V \left[ (\Delta \vec{D}) \right] + (\Delta \vec{D}) \cdot \vec{E},
\] (4.207)

Eq. (4.206) can be written as

\[
\Delta W = \int \left[ \nabla \cdot V \left( \Delta \vec{D} \right) \right] d\tau + (\Delta \vec{D}) \cdot \vec{E} d\tau
\]

\[
= \int \nabla \cdot V \left[ (\Delta \vec{D}) \right] d\tau + (\Delta \vec{D}) \cdot \vec{E} d\tau
\] (4.208)

Using the Divergence theorem

\[
\int \nabla \cdot \vec{A} d\tau = \oint \vec{A} \cdot d\vec{a}
\]

we may write Eq. (4.208) as

\[
\Delta W = \oint \left[ V \left( \Delta \vec{D} \right) \right] \cdot d\vec{a} + \int (\Delta \vec{D}) \cdot \vec{E} d\tau
\] (4.209)
If we integrate over all space the first integral would vanish. Hence

\[ \Delta W = \int_{all\ space} (\Delta \vec{D}) \cdot \vec{E} d\tau \]

(4.210)

For a linear dielectric material, using \( \vec{D} = \epsilon \vec{E} \), noting that the relation

\[ \Delta (\vec{D} \cdot \vec{E}) = \Delta (\epsilon \vec{E}) \cdot \vec{E} + \epsilon \Delta \vec{E} = 2 \Delta (\epsilon \vec{E}) \cdot \vec{E} \]

(4.211)

we may write

\[ \Delta (\vec{D} \cdot \vec{E}) = \Delta (\epsilon \vec{E}) \cdot \vec{E} + \epsilon \Delta \vec{E} = 2 \Delta (\epsilon \vec{E}) \cdot \vec{E} \]

\[ \Rightarrow \Delta (\vec{D} \cdot \vec{E}) = \frac{1}{2} \Delta (\vec{D} \cdot \vec{E}) \]

(4.212)

so that

\[ \Delta W = \frac{1}{2} \int_{all\ space} \Delta (\vec{D} \cdot \vec{E}) d\tau = \Delta \left[ \frac{1}{2} \int_{all\ space} (\vec{D} \cdot \vec{E}) d\tau \right] \]

(4.213)

Therefore for a linear dielectric medium the energy can be determined using

\[ W = \frac{1}{2} \int_{all\ space} (\vec{D} \cdot \vec{E}) d\tau \]

(4.214)

Note: Eq. (4.214) is valid only for a linear dielectric material. For nonlinear, in general, we must start from

\[ \Delta W = \int_{all\ space} (\Delta \vec{D}) \cdot \vec{E} d\tau \]

(4.215)

Example 4.11 A spherical conductor, of radius \( a \), carries a charge \( Q \) (Fig. below). It is surrounded by linear dielectric of susceptibility \( \chi_c \), out to radius \( b \). Find the energy of this configuration.

Solution: The strength of the electric field is different inside the conductor, the dielectric, and outside the dielectric. It is given by

\[ \vec{E} = \begin{cases} 0 & r < a \\ \frac{Q}{4\pi r^2} \hat{r} & a < r < b \\ \frac{Q}{4\pi \epsilon_0 r^2} \hat{r} & r > b \end{cases} \]

(4.216)

and the corresponding electric displacement

\[ \vec{D} = \epsilon \vec{E} = \begin{cases} 0 & r < a \\ \frac{Q}{4\pi \epsilon} \hat{r} & a < r < b \\ \frac{Q}{4\pi \epsilon_0} \hat{r} & r > b \end{cases} \]

(4.217)
The energy of the configuration can then be expressed as

\[ W = \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} \, d\tau = \frac{1}{2} \int_{\text{all space}} \left( \vec{D} \cdot \vec{E} \right) 4\pi r^2 \, dr \]  

(4.218)

\[ \Rightarrow W = 2\pi \int_{0}^{\infty} \left( \vec{D} \cdot \vec{E} \right) r^2 \, dr = 2\pi \int_{0}^{a} \left( \vec{D} \cdot \vec{E} \right) r^2 \, dr \]

\[ + 2\pi \int_{a}^{b} \left( \vec{D} \cdot \vec{E} \right) r^2 \, dr + 2\pi \int_{b}^{\infty} \left( \vec{D} \cdot \vec{E} \right) r^2 \, dr \]

\[ = 2\pi \int_{a}^{b} \left( \frac{Q}{4\pi r^2} \frac{Q}{4\pi \epsilon r^2} \right) r^2 \, dr + 2\pi \int_{b}^{\infty} \left( \frac{Q}{4\pi \epsilon_0 r^2} \frac{Q}{4\pi r^2} \right) r^2 \, dr \]

\[ = \frac{Q^2}{8\pi \epsilon} \int_{a}^{b} \frac{1}{r^2} \, dr + \frac{Q^2}{8\pi \epsilon_0} \int_{b}^{\infty} \frac{1}{r^2} \, dr \]

\[ \Rightarrow W = \frac{Q^2}{8\pi \epsilon} \left( \frac{1}{a} - \frac{1}{b} \right) + \frac{Q^2}{8\pi \epsilon_0} \frac{1}{b} = \frac{Q^2}{8\pi} \left( \frac{b - a}{ab\epsilon} + \frac{1}{b\epsilon_0} \right) \]  

(4.219)

using \( \epsilon = \epsilon_0 (1 + \chi_\epsilon) \), one finds

\[ W = \frac{Q^2}{8\pi \epsilon_0} \left( \frac{b - a}{ab(1 + \chi_\epsilon)} + \frac{1}{b} \right). \]  

(4.220)

### 4.10 Forces on a dielectric

Consider a parallel plate capacitor filled with a dielectric. The capacitor is maintained at a constant potential difference (Fig. 4.6) You begin to pull out the dielectric as shown in the figure. The work \( dW \) you do to pull the dielectric a distance \( dx \) can be expressed as

\[ dW = F_{ap} \, dx \Rightarrow F_{ap} = \frac{dW}{dx}, \]  

(4.221)
where $F_{ap}$ is the force you applied to do the work. This force is needed to overcome the electrical force $F_e$. Thus, we can express the electrical force as

$$F_e = -F_{ap} = -\frac{dW}{dx}. \quad (4.222)$$

The energy stored in a capacitor is given by

$$W = \frac{1}{2}CV^2 = \frac{Q^2}{2C}. \quad (4.223)$$

If we assume the surface charge density on the plate is $\sigma$ and consider a pill-box with area, $A = lw$, then

$$Q = \sigma lw. \quad (4.224)$$

Recalling that the potential difference between a parallel plate capacitors separated by a distance, $d$, is

$$V = \frac{\sigma d}{\varepsilon_0}, \quad (4.225)$$
4.10. FORCES ON A DIELECTRIC

the capacitance

\[ C = \frac{Q}{V} \]  

(4.226)

for a parallel plate capacitor, in the absence of the dielectric becomes

\[ C = \frac{\sigma lw}{\varepsilon_0} = \frac{\varepsilon_0 lw}{d} \]  

(4.227)

Now if it is completely filled by a linear dielectric material with, \( \varepsilon = \varepsilon_0 (1 + \chi_e) \), we can write

\[ C = \frac{\varepsilon_0 (1 + \chi_e) lw}{d} \]  

(4.228)

For the capacitor shown above only the region \( (l - x) \) is filled with a dielectric as we pull the dielectric out by a distance \( x \). In this part of the capacitor the capacitance is

\[ C_d = \frac{\varepsilon_0 (1 + \chi_e) (l - x) w}{d} \]  

(4.229)

and in the rest part (the vacuum part) the capacitance in this region, \( C_v \), is given by

\[ C_v = \frac{\varepsilon_0 xw}{d} \]  

(4.230)

Then the total capacitance

\[ C = C_d + C_v = \frac{\varepsilon_0 (1 + \chi_e) (l - x) w}{d} + \frac{\varepsilon_0 xw}{d} \]

\[ \Rightarrow C = \frac{\varepsilon_0 w l (1 + \chi_e) - x \chi_e}{d} = \frac{\varepsilon_0 w (\varepsilon_r - x \chi_e)}{d}, \]  

(4.231)

where \( \varepsilon_r = 1 + \chi_e \). Now coming back to the force on the dielectric

\[ F_e = -F_{ap} = -\frac{dW}{dx}; \]  

(4.232)

using

\[ W = \frac{Q^2}{2C} \]  

(4.233)

we can write

\[ F_e = -\frac{d}{dx} \left[ \frac{Q^2}{2C} \right] = \frac{Q^2}{2C^2} \frac{dC}{dx} \]

\[ Q = CV \Rightarrow F_e = \frac{1}{2} \frac{V^2 dC}{dx} \]  

(4.234)

Substituting the result for the capacitance in Eq. (4.231), one finds

\[ F_e = \frac{1}{2} \frac{V^2 dC}{dx} = \frac{1}{2} \frac{V^2 d}{dx} \left[ \frac{\varepsilon_0 w (\varepsilon_r - x \chi_e)}{d} \right] \]

\[ \Rightarrow F_e = -\frac{\varepsilon_0 w \chi_e V^2}{2d}. \]  

(4.235)

The negative sign shows the force is in the negative \( x \) direction.
Example 4.12 Two long coaxial cylindrical metal tubes (inner radius $a$, outer radius $b$) stand vertically in a tank of oil (Susceptibility $\chi$, mass density $\rho$). The inner one is maintained at a potential $V$, and the outer one is grounded (see Figure below). To what height ($h$) does the oil rise in the space between the tubes?

Solution: The oil is a dielectric and it experiences a force due to the electric field and it will rise up until it is balanced by the gravitational force. Let’s assume the oil has risen a height $h$, then the corresponding mass to this volume of oil is

$$m = \rho \pi (b^2 - a^2) h.$$  \hspace{1cm} (4.236)

and the weight, $w$, for this mass is

$$w = \rho g \pi (b^2 - a^2) h.$$  \hspace{1cm} (4.237)

At the maximum height, $h_{\text{max}}$, the weight of the oil is the same as the electrical force, $F_e$,

$$F_e = \frac{dW}{dh}.$$  \hspace{1cm} (4.238)

The energy stored in a capacitor is given by

$$W = \frac{1}{2} CV^2.$$  \hspace{1cm} (4.239)

As the oil rises what is changing is the capacitance of the capacitor. Since the potential difference is kept constant

$$F_e = -\frac{d}{dh} \left[ \frac{1}{2} CV^2 \right] = \frac{1}{2} V^2 \frac{dC}{dh}.$$  \hspace{1cm} (4.240)

We recall that for a coaxial cylindrical capacitor of inner radius $a$ and outer radius $b$ filled by a dielectric with dielectric constant $\epsilon_r$, the capacitance
per unit length is given by

\[ \frac{C}{l} = \frac{2\pi \varepsilon_0}{\ln \frac{b}{a}}. \]  

(4.241)

For the part of the cylinder filled by the oil (i.e. \( l = h, \varepsilon_r = \varepsilon/\varepsilon_0 = 1+\chi_r \)), we have

\[ C_d = \frac{2\pi \varepsilon_0 (1+\chi_r) h}{\ln \frac{b}{a}}. \]  

(4.242)

and for the part that is not filled (i.e. \( l = L - h, \varepsilon_r = 1 \))

\[ C_v = \frac{2\pi \varepsilon_0 (L-h)}{\ln \frac{b}{a}}. \]  

(4.243)

The total capacitance will then be

\[ C = \frac{2\pi \varepsilon_0 (1+\chi_r) h}{\ln \frac{b}{a}} + \frac{2\pi \varepsilon_0 (L-h)}{\ln \frac{b}{a}}. \]  

(4.244)

Using this result one finds for the electrical force

\[ F_e = \frac{1}{2} V^2 \frac{dC}{dh} = \frac{2\pi \varepsilon_0 \chi_r V^2}{\ln \frac{b}{a}}, \]  

(4.245)

that is the same as the weight of the oil

\[ w = \rho g \pi (b^2 - a^2) h. \]  

(4.246)

Then the maximum height, \( h_{\text{max}} \) will be

\[ F_e = w \Rightarrow \rho g \pi (b^2 - a^2) h_{\text{max}} = \frac{2\pi \varepsilon_0 \chi_r V^2}{\ln \frac{b}{a}}, \]

\[ \Rightarrow h_{\text{max}} = \frac{2\varepsilon_0 \chi_r V^2}{\rho g (b^2 - a^2) \ln \frac{b}{a}}. \]  

(4.247)
Magnetostatics is the study of static magnetic fields. In electrostatics, the charges are stationary, whereas here, the currents are steady or dc (direct current) or the currents do not alternate rapidly.

5.1 The Lorentz Force Law

Magnetic fields: Two conducting wires carrying a current in the same direction attract each other while those carrying in the opposite direction repel each other. In each of the wires the net free charge is zero. Consequently, there is no electric field created by one wire at the position of the other. This mean the attraction nor the repulsion force is electrostatic in nature. Thus what could be the cause of this force? It is magnetic field that is resulted from the motion of the charges in the wire (current)! [Refer to the online video]
**Magnetic forces:** A charge $Q$ moving with a velocity $\vec{V}$ in a uniform magnetic field experiences a magnetic force $\vec{F}_{\text{mag}}$ given by

$$\vec{F}_{\text{mag}} = Q(\vec{V} \times \vec{B})$$

(5.1)

**Magnetic and electrical force:** A charge $Q$ moving with a velocity $\vec{V}$ in a uniform magnetic field and electric field experiences a magnetic force and electrical force given by

$$\vec{F} = Q(\vec{E} + \vec{V} \times \vec{B})$$

(5.2)

**Cyclotron motion:** When the magnetic field is normal to the velocity, the charged particle moves in a circular path of fixed radius. The radius of the circle is determined by using Newton’s second law for circular motion.

$$\frac{mv^2}{R} = qvB$$

**Helical motion:** When the particle has a velocity components that are normal and tangential to the magnetic field, the charged particle moves in a helix
Example 5.1 \textit{(Cycloid Motion)} Consider a particle of charge $Q$ and mass $m$ sitting at the origin. Suppose a uniform electric field $\vec{E}$ is applied along the positive $x-$direction and a uniform magnetic field $\vec{B}$ is also applied simultaneously along the positive $y-$direction. Describe the trajectory of the charge qualitatively and quantitatively. (Refer to the online video)

The net force acting on the charge at a given instant of time is the sum of the electric and magnetic force and it is given by

$$\vec{F} = Q(\vec{E} + \vec{V} \times \vec{B}) \tag{5.3}$$

Let’s the position of the charge at a given time $t$ be

$$r = x\hat{x} + y\hat{y} + z\hat{z} \tag{5.4}$$

then the velocity can be expressed as

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}. \tag{5.5}$$

Noting that $\vec{B} = B\hat{y}$ and $\vec{E} = E\hat{x}$, we may write the net force acting on the charge as

$$\vec{F} = Q(E\hat{x} + (x\hat{x} + y\hat{y} + z\hat{z}) \times B\hat{y})$$

$$\Rightarrow \vec{F}_{mag} = Q(E\hat{x} + B(\dot{x}\hat{z} - \dot{z}\hat{x}))$$

$$\Rightarrow \vec{F}_{mag} = Q((E - B\dot{z})\hat{x} + B\dot{x}\hat{z}) \tag{5.6}$$
Now applying Newton’s second law
\[
\vec{F} = Q((E - B\dot{z}) \hat{x} + B\dot{x}\hat{z}) = m\ddot{x} = m\ddot{x} + m\ddot{z}
\]
\[
\Rightarrow Q((E - B\dot{z}) \hat{x} + B\dot{x}\hat{z}) = m\ddot{x} + m\ddot{z}
\]
\[
\Rightarrow Q(E - B\dot{z}) = m\ddot{x}, \quad QB\dot{x} = m\ddot{z}
\]
\[
\Rightarrow \frac{QB}{m} \left( \frac{E}{B} - \dot{z} \right) = \ddot{x}, \quad \ddot{z} = \frac{QB}{m} \dot{x}
\] (5.7)

We define the Cyclotron frequency, \( \omega \)
\[
\omega = \frac{QB}{m}
\] (5.8)
so that
\[
\ddot{z} = \omega \dot{x}, \Rightarrow \dot{z} = \omega x + C_1 = \omega \int x dt + C_1 t + C_2
\] (5.9)
and
\[
\omega \left( \frac{E}{B} - \dot{z} \right) = \ddot{x} \Rightarrow \omega \left( \frac{E}{B} - \omega x - C_1 \right) = \ddot{x}
\]
\[
\Rightarrow \ddot{x} + \omega^2 x = \omega \left( \frac{E}{B} - C_1 \right)
\] (5.10)
The general solution is given by
\[
x(t) = A \cos(\omega t) + D \sin(\omega t) + \frac{1}{\omega} \left( \frac{E}{B} - C_1 \right)
\] (5.11)
Substituting this into the expression for \( z(t) \),
\[
z(t) = \omega \int x dt + C_1 t + C_2
\]
\[
\Rightarrow z(t) = A \sin(\omega t) - D \cos(\omega t) + \frac{E}{B} t + C_2
\] (5.12)
The charge initially is at rest at the origin,
\[
x(0) = z(0) = 0, \quad \dot{x}(0) = \dot{z}(0) = 0
\]
so that using Eqs. (5.11) and (5.12), we find
\[
x(0), \dot{z}(0) = 0 \Rightarrow \dot{z}(0) = \omega x(0) + C_1 = 0 \Rightarrow C_1 = 0
\]
and
\[
\dot{x}(0) = 0 \Rightarrow D\omega = 0 \Rightarrow D = 0
\] (5.13)
which leads to
\[
x(t) = A \cos(\omega t) + \frac{1}{\omega} \frac{E}{B}, \quad z(t) = A \sin(\omega t) + \frac{E}{B} t + C_2
\] (5.14)
5.1. THE LORENTZ FORCE LAW

Now using

\[ z(0) = 0 \Rightarrow C_2 = 0, \ x(0) = 0 \Rightarrow A = -\frac{1}{\omega B} \]  \hspace{1cm} (5.15)

Therefore

\[ x(t) = A \cos(\omega t) + \frac{E}{\omega B} t \]
\[ z(t) = A \sin(\omega t) + \frac{E}{\omega B} t + C_2 \]  \hspace{1cm} (5.16)

\[ z(t) = -\frac{E}{\omega B} \sin(\omega t) + \frac{E}{B} t = \frac{E}{\omega B} (\omega t - \sin(\omega t)) \]  \hspace{1cm} (5.17)
\[ x(t) = -\frac{E}{\omega B} \cos(\omega t) + \frac{E}{\omega B} = \frac{E}{\omega B} (1 - \cos(\omega t)) \]  \hspace{1cm} (5.18)

or

\[ z(t) - R\omega t = -R \sin(\omega t), \ x(t) - R = -R \cos(\omega t) \]

where

\[ R = \frac{E}{\omega B}. \]  \hspace{1cm} (5.19)

The trajectory of the charge is given by

\[
(z(t) - R\omega t)^2 + (x(t) - R)^2 = R^2 \sin^2 (\omega t) + R^2 \cos^2 (\omega t)
\]

\[ \Rightarrow (z(t) - R\omega t)^2 + (x(t) - R)^2 = R^2 \]  \hspace{1cm} (5.20)

This is equation of a circle whose center \((R, 0, R\omega t)\) moves with a velocity \(v = R\omega = \frac{E}{B}\) along the positive \(z\) axis. The charged particle moves as if it were a spot on the rim of the wheel, rolling down the \(z\)-axis at speed \(v\). The curve generated in this way is called a cycloid.

Figure 5.1: Cycloid motions.
Magnetic forces do no work!

**Currents:** charge per unit time passing a given point inside a conductor. It is measured in Amperes (A), $1A = 1C/s$.

*A moving line charge:* For a line charge with line charge density $\lambda$ moving with a velocity $\vec{v}$, the current

$$I = \lambda \vec{v}$$  \hspace{1cm} (5.21)

If this line charge is placed in a magnetic field $\vec{B}$ the magnetic force

$$d\vec{F}_{mag} = (\vec{V} \times \vec{B})dq = (\vec{V} \times \vec{B})\lambda dl$$

$$\Rightarrow d\vec{F}_{mag} = I \times \vec{B}dl = I (d\vec{l} \times \vec{B}) \Rightarrow \vec{F}_{mag} = \int I (d\vec{l} \times \vec{B})$$  \hspace{1cm} (5.22)

**Example 5.2** A rectangular loop of wire, supporting a mass $m$, hangs vertically with one end in a uniform magnetic field $B$, which points into the page in the shaded region (Fig. below).

(a) For what current $I$, in the loop, would the magnetic force upward exactly balance the gravitational force downward?
5.1. THE LORENTZ FORCE LAW

**Sol:** The current that flows through the wire must produce a magnetic force that is strong enough to balance the gravitational force. The magnetic force is given by

\[
\vec{F}_{\text{mag}} = \int I (d\vec{l} \times \vec{B}) = IBa \hat{z}
\]

and the gravitational force

\[
\vec{F}_g = -mg \hat{z}
\]

Then

\[
F_g = F_{mag} \Rightarrow I = \frac{mg}{Ba}
\]

(b) If the current exceeds the current in part (a) what would happen to the wire. Does it move up or down or remain static. If it moves, is there a work done by the magnetic force? Explain.

**Sol:** If the current \( I > \frac{mg}{Ba} \), since magnetic force becomes greater than gravitational force, the wire moves up lifting the weight. Though it seems that the magnetic force is the one lifting the object and there is work done by the magnetic force, in reality it is not lifting the wire and it does no work. It just simply redirect the lifting force which is supplied by the source responsible for increasing the current. Let’s say the velocity of the charge carrier in the wire in the vertical direction resulting from increase in the current in the wire is \( u \) and the velocity of the charges moving along the wire be \( w \), since both these velocities are normal to the magnetic field, it produce there will be a magnetic force as shown in the figure below.

This causes the net magnetic force to be normal to the net velocity (net displacement) of the charge carriers inside the wire. This results in the work done due to the magnetic field to be zero. Then if the work done by \( \vec{F}_{mag} \) is zero, then who actually lifting the wire? The external source responsible for the increase in current is the one who actually does the work. Let say this source
causes a velocity component, \( \vec{v} \), along the vertical direction, which results in a force in a horizontal direction given by

\[
F_{\text{horz}} = \lambda auB
\]  

(5.26)

The work done by this force is

\[
W = \int F_{\text{horz}} \, dl = \int \lambda auB \, w \, dt = aB \int (\lambda w) \, u \, dt
\]

\[\Rightarrow W = aB \int (\lambda w) \, (u \, dt) = IBah \]

(5.27)

**Surface current density, \( \vec{K} \):** The surface current density is defined as

\[
\vec{K} = \frac{d\vec{I}}{dl \perp} = \sigma \vec{v}
\]

(5.28)

where \( \sigma \) is the surface charge density. The magnetic force

\[
\vec{F}_{\text{mag}} = \int (\vec{v} \times \vec{B}) \, dq = \int (\vec{v} \times \vec{B}) \, \sigma \, da
\]

\[= \int (\sigma \vec{v} \times \vec{B}) \, da = \int (\vec{K} \times \vec{B}) \, da
\]

(5.29)

**Volume current density:**

\[
\vec{J} = \frac{d\vec{I}}{da \perp} = \rho \vec{v}
\]

(5.30)

where \( \rho \) is the volume charge density. The magnetic force

\[
\vec{F}_{\text{mag}} = \int (\vec{v} \times \vec{B}) \, dq = \int (\vec{v} \times \vec{B}) \, \rho \, d\tau
\]

\[= \int (\rho \vec{v} \times \vec{B}) \, d\tau = \int (\vec{J} \times \vec{B}) \, d\tau.
\]

(5.31)
Continuity equation: using the unit vector $\hat{n}$ normal to the infinitesimal area $da$ where the infinitesimal current $dI$ crosses

$$d\hat{a} = \hat{n} da$$

the total current flowing through a closed surface $S$, in terms of the volume current density $\vec{J}$

$$\vec{J} = \frac{d\vec{I}}{da.}$$

can be expressed as

$$I = \oint S \vec{J} \cdot d\hat{a}.$$  

Using the divergence theorem, we may write

$$I = \int_S \vec{J} \cdot d\hat{a} = \int_V (\nabla \cdot \vec{J}) \, d\tau$$

where $V$ is the volume bounded by the surface $S$. The net current coming out of this volume is the result of net flow of the charges from the volume $V$. If the total charge inside the volume at time $t$ is $Q(t)$, then due to the current flow (charges moving out) at a later time $t + \delta t$, the charge $Q(t + \delta t)$ inside the volume will be less $Q(t)$. Therefore, the net current flow $I$ is

$$I = -\lim_{\delta t \to 0} \frac{Q(t + \delta t) - Q(t)}{\delta t} = -\frac{dQ}{dt}$$

If the density of the charge inside the volume is described by the volume charge
density \( \rho(\vec{r}) \), then

\[
Q = \int _{V} \rho(\vec{r}) \, d\tau \Rightarrow I = -\frac{d}{dt} \int _{V} \rho(\vec{r}) \, d\tau \tag{5.37}
\]

and this can be expresses as

\[
I = -\int _{V} \frac{\partial \rho(\vec{r})}{\partial t} \, d\tau. \tag{5.38}
\]

Now combining the two expressions for I, we may write

\[
\int _{V} \left( \nabla \cdot \vec{J} \right) \, d\tau = -\int _{V} \frac{\partial \rho(\vec{r})}{\partial t} \, d\tau. \tag{5.39}
\]

which leads to the continuity equation

\[
\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}. \tag{5.40}
\]

This equation is just a consequence of conservation of charges.

### 5.2 The Bio-Savart Law

*Steady currents:* the magnitude of the current \( I \) is the same all along the line. That means for a steady-state the charge would not be piling up somewhere.

\[
\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J} = 0 \tag{5.41}
\]

*The magnetic field of a steady current:* The magnetic field of a steady current is given by the Bio-Savart law:

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\tau' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \tag{5.42}
\]

where \( \mu_0 = 4\pi \times 10^{-7} \, N/A^2 \) is the permeability of free space. The units of a magnetic field is *tesla* (T).

\[
T = N/(A.m) \tag{5.43}
\]

*Note:* when the charges are stationary in the previous chapters, we said electrostatics. In this chapter the motion of charges gives current and current gives magnetic field. If the current is steady current,

\[
\nabla \cdot \vec{J} = 0 \tag{5.44}
\]

then we are in Magnetostatic.
Example 5.3 Find the magnetic field a distance $s$ from a long straight wire carrying a steady current $I$ (see Fig. below)

Solution: The magnetic field is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\vec{l}' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

If we define the positive $z$-axis along the direction for the current flow and point $P$ is on the $x-z$ plane and the center of the wire be at the origin, we have $d\vec{l}' = dz\hat{z}, \vec{r}' = z\hat{z}, \vec{r} = s\hat{x}$ so that

$$\vec{r} - \vec{r}' = s\hat{x} + (z - z')\hat{z} \Rightarrow |\vec{r} - \vec{r}'| = \sqrt{s^2 + z'^2}$$

$$d\vec{l}' \times (\vec{r} - \vec{r}') = dz\hat{z} \times s\hat{x} = sdz\hat{y} \quad (5.45)$$

Then

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{-z_0}^{z_0} \frac{sdz'\hat{y}}{(s^2 + z'^2)^{3/2}} \quad (5.46)$$

Noting that

$$\tan \theta = \frac{z'}{s} \Rightarrow z' = s \tan \theta \Rightarrow dz' = s \sec^2 \theta d\theta$$

$$s^2 + z'^2 = s^2(1 + \tan^2 \theta) = s^2 \sec^2 \theta \quad (5.47)$$

we have

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{-z_0}^{z_0} \frac{ss \sec^2 \theta d\theta \hat{y}}{(s^2 \sec^2 \theta)^{3/2}} = \frac{\mu_0 I}{4\pi s} \int_{-z_0}^{z_0} d\theta \hat{y}$$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi s} \int_{-\theta_0}^{\theta_0} \cos \theta d\theta \hat{y} = \frac{\mu_0 I}{2\pi s} \sin \theta_0 \hat{y} \quad (5.48)$$
For an infinite wire $\theta_0 = \pi/2$

$$\vec{B}(r) = \frac{\mu_0 I}{2\pi s} \hat{y}$$  \hspace{1cm} (5.49)

**Example 5.4** Find the force of attraction (magnitude and direction) per unit length between two long, parallel wires a distance $d$ apart, carrying currents $I_1$ and $I_2$. (See Fig. Below)

**Example 5.5** Find the magnetic field a distance $z$ above the center of a circular loop of radius $R$ which carries a steady current $I$. (See Fig. below)
5.2. THE BIO-SAVART LAW

Solution: The magnetic field is given by

\[ \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\vec{l}' = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \]  

(5.50)

For a circular loop of radius \( R \) on the x-y plane centered at the origin, we have

\[ \vec{r}' = R \cos (\varphi') \hat{x} + R \sin (\varphi') \hat{y}, \]
\[ d\vec{l}' = d\vec{r}' = -R \sin (\varphi') \, d\varphi' \hat{x} + R \cos (\varphi') \, d\varphi' \hat{y}, \]
\[ \vec{r} = z \hat{z}, \vec{r} - \vec{r}' = -R \cos (\varphi') \hat{x} - R \sin (\varphi') \hat{y} + z \hat{z}, \]
\[ \Rightarrow |\vec{r} - \vec{r}'| = \sqrt{R^2 + z^2}, \]  

(5.51)

so that

\[ \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \]
\[ = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} [-R \sin (\varphi') \, d\varphi' \hat{x} + R \cos (\varphi') \, d\varphi' \hat{y}] \times [-R \cos (\varphi') \hat{x} - R \sin (\varphi') \hat{y} + z \hat{z}] / (R^2 + z^2)^{3/2} \]  

(5.52)

\[ \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} \left[ \sin^2 (\varphi') + \cos^2 (\varphi') \right] \hat{z} \, d\varphi' \]
\[ + \frac{\mu_0 I R z}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} \left[ \cos (\varphi') \hat{x} + \sin (\varphi') \hat{y} \right] d\varphi'. \]  

(5.53)

\[ \vec{B}(\vec{r}) = \frac{\mu_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} d\varphi' \hat{z} \]
\[ + \frac{\mu_0 I R z}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} \left[ \cos \varphi' \hat{x} + \sin \varphi' \hat{y} \right] d\varphi'. \]  

(5.54)

Noting that

\[ \int_0^{2\pi} \cos (\varphi') \, d\varphi' = \int_0^{2\pi} \sin (\varphi') \, d\varphi' = 0, \]

(5.55)

we find the magnetic field to be

\[ \vec{B}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z}. \]  

(5.56)

Example 5.6 Find the magnetic field at point \( p \) for the steady current configurations shown below
CHAPTER 5. MAGNETOSTATIC

Solution: For part (a) the net magnetic field is the magnetic field of the curved parts with radius $a$ and $b$ neglecting the magnetic field due to the current in the two small line segments. To find this magnetic field we can use the expression

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int d\varphi' \hat{z} + \frac{\mu_0 I R z}{4\pi (R^2 + z^2)^{3/2}} \int (\cos \varphi' \hat{x} + \sin \varphi' \hat{y}) d\varphi'.
$$

from the previous example. In this case, since we are interested in the magnetic field at point $p$ which is the center for both arcs of radius $a$ and $b$, we set $z = 0$. We integrate from $\varphi = 0$ to $\varphi = \pi/2$ since it is a quadrant of a circle. We also take into account that the currents are opposite in the two quadrant circles, we can then write the magnetic field as

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a} \int_0^{\pi/2} d\varphi' \hat{z} - \frac{\mu_0 I}{4\pi b} \int_0^{\pi/2} d\varphi' \hat{z}.
$$

For part (b) the magnetic field is the vector sum of the magnetic field of the current through the semicircular part and the two semi-infinite parts. For the semicircular part the magnetic field, applying the previous examples, is given by

$$
\mathbf{B}_1(\mathbf{r}) = -\frac{\mu_0 I}{4\pi R} \int_0^{\pi} d\varphi' \hat{z} \Rightarrow \mathbf{B}_1(\mathbf{r}) = -\frac{\mu_0 I}{4R} \hat{z}.
$$

From example 5.3 for a current carrying wire, the magnitude of the magnetic field at a distance $s$ from the center of the wire is given by

$$
B(\mathbf{r}) = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos (\theta) d\theta.
$$
5.3. Ampère’s Law, the divergence, and the curl of $\vec{B}$

Consider the magnetic field of a long current carrying wire. The magnetic field lines form a concentric circles as shown in the figure below.

Consider an arbitrary closed loop and evaluate $\oint \vec{B} \cdot d\vec{l}$. If we assume the current be parallel to the positive $z$–axis, then in cylindrical coordinates

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\varphi}, \quad dl = ds \hat{s} + sd\varphi \hat{\varphi} + dz \hat{z}$$  

(5.64)
which leads to
\[
\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} \cdot (ds\hat{s} + sd\varphi \hat{\varphi} + dz\hat{z})
\]
\[
\Rightarrow \oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi} d\varphi = \mu_0 I
\]  (5.65)

Ampère’s Law:
\[
\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}, \text{ Integral form} \quad (5.66)
\]

The curl of the magnetic field: If we apply Stokes Theorem, we may write
\[
\oint_C \vec{B} \cdot d\vec{l} = \int_S \left( \nabla \times \vec{B} \right) \cdot d\vec{a} = \mu_0 I_{enc},
\]  (5.67)
where \( S \) is the surface bounded by the closed curve \( C \). The electrical current enclosed by this curve and flowing normal to the surface bounded by the curve can be expressed in terms of the volume current density \( \vec{J} \),
\[
I_{enc} = \int_S \vec{J} \cdot d\vec{a}.
\]  (5.68)

Using this expression Ampère’s Law can be expressed as
\[
\oint_C \vec{B} \cdot d\vec{l} = \int_S \left( \nabla \times \vec{B} \right) \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}
\]  (5.69)
which leads to the differential form of Ampère’s Law
\[
\nabla \times \vec{B} = \mu_0 \vec{J}_0.
\]  (5.70)

The divergence of a magnetic field: We recall from the Bio-Savart Law, the magnetic field of a steady current is given by
\[
\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\vec{l}' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.
\]  (5.71)

The divergence of the \( \vec{B} \) field can then be expressed as
\[
\nabla \cdot \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \nabla \cdot \left[ \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right].
\]  (5.72)
and applying the relation
\[
\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})
\]  (5.73)
one can write
\[
\nabla \cdot \left[ \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] = \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \cdot \left( \nabla \times d\vec{l}' \right)
\]
\[
- d\vec{l} \cdot \left( \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right)
\]  (5.74)
5.3. AMPÈRE’S LAW, THE DIVERGENCE, AND THE CURL OF $\mathbf{B}$

Using the result we derived in chapter 2

$$\nabla \times \left[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{r} \right] \left( \mathbf{r} - \mathbf{r}' \right) = \left( \mathbf{r} - \mathbf{r}' \right) \times \frac{3\left( \mathbf{r} - \mathbf{r}' \right)}{|\mathbf{r} - \mathbf{r}'|^3} = 0. \quad (5.75)$$

and noting that

$$\nabla \times d\mathbf{l} = 0 \quad (5.76)$$

we find

$$\nabla \cdot \left[ \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] = 0. \quad (5.77)$$

Thus

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \nabla \cdot \left[ \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] = 0. \quad (5.78)$$

The divergence of the magnetic field for a steady current is zero.

**Example 5.7** Find the magnetic field a distance $s$ from a long straight wire (See Fig. below), carrying a steady current $I$ (the same problem we solved in Example 5.3.)

**Solution:** If the wire carrying the current is long, the magnetic field lines form a concentric circle. Therefore, for a circular Ampèreian loop of radius $s$ shown in the figure above, the magnetic field is tangential to the circle and the integral form of Ampère’s Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \quad (5.79)$$

for this loop gives

$$B2\pi = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi s} \quad (5.80)$$

**Example 5.8** Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\hat{x}$, flowing over the $x - y$ plane (Fig below)
CHAPTER 5. MAGNETOSTATIC

Solution: For the rectangular Ampèreian loop shown in the figure above, we can write

\[ I = \oint \mathbf{B} \cdot d\mathbf{l} = \int_{\text{bottom}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{top}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{left}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{right}} \mathbf{B} \cdot d\mathbf{l} \]

(5.81)

For an infinite uniform surface current that flows along the positive \( x \) direction, using the right-hand rule we note that the magnetic field on the top side of the surface points along the \( +y \) direction and on the bottom side along the \( -y \). As a result the integral on the left and right side of the loop vanish since \( \mathbf{B} \cdot d\mathbf{l} = 0 \). We can then write

\[ I = \oint \mathbf{B} \cdot d\mathbf{l} = \int_{\text{bottom}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{top}} \mathbf{B} \cdot d\mathbf{l} \]

(5.82)

Noting that the total current enclosed by the rectangular loop is \( I_{\text{enc}} = Kl \).

(5.83)

Using Ampère’s law

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \]

(5.84)

we find the magnitude of the magnetic field to be

\[ B = (\mu_0/2)K \]

(5.85)
Then the magnetic field above and below the surface current can be expressed as
\[ \vec{B} = \begin{cases} 
+ (\mu_0/2) K \hat{y} & \text{for } z < 0 \\
- (\mu_0/2) K \hat{y} & \text{for } z > 0 
\end{cases} \] (5.86)

**Example 5.9** Find the magnetic field of a very long solenoid, consisting of \( n \) closely wound turns per unit length on a cylinder of radius \( R \) carrying a current \( I \) (Fig. below). You can picture this like a sheet of aluminium foil wrapped around the cylinder, carrying the equivalent uniform surface current \( K = nI \)

**Solution:** Let’s first consider a rectangular Amperian loop of length \( l \) and width \( w \) with all its sides outside the solenoid as shown in the figure below.

Applying Ampere’s law
\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} . \] (5.87)

We know the direction of the magnetic field is along the \( z \) direction and the magnitude of the field must vanish as we go far away from the solenoid. Taking these into account for the rectangular loop shown in the figure above (if we go in a counterclockwise direction for the loop, we may write
\[ \oint \vec{B} \cdot d\vec{l} = B(b) l - B(a) l = 0 \Rightarrow B(b) = B(a) \] (5.88)

This shows that the \( \vec{B} \) field is independent of the distance from the solenoid. Therefore since we know the \( B \) field is zero far from the solenoid, we conclude that \( \vec{B} = 0 \) for \( s > R \).

Now to find the field inside the solenoid we consider the rectangular loop shown in the figure above with half side of the loop inside the solenoid and the other half outside the solenoid. Since we already seen that the field outside is zero we can write
\[ \oint \vec{B} \cdot d\vec{l} = B(a) l - B(b) l = B(a) l = \mu_0 I_{\text{enc}} \] (5.89)
Now using the current enclosed by this rectangular loop

\[ I_{enc} = Kl = nIl \]  \hspace{1cm} (5.90)

we find the magnitude of the magnetic field inside the solenoid is given by

\[ B(a) = \mu_0 nI. \]  \hspace{1cm} (5.91)

Thus

\[ \vec{B} = \begin{cases} \mu_0 nI \hat{z} & \text{Inside the solenoid} \\ 0 & \text{Outside the solenoid} \end{cases} \]  \hspace{1cm} (5.92)

**Example 5.10** A steady current \( I \) flows down a long cylindrical wire of radius \( a \) (Fig. below).

Find the magnetic field, both outside and inside the wire when

(a) the current is uniformly distributed over the outside of the wire.

(b) the current is distributed in such a way that \( J \) is proportional to \( s \), the distance from the axis.

**Solution:**

(a) Using Ampere’s law the \( \vec{B} \) field inside is zero since there is no current enclosed for any Amperian loop inside the cylinder when the current is
restricted to flow on the surface. However, outside the cylinder, the \( \vec{B} \) field form a concentric circle. If we consider an Amperian loop of radius \( s \), we have
\[
\oint \vec{B} \cdot d\vec{l} = B \int_0^{2\pi} s d\varphi = B2\pi s
\] (5.93)
and noting that
\[
I_{\text{enc}} = I
\] (5.94)
we find the magnetic field to be
\[
\vec{B} = \begin{cases} 0 & \text{for } s < a \\ (\mu_0 I / 2\pi s) \hat{\varphi} & \text{for } s > a \end{cases}
\] (5.95)

(b) If the current is distributed in such a way that \( J \) is proportional to \( s \), the distance from the axis, which means
\[
\vec{J} = ks \hat{z}.
\] (5.96)
Using Amperes Law for a circular loop inside the cylinder and noting that the total current enclosed becomes
\[
I_{\text{enc}} = \int_S \vec{J} \cdot d\vec{a} = \int_0^s ks (2\pi s ds) = \frac{2\pi}{3} ks^3
\] (5.97)
we may write
\[
\oint \vec{B} \cdot d\vec{l} = B \int_0^{2\pi} s d\varphi = B2\pi s = \frac{2\pi}{3} ks^3 \\
\Rightarrow B = \frac{1}{3} ks^2
\] (5.98)
Now if we integrate
\[
I_{\text{enc}} = \int_S \vec{J} \cdot d\vec{a}
\] (5.99)
over the entire area, we find the total current, which is \( I \). Thus
\[
I = \int_0^a ks (2\pi s ds) = \frac{2\pi}{3} ka^3 \Rightarrow k = \frac{3I}{2\pi a^3}
\] (5.100)
Using this expression for \( k \) the magnetic field inside the cylinder can be expressed as
\[
B = \frac{\mu_0 I s^2}{2\pi a^3}
\] (5.101)
The magnetic field outside would not change since the current enclosed by the Amperian loop would not change. Therefore the magnetic field vector can be written as
\[
\vec{B} = \begin{cases} (\mu_0 I s^2 / 2\pi a^3) \hat{\varphi} & \text{for } s < a \\ (\mu_0 I / 2\pi s) \hat{\varphi} & \text{for } z > 0 \end{cases}
\] (5.102)
5.4 Magnetic vector potential

The vector potential: Like the electric field which we expressed in terms of the electric potential ($\vec{E} = -\nabla V$), we can also express the magnetic field in terms of the vector potential, $\vec{A}$. The magnetic field is expressed in terms of the curl of the vector potential

$$\vec{B} = \nabla \times \vec{A}. \quad (5.103)$$

To find the expression for the vector potential we use Ampere’s Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$$

$$\Rightarrow \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \quad (5.104)$$

we can choose the vector potential $\vec{A}$ with zero divergence (i.e. $\nabla \cdot \vec{A} = 0$) and still get the right expression for the magnetic field. What this mean is that if $\vec{A}_0$ is a vector with none zero divergence (i.e. $\nabla \cdot \vec{A}_0 \neq 0$), we can still choose $\vec{A} = \vec{A}_0 + \nabla \lambda$ with zero divergence (i.e. $\nabla \cdot \vec{A} = \nabla \cdot (\vec{A}_0 + \nabla \lambda) = 0$) so that

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}_0. \quad (5.105)$$

This can be proved as follows. If we chose $\vec{A}_0$ as the vector potential, we have

$$\vec{B} = \nabla \times \vec{A}_0 \quad (5.106)$$

so that in terms of a vector potential with zero divergence, $\vec{A} = \vec{A}_0 + \nabla \lambda$ $\Rightarrow \vec{A}_0 = \vec{A} - \nabla \lambda$, we can write

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \bigg( \vec{A} - \nabla \lambda \bigg)$$

$$\Rightarrow \vec{B} = \nabla \times \vec{A} - \nabla \times (\nabla \lambda) \quad (5.107)$$

Noting that (using Cartesian coordinates)

$$\nabla \times (\nabla \lambda) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \quad (5.108)$$

$$\Rightarrow \nabla \times (\nabla \lambda) = \hat{x} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{vmatrix} - \hat{y} \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \hat{z} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{vmatrix} \Rightarrow \nabla \times (\nabla \lambda) = \hat{z} \left( \frac{\partial \lambda}{\partial x} \frac{\partial y}{\partial z} - \frac{\partial \lambda}{\partial z} \frac{\partial y}{\partial x} \right)$$

$$-\hat{y} \left( \frac{\partial \lambda}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial \lambda}{\partial y} \frac{\partial z}{\partial x} \right) - \hat{z} \left( \frac{\partial \lambda}{\partial x} \frac{\partial y}{\partial z} - \frac{\partial \lambda}{\partial z} \frac{\partial y}{\partial x} \right) \quad (5.109)$$
for any differentiable and non-singular function $\lambda(x, y, z)$, we have

$$\frac{\partial \lambda}{\partial y \partial z} = \frac{\partial \lambda}{\partial z \partial y} = \frac{\partial \lambda}{\partial z \partial x} = \frac{\partial \lambda}{\partial x \partial y} = \frac{\partial \lambda}{\partial y \partial z}$$

so that

$$\nabla \times (\nabla \lambda) = 0.$$  \hfill (5.111)

Therefore in view of the above result we can conclude that

$$\vec{B} = \nabla \times \vec{A}_0 = \nabla \times \vec{A}$$

with $\nabla \cdot \vec{A} = 0$. Then we can set $\nabla \cdot \vec{A} = 0$ in Eq. (5.104) which leads to

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Comparing this with the Poisson’s equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

with the corresponding solution

$$V = \frac{1}{4\pi \varepsilon_0} \int_{vol} \frac{\rho (\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'$$

we can write the solution to Eq. (5.113) as

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{vol} \frac{\vec{J} (\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'.$$ \hfill (5.116)

For surface and line current density we can write

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{sur} \frac{\vec{K} (\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{a}'.$$ \hfill (5.117)

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{line} \frac{\vec{I} (\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{l}'.$$ \hfill (5.118)

**Example 5.11** A spherical shell, of radius $R$, carrying a uniform surface charge $\sigma$, is set spinning at angular velocity $\omega$ about the $z$-axis. Find the vector potential it produces at point $\vec{r}$ (see Fig. below)

**Solution:** Let’s rotate the position $\vec{r}$ where we want to determine the vector potential by an angle $\psi$ in a counter clockwise direction so that it coincides with the positive $z-$axis as shown in the figure below.
The angular frequency $\omega$ lies on the $x-z$ plane and can be expressed as

$$\omega = \omega \sin \psi \hat{x} + \omega \cos \psi \hat{z}.$$  

The velocity of a charge $q$ located at $\vec{r} = R \sin \theta' \cos \varphi' \hat{x} + R \sin \theta' \sin \varphi' \hat{y} + R \cos \theta' \hat{z}$ on the surface of the sphere can then be expressed as

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega \sin \psi & 0 & \omega \cos \psi \\ R \sin \theta' \cos \varphi' & R \sin \theta' \sin \varphi' & R \cos \theta' \end{vmatrix}$$  

$$= -R\omega \cos \psi \sin \theta' \sin \varphi' \hat{x} + R\omega \sin \psi \sin \theta' \sin \varphi' \hat{y} + R\omega \sin \psi \sin \theta' \sin \varphi' \hat{z}.$$  

(5.119)

The surface current density will then be

$$\vec{K}' = \sigma \vec{v} = -\sigma R\omega \cos \psi \sin \theta' \sin \varphi' \hat{x} + \sigma R\omega \sin \psi \sin \theta' \sin \varphi' \hat{y} + \sigma R\omega \sin \theta' \sin \varphi' \hat{z}.$$  

(5.120)

Noting that

$$|\vec{r} - \vec{r}'| = \sqrt{R^2 + r^2 - 2rr' \cos \theta'},$$

$$d\sigma = R^2 \sin \theta' d\theta' d\varphi'$$  

(5.121)
the vector potential

\[ \vec{A} = \frac{\mu_0}{4\pi} \int_{\text{sur}} \frac{\vec{K}(r')}{|\vec{r} - \vec{r}'|} \, d\alpha' \]  \tag{5.123}

becomes

\[ \vec{A} = \frac{\mu_0 \sigma}{4\pi} \left\{ - \int_0^{2\pi} \int_0^{2\pi} \frac{R^3 \omega \cos \psi \sin^2 \theta' \sin \varphi' d\theta' d\varphi'}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}} \hat{x} \\
+ \int_0^{2\pi} \int_0^{2\pi} \frac{R^3 \omega (\cos \psi \sin^2 \theta' \cos \varphi' - \sin \psi \sin \theta' \cos \theta')}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}} \, \hat{y} \\
\times d\theta' d\varphi' \hat{y} + \int_0^{2\pi} \int_0^{2\pi} \frac{R^3 \omega \sin \psi \sin^2 \theta' \sin \varphi' d\theta' d\varphi'}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}} \hat{z} \right\}, \tag{5.124} \]

Because of the integrals

\[ \int_0^{2\pi} \sin \varphi' d\varphi' = \int_0^{2\pi} \cos \varphi' d\varphi' = 0 \]  \tag{5.125}

the terms involving \( \sin \varphi' \) and \( \cos \varphi' \) vanish when we integrate over \( \varphi' \).
Hence

\[
\vec{A} = -\frac{\mu_0 \sigma}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{R^3 \omega \sin \psi \sin \theta' \cos \theta' d\theta' d\phi'}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}} \hat{y}
\]

\[\Rightarrow \vec{A} = -\frac{\mu_0 \sigma R^3 \omega \sin \psi}{2} \int_0^\pi \frac{\sin \theta' \cos \theta' d\theta'}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}} \hat{y}.
\] (5.126)

Introducing the transformation of variable defined by

\[u = \sqrt{R^2 + r^2 - 2rR \cos \theta'} \Rightarrow \frac{du}{rR} = \frac{\sin \theta' d\theta'}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}},\]

\[\cos \theta' = \frac{R^2 + r^2 - u^2}{2rR},\] (5.127)

\[\Rightarrow \frac{\sin \theta' \cos \theta' d\theta'}{\sqrt{R^2 + r^2 - 2rR \cos \theta'}} = \frac{(R^2 + r^2 - u^2) du}{2 (rR)^2} \] (5.128)

\[\theta = 0 \Rightarrow u = \sqrt{R^2 + r^2 - 2rR} = |r - R|,\]

\[\theta = \pi \Rightarrow u = \sqrt{R^2 + r^2 + 2rR} = r + R,\] (5.129)

we find

\[\vec{A} = -\frac{\mu_0 \sigma R^3 \omega \sin \psi}{2} \int_{|r-R|}^{r+R} \frac{(R^2 + r^2 - u^2) du}{2 (rR)^2} \hat{y}
\] (5.130)

\[\Rightarrow \vec{A} = -\frac{\mu_0 \sigma R \omega \sin \psi}{4r^2} \left[ (R^2 + r^2) u - \frac{u^3}{3} \right]_{|r-R|}^{r+R} \hat{y}
\] (5.131)

\[\Rightarrow \vec{A} = -\frac{\mu_0 \sigma R \omega \sin \psi}{4r^2} \left[ (R^2 + r^2) u - \frac{u^3}{3} \right]_{|r-R|}^{r+R} \hat{y}
\] (5.132)

We need to consider two cases. The first is when we are outside the sphere (i.e. \(r > R \Rightarrow |r - R| = r - R\)), which gives

\[\vec{A} = -\frac{\mu_0 \sigma R \omega \sin \psi}{4r^2} \left\{ \left[ R^2 + r^2 - \frac{(r + R)^2}{3} \right] (r + R) \right. \\
- \left. \left[ R^2 + r^2 - \frac{(r - R)^2}{3} \right] (r - R) \right\} \hat{y}
\] (5.133)

Using
the vector potential outside the sphere becomes

\[ \vec{A} = -\frac{\mu_0 \sigma R^4 \omega \sin \psi}{3r^2} \hat{y} \]  \hspace{1cm} (5.134)

The second case is when we are inside the sphere (i.e. \( r < R \Rightarrow |r - R| = -(r - R) \)), the vector potential becomes

\[
\vec{A} = -\frac{\mu_0 \sigma R \omega \sin \psi}{4r^2} \left\{ \left[ R^2 + r^2 - \frac{(r + R)^2}{3} \right] (r + R) + \left[ R^2 + r^2 - \frac{(r - R)^2}{3} \right] (r - R) \right\} \hat{y} \]  \hspace{1cm} (5.135)

so that using

\[
\text{ln}[2] = \text{Simplify} \left[ \left( (R^2 + r^2) - \left( \frac{(r + R)^2}{3} \right) \right) (r + R) + \left( (R^2 + r^2) - \left( \frac{(r - R)^2}{3} \right) \right) (r - R) \right]
\]

\[ \text{Out}[2] = \frac{4 R^2}{3} \]

we find

\[ \vec{A} = -\frac{\mu_0 \sigma R \omega \sin \psi}{3} \hat{y}. \]  \hspace{1cm} (5.136)

Therefore the vector potential is given by

\[ \vec{A} = \begin{cases} 
-\frac{\mu_0 \sigma R \omega \sin \psi}{3r^2} \hat{y} & r < R \\
-\frac{\mu_0 \sigma R \omega \sin \psi}{3r^2} \hat{y} & r > R 
\end{cases} \]  \hspace{1cm} (5.137)

Referring to the figure above we note that

\[ \vec{\omega} \times \vec{r} = -r \omega \sin \psi \hat{y} \]  \hspace{1cm} (5.138)

or

\[ \Rightarrow \vec{A} = \begin{cases} 
\frac{\mu_0 \sigma R}{3} \vec{\omega} \times \vec{r} & r < R \\
\frac{\mu_0 \sigma R^4}{3r^2} \vec{\omega} \times \vec{r} & r > R 
\end{cases} \]  \hspace{1cm} (5.139)
Example 5.12 Find the vector potential of an infinite solenoid with \( n \) turns per unit length, radius \( R \), and current \( I \) using the Magnetic field of a solenoid

\[
\vec{B} = \begin{cases} 
0 & s > R \\
(\mu_0 n I) \hat{z} & s < R 
\end{cases} 
\]  
(5.140)

Solution: Although we have a surface current here, we can not use the relation

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_{\text{sur}} \frac{\vec{K}(r')}{\left| \vec{r} - \vec{r}' \right|} d\sigma',
\]  
(5.141)

since the solenoid extends all the way to infinity. We use the magnetic field to find the vector potential. To this end we note that

\[
\oint \vec{A} \cdot d\vec{l} = \int \nabla \times \vec{A} \cdot d\vec{a}'
\]
\[
\Rightarrow \oint \vec{A} \cdot d\vec{l} = \int_{S} \vec{B} \cdot d\vec{a}'.
\]  
(5.142)

Recalling that the magnetic field of an infinitely long solenoid is given by

\[
\vec{B} = \begin{cases} 
0 & s > R \\
(\mu_0 n I) \hat{z} & s < R 
\end{cases} 
\]  
(5.143)

and from symmetry of the problem the vector potential is in the \( \hat{\phi} \), we can write

\[
\oint \vec{A} \cdot d\vec{l} = \int_{S} \vec{B} \cdot d\vec{a}'.
\]  
(5.144)

for \( s < R \)

\[
A2\pi s = \mu_0 n I \pi s^2 \Rightarrow \vec{A} = \frac{\mu_0 n I}{2} s \hat{\phi}. 
\]  
(5.145)

and for \( s > R \)

\[
A2\pi s = \mu_0 n I \pi R^2 \Rightarrow \vec{A} = \frac{\mu_0 n I}{2} \frac{R^2}{s} \hat{\phi}. 
\]  
(5.146)

5.5 Magnetostatic boundary conditions

Just like the electric field suffering discontinuity at a surface charge, the magnetic field also suffers discontinuity at a surface current. Unlike the electric field, however, it is the tangential component that suffers the discontinuity. We prove this starting from the two fundamental equations in magnetostatic.

\[
\nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = \mu_0 \vec{J}
\]
\[
\Rightarrow \oint \nabla \times \vec{B} \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l}' = \mu_0 \oint \vec{J} \cdot d\vec{a}
\]
\[
\oint \vec{B} \cdot d\vec{l}' = \mu_0 I_{\text{enclosed}}
\]  
(5.147)
The normal component is continuous: Consider the pill box shown in the figure below

\[ \nabla \cdot \vec{B} = 0 \Rightarrow \int \nabla \cdot \vec{B} \, d\tau = \oint \vec{B} \cdot d\vec{a} = 0 \]

\[ \Rightarrow B_{\text{above}}^\perp A - B_{\text{below}}^\perp A = 0 \Rightarrow B_{\text{above}}^\perp = B_{\text{below}}^\perp \] (5.148)

The tangential component is discontinuous: Consider the Amperian loop shown in the figure below

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}} \Rightarrow B_{\text{above}}^\parallel l - B_{\text{below}}^\parallel l = \mu_0 K l \]

\[ \Rightarrow B_{\text{above}}^\parallel - B_{\text{below}}^\parallel = \mu_0 K \] (5.149)

The continuity equation the normal and tangential components combined:

\[ \vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 \vec{K} \times \vec{n} \] (5.150)

The vector potential is continuous but not its derivative:

\[ \vec{A}_{\text{above}} = \vec{A}_{\text{below}} \]

\[ \frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K} \] (5.151)
5.6 Multipole expansion of the vector potential

Let’s consider the vector potential of a current loop shown in the figure below. The loop carries a steady current $I$.

The vector potential is given by

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}}{|\vec{r} - \vec{r}'|}. \quad (5.152)$$

From what we studied in multipole expansion in electrostatic, we know that

$$\frac{1}{|\vec{r} - \vec{r}|} = \frac{1}{r} \left[ P_0(\cos \theta') + \frac{r'}{r} P_1(\cos \theta') + \frac{r'^2}{r^2} P_2(\cos \theta') + \ldots \right]$$

$$\Rightarrow 1 \frac{1}{|\vec{r} - \vec{r}|} = \frac{1}{r} \sum_{l=0}^{\infty} \frac{r'^l}{r^l} P_l(\cos \theta') = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \theta'), \quad (5.153)$$

where $P_l(\cos \theta')$ is the Legendre’s polynomials. Using this relation the vector potential can be expressed as a

$$\vec{A} = \frac{\mu_0 I}{4\pi} \left\{ \frac{1}{r} \oint d\vec{l} + \frac{1}{r^2} \oint r' P_1(\cos \theta') d\vec{l} \right. \right.$$

$$+ \frac{1}{r^3} \oint r'^2 P_2(\cos \theta') d\vec{l} + \left. \frac{1}{r^4} \oint r'^3 P_3(\cos \theta') d\vec{l} \right. \right.$$n

$$+ \ldots \right\} = \frac{\mu_0 I}{4\pi} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \oint r'^l P_l(\cos \theta') d\vec{l}. \quad (5.154)$$
5.6. MULTIPOLe EXPANSION OF THE VECTOR POTENTIAL

For a closed loop since \( \oint d\vec{l} = 0 \), the monopole term becomes zero. Therefore, the lowest term in magnetostatic is the dipole term. There is no monopole in magnetostatic unlike in electrostatic. The dipole term is given by

\[
\vec{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi r^2} \int r' P_1 (\cos \theta') d\vec{l}' = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \theta' d\vec{l}'.
\]

\[
\Rightarrow \vec{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \hat{r}') d\vec{l}'. \tag{5.155}
\]

Using Stoke’s theorem

\[
\oint \vec{G} (\hat{r}') \cdot d\vec{r}' = \int_S \nabla' \times \vec{G} (\hat{r}') \cdot d\vec{a}' \tag{5.156}
\]

for \( \vec{G} (\hat{r}') = \vec{c} T (\hat{r}') \), a constant vector \( \vec{c} \) which does not depend on \( \hat{r}' \), we have

\[
\oint (\vec{c} T) \cdot d\vec{r}' = \int_S \nabla' \times (\vec{c} T) \cdot d\vec{a}'. \tag{5.157}
\]

so that applying

\[
\nabla' \times (\vec{c} T) = T (\nabla' \times \vec{c}) - \vec{c} \times \nabla' T = -\vec{c} \times \nabla' T \tag{5.158}
\]

we get

\[
\oint (\vec{c} T) \cdot d\vec{r}' = -\int_S \vec{c} \times \nabla' T \cdot d\vec{a}'. \tag{5.159}
\]

Since we know that

\[
\vec{c} \times \nabla' T \cdot d\vec{a}' = (d\vec{a}' \times \vec{c}) \cdot \nabla' T = (\nabla' T \times d\vec{a}') \cdot \vec{c} \tag{5.160}
\]

we may write

\[
\oint (\vec{c} T) \cdot d\vec{r}' = -\int_S (\nabla' T \times d\vec{a}') \cdot \vec{c} \Rightarrow \left( \oint T d\vec{r}' \right) \cdot \vec{c} = \left( -\int_S (\nabla' T \times d\vec{a}') \right) \cdot \vec{c} \tag{5.161}
\]

which leads to

\[
\oint T d\vec{r}' = -\int_S (\nabla' T \times d\vec{a}') . \tag{5.162}
\]

For

\[
T = \hat{r} \cdot \hat{r}' \tag{5.163}
\]

we find

\[
\oint (\hat{r} \cdot \hat{r}') d\vec{r}' = -\int_S \nabla' (\hat{r} \cdot \hat{r}') \times d\vec{a}' \tag{5.164}
\]
Using the relation

\[
\nabla' \left( \vec{E} \cdot \vec{F} \right) = \nabla' \left( \vec{E} \times \vec{F} \right) + \vec{F} \times \left( \nabla' \times \vec{E} \right)
\]

\[
+ \left( \vec{E} \cdot \nabla' \right) \vec{F} + \left( \vec{F} \cdot \nabla' \right) \vec{E}
\]

(5.165)

for \( \vec{E} = \vec{r}', \vec{F} = \hat{r} \), we get

\[
\nabla' \left( \hat{r} \cdot \vec{r}' \right) = \hat{r} \times \left( \nabla' \times \hat{r} \right) + \hat{r} \times \left( \nabla' \times \vec{r}' \right)
\]

\[
+ \left( \vec{r}' \cdot \nabla' \right) \hat{r} + \left( \vec{r}' \cdot \nabla' \right) \vec{r}'
\]

\[
\Rightarrow \nabla' \left( \hat{r} \cdot \vec{r}' \right) = \hat{r} \times \left( \nabla' \times \vec{r}' \right) + \left( \vec{r}' \cdot \nabla' \right) \hat{r}
\]

(5.166)

\[
\int_S (\hat{r} \cdot \vec{r}') \, d\vec{r}' = -\int_S \left( \nabla' (\hat{r} \cdot \vec{r}') \times d\vec{a}' \right)
\]

\[
= -\int_S \hat{r} \times (\nabla' \times \vec{r}') \times d\vec{a}' - \int_S (\vec{r}' \cdot \nabla') \vec{r}' \times d\vec{a}'
\]

(5.167)

Since

\[
\nabla' \times \vec{r}' = \left( \frac{\partial}{\partial x'} \hat{x} + \frac{\partial}{\partial y'} \hat{y} + \frac{\partial}{\partial z'} \hat{z} \right) = 0
\]

and

\[
\hat{r} \cdot \nabla' = \frac{\vec{r}}{r} \cdot \left( \frac{\partial}{\partial x'} \hat{x} + \frac{\partial}{\partial y'} \hat{y} + \frac{\partial}{\partial z'} \hat{z} \right)
\]

\[
= \frac{1}{r} \cdot \left( x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} \right)
\]

(5.169)

\[
\Rightarrow \hat{r} \cdot \nabla' \vec{r}' = \frac{1}{r} \cdot \left( x' \hat{x} + y' \hat{y} + z' \hat{z} \right)
\]

\[
\times (x' \hat{x} + y' \hat{y} + z' \hat{z})
\]

\[
\Rightarrow \hat{r} \cdot \nabla' \vec{r}' = \frac{1}{r} \cdot (x' \hat{x} + y' \hat{y} + z' \hat{z}) = \hat{r}
\]

(5.170)

the integral becomes

\[
\int_S (\hat{r} \cdot \vec{r}') \, d\vec{a}' = -\int_S \hat{r} \times d\vec{a}' = -\hat{r} \times \int_S d\vec{a}'
\]

(5.171)

Then the vector potential of the dipole contribution is given by

\[
\vec{A}_{dip} = -\frac{\mu_0 I}{4\pi r^2} \hat{r} \times \int_S d\vec{a}' = -\frac{\mu_0 I}{4\pi r^2} \hat{r} \times I \int_S d\vec{a}'
\]

\[
\Rightarrow \vec{A}_{dip} = -\frac{\mu_0}{4\pi r^2} \hat{r} \times \vec{m} \Rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2},
\]

(5.172)
where
\[ \vec{m} = I \int d\vec{a} = I \hat{a} \]  
(5.173)
is the \textit{magnetic dipole moment}.

\textbf{Example 5.13} Find the magnetic dipole moment of the "bookend-shaped" loop shown in Fig. below. All sides have length \( w \), and it carries a current \( I \).

\textbf{Solution}: We recall the magnetic dipole moment is given by
\[ \vec{m} = I \int_S d\vec{a} = I \hat{a} \]  
(5.174)
where the integral is carried over the surface bounded by the current loop.

The current loop in this case, as shown in the figure above, can be considered as a sum of two loops forming two surfaces, one on the x-y plane and the
other on the x-z plane. This leads to
\[ \vec{m} = I \left[ \int_S d\vec{a}_1' + \int_S d\vec{a}_2' \right] = I \left[ \int_S d\vec{a}_1' \hat{z} + \int_S d\vec{a}_1' \hat{y} \right] = I \left[ w^2 \hat{z} + w^2 \hat{y} \right] \] (5.175)

and the magnetic dipole moment will then be
\[ \vec{m} = I w^2 (\hat{z} + \hat{y}) \] (5.176)

The magnetic field of a pure dipole: Consider a pure magnetic dipole with magnetic dipole moment $\vec{m} = m \hat{z}$ as shown in the figure below. We want to determine the magnetic field of this dipole. We proceed by first finding the vector potential of the dipole.

The vector potential is given by
\[ \vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^2} \hat{\varphi}, \] (5.177)

where we used
\[ \vec{m} = m \cos \theta \hat{r} - m \sin \theta \hat{\theta} \]
\[ \Rightarrow \vec{m} \times \hat{r} = \left( m \cos \theta \hat{r} - m \sin \theta \hat{\theta} \right) \times \hat{r} = m \sin \theta \hat{\varphi}. \] (5.178)

Then the magnetic field can be expressed as
\[ \vec{B}_{dip} = \nabla \times \vec{A}_{dip} = \nabla \times \left( \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^2} \hat{\varphi} \right) \]
\[ \Rightarrow \vec{B}_{dip} = \nabla \times (A_{\varphi} \hat{\varphi}), \] (5.179)

where
\[ A_{\varphi} = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^2}. \] (5.180)
5.6. MULTIPOLE EXPANSION OF THE VECTOR POTENTIAL

In spherical coordinates, we recall that the curl of a given vector \( \vec{A} \) is given by

\[
\nabla \times \vec{A} = \hat{r} \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_{\varphi}) - \frac{\partial A_{\theta}}{\partial \varphi} \right) + \hat{\varphi} \frac{1}{r} \left( \frac{\partial A_{r}}{\partial \varphi} - \frac{\partial}{\partial r} (r A_{\varphi}) \right) + \hat{\theta} \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_{\theta}) - \frac{\partial A_{\theta}}{\partial \theta} \right).
\]

which gives

\[
\nabla \times (\vec{A}_{dip}) = \hat{r} \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_{dip,\varphi}) - \frac{1}{r} \frac{\partial}{\partial r} (r A_{dip,\varphi}) \right) \tag{5.182}
\]

\[
\Rightarrow \nabla \times (\vec{A}_{dip}) = \frac{\mu_0 m}{4\pi} \left\{ \hat{r} \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\sin^2 \theta}{r^2} \right) \right) - \hat{\theta} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\sin \theta}{r} \right) \right\} \
\Rightarrow \vec{B}_{dip} = \nabla \times (\vec{A}_{dip}) = \frac{\mu_0 m}{4\pi r^3} \left\{ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right\} \tag{5.183}
\]

**Example 5.14** A circular loop of wire, with radius \( R \), lies in the \( xy \) plane, centered at the origin, and carries a current \( I \) running counterclockwise as viewed from the positive \( z \)-axis.

(a) What is its magnetic dipole moment?

(b) What is the (approximate) magnetic field at points far from the origin?

(c) Show that, for points on the \( z \)-axis, your answer is consistent with the exact field (Example 5.4)

**Solution:**
(a) The magnetic dipole moment is
\[ \vec{m} = I \int d\vec{a}' = I \int d\alpha' \hat{\mathbf{z}} = I \pi R^2 \hat{\mathbf{z}}. \]  
(5.184)

(b) Far from the origin the magnetic field is approximately the same as the magnetic field of a magnetic dipole, \( \vec{m} = m \hat{\mathbf{z}} \), which is given by
\[ \vec{B} \simeq \nabla \times \vec{A}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3} \left\{ 2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\mathbf{\theta}} \right\}. \]  
(5.185)

Using \( m = 2\pi R^2 \), we find
\[ \vec{B} \simeq \nabla \times \vec{A}_{\text{dip}} = \frac{\mu_0 IR^2}{4\pi^3} \left\{ 2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\mathbf{\theta}} \right\}. \]  
(5.186)

(c) For a far point on the \( z \)-axis, we can set \( \theta = 0 \) and \( r = z, \hat{\mathbf{r}} = \hat{\mathbf{z}} \). Thus
\[ \vec{B} \simeq \nabla \times \vec{A}_{\text{dip}} = \frac{\mu_0 IR^2}{2z^3} \hat{\mathbf{z}}. \]  
(5.187)

We recall that the magnetic field of a current \( \mathcal{I} \) owing in the counterclockwise direction in a circular conducting wire of radius \( R \) on the \( x-y \) plane is given by
\[ \vec{B}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}. \]  
(5.188)

so that for \( z >> R \)
\[ \vec{B}(z) \simeq \frac{\mu_0 IR^2}{2z^3} \hat{\mathbf{z}} \]  
(5.189)

which agrees with the result in part \( b \).
Part II

Electricity and Magnetism
II (PHYS 4330)
Chapter 6

Magnetic Fields in Matter

6.1 Magnetostatic review

*Magnetic fields:* Two conducting wires carrying a current in the same direction attract each other while those carrying in the opposite direction repel each other.

*Direction of magnetic field and magnetic force-Right-hand rule*

*Magnetic forces:* A charge, $Q$, moving with a velocity, $\vec{V}$, in a uniform magnetic field experiences a magnetic force, $\vec{F}_{\text{mag}}$, given by

$$\vec{F}_{\text{mag}} = Q(\vec{V} \times \vec{B}).$$  \hfill (6.1)

*Magnetic and electrical force:* A charge, $Q$, moving with a velocity, $\vec{V}$, in a uniform magnetic field and electric field experiences a magnetic force and electric force given by

$$\vec{F}_{\text{mag}} = Q(\vec{E} + \vec{V} \times \vec{B}) .$$  \hfill (6.2)

*Cyclotron motion:* When the charged particle has a velocity normal to the magnetic field, the particles moves in a circular orbit. The radius of the circle is determined by using Newton’s second law for circular motion.

$$\frac{m v^2}{R} = qvB$$  \hfill (6.3)

*Helical motion:* When the charged particle travels with a velocity that has components parallel and normal to the magnetic field, the charged particle moves in a helix.

*Surface current density, $\vec{K}$:* The surface current density is defined as

$$\vec{K} = \frac{d\vec{I}}{dl_{\perp}} = \sigma \vec{v}.$$  \hfill (6.4)

*Volume current density, $\vec{J}$:* 

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}} = \rho \vec{v},$$  \hfill (6.5)
where \( \rho \) is the volume charge density. The magnetic force

\[
\vec{F}_{\text{mag}} = \int \left( \vec{J} \times \vec{B} \right) d\tau.
\]  

(6.6)

The Bio-Savart law: The magnetic field of a steady current is given

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau'.
\]  

(6.7)

where \( \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \) is the magnetic permeability of free space. The SI unit for a magnetic field is Tesla (T).

\[
T = \frac{N}{(A.m)}.
\]  

(6.8)

Ampère’s Law:

\[
\oint \vec{B}.d\vec{l} = \mu_0 I_{\text{enc}} \Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}.
\]  

(6.9)

The divergence of a magnetic:

\[
\nabla \cdot \vec{B} = 0.
\]  

(6.10)

The vector potential: Like the electric field that we expressed in terms of the electric potential \( \vec{E} = -\nabla V \) we can also express the magnetic field in terms of the vector potential \( \vec{A} \). It is expressed as

\[
\vec{B} = \nabla \times \vec{A}.
\]  

(6.11)

Ampère’s Law \& terms of the vector:

\[
\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}.
\]  

(6.12)

Comparing this with the Poisson’s equation

\[
\nabla^2 V = -\frac{\rho}{\epsilon_0}
\]  

(6.13)

with the corresponding solution

\[
V = \frac{1}{4\pi \epsilon_0} \int_{\text{vol}} \frac{\rho(r')}{|\vec{r} - \vec{r}'|} d\tau'
\]  

(6.14)

we can write the solution to Eq. (5.113) as

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_{\text{vol}} \frac{\vec{J}(r')}{|\vec{r} - \vec{r}'|} d\tau'.
\]  

(6.15)

For surface and line current density we can write

\[
\vec{A} = \frac{\mu_0}{4\pi} \int_{\text{sur}} \frac{\vec{K}(r')}{|\vec{r} - \vec{r}'|} dd'.
\]  

(6.16)
\[ \vec{A} = \frac{\mu_0}{4\pi} \int_{\text{line}} \frac{I(r')}{|\vec{r} - \vec{r}'|} d\ell'. \] (6.17)

**Boundary conditions:**

The normal component
\[ \nabla \cdot \vec{B} = 0 \Rightarrow \int \nabla \cdot \vec{B} d\tau = \int \nabla \cdot \vec{B} \cdot d\vec{a} = 0 \]
\[ \Rightarrow B_{\text{above}}^\perp A - B_{\text{below}}^\perp A = 0 \Rightarrow B_{\text{above}}^\perp = B_{\text{below}}^\perp \] (6.18)

The tangential component
\[ \oint \vec{B} \cdot d\ell = \mu_0 I_{\text{enclosed}} \Rightarrow B_{\text{above}}^\parallel l - B_{\text{below}}^\parallel l = \mu_0 K l \]
\[ \Rightarrow B_{\text{above}}^\parallel - B_{\text{below}}^\parallel = \mu_0 K \] (6.19)

**Generally the magnetic field**
\[ \vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 \left( \vec{K} \times \vec{n} \right) \] (6.20)

**In terms of the vector potential**
\[ \vec{A}_{\text{above}} = \vec{A}_{\text{below}}, \quad \frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K} \] (6.21)

**Multipole Expansion:**
\[ \vec{A} = \frac{\mu_0 I}{4\pi} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int r' P_l(\cos \theta') d\ell' \] (6.22)

**Magnetic dipole:** the vector potential of a dipole
\[ \vec{A}_{\text{dip}} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}, \] (6.23)

where
\[ \vec{m} = I \int d\vec{a}' = I \hat{a} \] (6.24)

is the magnetic dipole moment.

**Magnetic field of a dipole:**
\[ \vec{B}(r, \theta) = \frac{\mu_0 m}{4\pi r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3 (\vec{m} \cdot \hat{r}) \hat{r} - \vec{m} \right] \] (6.25)

we recall for an electric dipole:
\[ \vec{E}(r, \theta) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r^3} \left[ 3 (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p} \right] \] (6.26)
6.2 Magnetic dipole in a magnetic field

Due to the motion of the electron around the nucleus or spinning of the electrons there is a tiny current loops. In the macroscopic scale these tiny current loops form a magnetic dipole. In the absence of external field, these dipole moments are randomly oriented and the net dipole moment is zero. However, if external magnetic field is applied all the dipole moments align either parallel or antiparallel to the direction of the external magnetic field and we say the material is magnetized.

- **Paramagnets**: parallel to the external magnetic field
- **Diamagnets**: antiparallel to the external magnetic field
- **Ferromagnets**: keep their magnetization even after the removal of the external magnetic field

**Torques and Forces on Magnetic Dipoles:** A magnetic dipole experiences a torque in a magnetic field.

Consider a rectangular current loop with a magnetic dipole moment, \( \mathbf{m} \) pointing in the direction shown below (it makes an angle \( \theta \) from the \( z \)-axis toward the positive \( y \)-axis). It is placed in a uniform magnetic field \( \mathbf{B} \) pointing along the positive \( z \)-axis. We are interested in the net force and the torque on this loop. The force on the two slopping sides (i.e. the sides with length \( a \)) cancel each other. This can be verified using the right hand rule. The same is true for the other two sides (side with length \( b \)) and the net force is zero on the loop. However, the torque is not zero since the two forces on the none slopping sides (sides with length \( b \)) tends to rotate the loop in the counterclockwise direction since they do not have the same line of action (see Fig. 6.2).

The force on the upper side of the loop (side with length \( b \)) is

\[
\mathbf{F}_1 = -F\mathbf{\hat{y}} = -IB\mathbf{\hat{y}}
\]

and the corresponding position vector for center of this side (i.e. its center)

\[
\mathbf{r}_1 = -\frac{a}{2} \cos \theta \mathbf{\hat{y}} + \frac{a}{2} \sin \theta \mathbf{\hat{z}}.
\]
Similarly for the bottom side, the force is

\[ \vec{F}_2 = F\hat{y} = Ib\hat{y} \quad (6.29) \]

and the position vector for its center is

\[ \vec{r}_2 = \frac{a}{2} \cos \theta \hat{y} - \frac{a}{2} \sin \theta \hat{z}. \quad (6.30) \]

Then the net torque on the loop can be expressed as

\[ \vec{N} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 \]

\[ \Rightarrow \vec{N} = \left( -\frac{a}{2} \cos \theta \hat{y} + \frac{a}{2} \sin \theta \hat{z} \right) \times (-Ib\hat{y}) + \left( \frac{a}{2} \cos \theta \hat{y} - \frac{a}{2} \sin \theta \hat{z} \right) \times (Ib\hat{y}) \]

\[ \Rightarrow \vec{N} = IabB \sin \theta \hat{x} \quad (6.31) \]

For a rectangular loop of sides \( a \) and \( b \) and carrying a current \( I \), we recall that the magnitude of the magnetic dipole moment, \( \vec{m} \), is given by

\[ m = IA = Iab. \quad (6.32) \]
so that the net torque becomes
\[ \vec{N} = mB \sin \theta \hat{\lambda} \Rightarrow \vec{N} = \vec{m} \times \vec{B} \] (6.33)

The torque causes a counterclockwise rotation about the x-axis. This results in a parallel alignment of the dipole moment and the external magnetic field. A material that shows this kind of response to external magnetic field is known as paramagnetic material. Paramagnetism occurs only in atoms or molecules with odd number of electrons. If the atoms or molecules are made of even number of electrons, according to the Pauli exclusion principle (as you recall from your modern physics class) the electrons form Pair of electrons, due to Pauli’s exclusion principle, occupying the same energy level must carry opposite spin which results in opposite magnetic dipole moments. As a result, the net magnetic moment is zero that lead to zero torque.

In a uniform field the net force on a current loop is zero:
\[ \vec{F} = I \oint d\vec{l} \times \vec{B} = I \left( \oint d\vec{l} \right) \times \vec{B} = 0. \] (6.34)

For a none uniform field, however, the net force is not zero. Consider a circular loop placed near the top end of a solenoid shown in Fig. 6.1

![Figure 6.1: A circular current carrying wire at one end of a solenoid centered about the axis of the solenoid.](image)

Using cylindrical coordinates we may express the magnetic field of the solenoid near top end of the solenoid where the dipole is positioned at as
\[ \vec{B}(s, z) = B \sin \theta \hat{z} + B \cos \theta \hat{s}. \] (6.35)

For an infinitesimal length, \(d\vec{l}\), on the dipole (circular loop of radius \(R\) carrying a current \(I\) in a counterclockwise direction), we have
\[ d\vec{l} = R \hat{\phi} d\varphi \] (6.36)
and the net force on the dipole can be determined using

\[ \mathbf{F} = I \oint d\mathbf{l} \times \mathbf{B} = I \left( \int_0^{2\pi} R \phi d\phi \right) (B \sin \theta \hat{z} + B \cos \theta \hat{s}) \]

\[ \Rightarrow \mathbf{F} = I R B \left( \sin \theta \int_0^{2\pi} \hat{\phi} \times \hat{z} d\phi + \cos \theta \int_0^{2\pi} \hat{\phi} \times \hat{s} d\phi \right). \tag{6.37} \]

Noting that

\[ \hat{\phi} \times \hat{z} = \hat{s} = \cos \phi \hat{x} + \cos \phi \hat{y}, \hat{\phi} \times \hat{s} = -\hat{z} \tag{6.38} \]

the net force is found to be

\[ \mathbf{F} = -2\pi I R B \cos \theta \hat{\phi}. \tag{6.39} \]

For an infinitesimal loop, with dipole moment \( \mathbf{m} \), in a field \( \mathbf{B} \), the force is

\[ \mathbf{F} = \nabla \left( \mathbf{m} \cdot \mathbf{B} \right). \tag{6.40} \]

**Example 6.1** Calculate the torque exerted on the square loop shown below due to the circular loop (assume \( r \) is much larger than \( a \) and \( b \)). If the square loop is free to rotate, what will its equilibrium orientation be?

**Solution:** In order to find the torque on the rectangular loop first we need to find the magnetic field of a circular loop at a distance far away (i.e. \( r \) is much bigger). The magnetic field in such case is predominantly due to the dipole contribution which is given by

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3(\mathbf{m} \cdot \hat{r}) \hat{r} - \mathbf{m} \right]. \tag{6.41} \]

Let the positive z-axis be along the direction of the magnetic dipole moment of the circular loop. Then magnetic moment of the rectangular loop
is going to be along the positive y-axis. We also note that for $r \gg b$, we can assume the magnetic field on each sides of the square loop is uniform and can be expressed as

$$\vec{B} (\vec{r}) \simeq -\frac{\mu_0}{4\pi} \frac{1}{r^2} \vec{m}_1 = -\frac{\mu_0}{4\pi} \frac{m_1}{r^3} \hat{z} = -\frac{\mu_0}{4\pi} \frac{I\pi a^2}{r^3} \hat{z} \Rightarrow \vec{B} (\vec{r}) = -\frac{\mu_0 I a^2}{4\pi r^3} \hat{z},$$

(6.42)

where $\vec{m}_1$ is the magnetic dipole moment of the circular loop and we have used

$$\vec{m}_1 \cdot \hat{r} \simeq 0,$$

(6.43)

The magnetic dipole moment of the square loop, $\vec{m}_2$, is given by

$$\vec{m}_2 = m_2 \hat{y} = Ib^2 \hat{y}.$$  

(6.44)

Then using the torque exerted on a magnetic dipole placed in a uniform magnetic field, we have

$$\vec{N} = \vec{m}_2 \times \vec{B} = (Ib^2 \hat{y}) \times \left( -\frac{\mu_0 I b^2 a^2}{4\pi r^3} \hat{z} \right) = -\frac{\mu_0 (Iab)^2}{4\pi r^3} \hat{y} \times \hat{z}$$

$$\Rightarrow \vec{N} = -\frac{\mu_0 (Iab)^2}{4br^3} \hat{x}$$

(6.45)

A torque in this direction results in rotation of the square loop towards the negative z-axis so that dipole moment of the square loop would be parallel to the direction of the dipole field at the location of the square loop.

### 6.3 Effect of a magnetic field on atomic orbitals:

We know that electrons rotates around the nucleus. Let’s consider the Bohr’s model for a hydrogen atom where the electron rotates with a speed $v$ in a circular orbit of radius, $R$, as shown in Fig. 6.2. Then the period $T$ can be expressed as

$$T = \frac{2\pi R}{v}$$

(6.46)

and the magnitude of the current generated due to the orbital motion of the electron can be expressed as

$$I = \frac{e}{T} = \frac{ev}{2\pi R}.$$  

(6.47)

Due to this current the atom possess orbital magnetic dipole moment given by

$$\vec{m} = -I\pi R^2 \hat{z} = -\frac{evR}{2} \hat{z}.$$  

(6.48)

The negative sign is due to the negative charge of the electron. This shows that any change in the speed of the electron leads to a change in orbital magnetic
dipole moment. How does the speed change? It changes when an external force is applied on the electron. If the only force acting on the electron is the coulombic force of attraction by the nucleus, then speed of the electron is given by

\[
\frac{m_e v^2}{R} = \frac{e^2}{4\pi\epsilon_0 R^2}. \tag{6.49}
\]

However, if the atom is placed in a uniform magnetic field, \( \vec{B} \) (see Fig.6.3), the magnitude of the net force acting on the electron becomes

\[
F_{\text{net}} = \frac{e^2}{4\pi\epsilon_0 R^2} + e\vec{v}B \tag{6.50}
\]

when \( \vec{v} \) is the speed of the electron in the presence of the magnetic force. Then the equation of motion under this condition can be written as

\[
F_{\text{net}} = \frac{e^2}{4\pi\epsilon_0 R^2} + e\vec{v}B = m_e \frac{\vec{v}^2}{R} \tag{6.51}
\]

Using Eq. (6.49), we find

\[
\frac{m_e v^2}{R} + e\vec{v}B = \frac{m_e \vec{v}^2}{R} \Rightarrow e\vec{v}B = \frac{m_e \vec{v}^2}{R} (\vec{v}^2 - v^2) \\
\Rightarrow e\vec{v}B = \frac{m_e}{R} (\vec{v} - v) (\vec{v} + v) = \frac{m_e}{R} \Delta v (\vec{v} + v) \tag{6.52}
\]

The change, \( \Delta v = \vec{v} - v \), is very small and we may replace \( \vec{v} \) in the left side by \( v \) and \( \vec{v} + v \) in the right side by \( 2v \) so that change in the speed of the electron
is given by
\[ \Delta v = \bar{v} - v = \frac{eBR}{2m_e}. \]  

When we substitute Eq. (6.53) into Eq. (6.48), we find the change in magnetic moment
\[ \Delta \bar{m} = -\frac{eR}{2} \hat{z} \Rightarrow \Delta \bar{m} = -\frac{eR}{2} \Delta \bar{v} \hat{z} \Rightarrow \Delta \bar{m} = -\left( \frac{e^2R^2}{4m_e} \right) B \hat{z}. \]  

**Note:** The change in magnetic moment ($\Delta \bar{m}$) is negative. Noting that $\Delta \bar{m}$ in terms of the initial ($\bar{m}_i$) and final ($\bar{m}_f$) magnetic moment
\[ \Delta \bar{m} = \bar{m}_f - \bar{m}_i \Rightarrow \bar{m}_f = \bar{m}_i + \Delta \bar{m} = -\left( \frac{evR}{2} + \frac{e^2R^2}{4m_e} B \right) \hat{z}. \]  

This means that dipole moment is antiparallel to the external magnetic field which is in the $+z$ direction (diamagnetism!). It is a universal property which exists in atoms or molecules but it is very weak and it can only be detected in a non-paramagnetic materials. On the other hand paramagnetism which is the result of the spin motion of the electron, it exist only in atoms with odd number of electrons.

**Magnetization ($\bar{M}$):** magnetic dipole moment per unit volume.

### 6.4 The field of a magnetized object

We recall that for a magnetic dipole with a magnetic moment, $\bar{m}$, the vector potential is given by
\[ \bar{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \]  

Figure 6.3: An electron rotating around the nucleus (Bohr model) in the presence of a uniform magnetic field.
A magnetized material with a magnetization $\mathbf{M}$ (total magnetic dipole moment per unit volume) in an infinitesimal volume $d\tau'$ will have a total magnetic dipole moment of $\mathbf{M} d\tau'$. The contribution to the vector potential by these magnetic dipoles can be expressed as

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{M} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'.$$

(6.57)

Then the vector potential of a magnetized object with a volume $V$ and magnetization $\mathbf{M}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{M} \times \left( \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\tau'.
$$

(6.58)

Using the relation

$$\nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3},$$

(6.59)

Eq. (6.58) can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{M} \times \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) d\tau'.
$$

(6.60)

Applying the relation

$$\nabla \times \left( f \mathbf{C} \right) = f \left( \nabla \times \mathbf{C} \right) - \mathbf{C} \times \left( \nabla f \right),$$

(6.61)

we have

$$\mathbf{M} \times \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) = \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) \left( \nabla' \times \mathbf{M} \right) - \nabla' \times \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{M} \right),$$

(6.62)
so that Eq. (6.60) can be expressed as

$$A(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \int_\mathcal{V} \nabla' \times \mathbf{M}(\mathbf{r'}) \, d\mathbf{r'} - \int_\mathcal{V} \nabla' \times \left( \frac{\mathbf{M}}{|\mathbf{r'} - \mathbf{r}|} \right) \, d\mathbf{r'} \right]. \quad (6.63)$$

**Exercise:** Show that

$$\int_\mathcal{V} \nabla' \times \mathbf{B} \, d\mathbf{r'} = - \oint_\mathcal{S} \mathbf{B} \times d\mathbf{a} \quad (6.64)$$

If we apply the relation in Eq. (6.64), we can express Eq. (6.63) as

$$A(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \int_\mathcal{V} \nabla' \times \mathbf{M}(\mathbf{r'}) \, d\mathbf{r'} + \oint_{\mathcal{S}} \frac{1}{|\mathbf{r'} - \mathbf{r}|} \mathbf{M}(\mathbf{r'}) \times d\mathbf{a} \right]. \quad (6.65)$$

or

$$A(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \int_\mathcal{V} \mathbf{J}_b(\mathbf{r'}) \, d\mathbf{r'} + \oint_{\mathcal{S}} \mathbf{K}_b(\mathbf{r'}) \, d\mathbf{a} \right]. \quad (6.66)$$

where

$$\mathbf{J}_b(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}), \quad (6.67)$$

is called the volume bound current density and

$$\mathbf{K}_b(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{n}. \quad (6.68)$$

is the bound surface current density.

**Example 6.2** Find the magnetic field of a uniformly magnetized sphere of radius $R$ and magnetization, $\mathbf{M} = M \hat{z}$.  

**Solution:** To find the magnetic field we first need to find the vector potential due to the bound currents. Since the material has a uniform magnetization the volume current density is zero

$$\mathbf{J}_b(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) = 0.$$  

The magnetization $\mathbf{M}$ pointing along the $z$ direction, in spherical coordinates (Fig. 6.4, can be expressed as

$$\mathbf{M} = M \hat{z} = M \cos(\theta) \hat{r} - M \sin(\theta) \hat{\theta}$$

and the normal unit vector to the area is

$$\hat{n} = \hat{r}. \quad (6.69)$$

Then the surface current

$$\mathbf{K}_b(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{n} = \left( M \cos(\theta) \hat{r} - M \sin(\theta) \hat{\theta} \right) \times \hat{r}$$

$$\Rightarrow \mathbf{K}_b(\mathbf{r}) = M \sin(\theta) \hat{\phi}. \quad (6.70)$$
We recall from Example 5.11 the vector potential for a spherical shell of radius, $R$, with surface charge density, $\sigma$, and spinning about the $z$ axis with angular velocity $\omega$, generates a surface current given by

$$\vec{K} = \sigma \vec{v} = \sigma R \omega \sin (\theta) \hat{\varphi}$$  \hspace{1cm} (6.71)

which lead us to a vector potential given by

$$\vec{A} = \begin{cases} \frac{\mu_0 \sigma R}{2} \omega r \sin (\theta) \hat{\varphi} & r < R \\ \frac{\mu_0 \sigma R^4}{3 \omega r^2} \omega \sin (\theta) \hat{\varphi} & r > R \end{cases}$$  \hspace{1cm} (6.72)

Comparing Eq. (6.70) with (6.71), we have $\sigma R \omega = M$ and the vector
Potential in Eq. (6.72) becomes
\[
\vec{A} = \begin{cases} 
\frac{\mu_0 M}{3} r \sin (\theta) \hat{\phi} & r < R \\
\frac{\mu_0 M r^3}{3 \pi} \sin (\theta) \hat{\phi} & r > R
\end{cases}
\]  
(6.73)

Then using the expression for the magnetic field in terms of the vector potential in spherical coordinates
\[
\vec{B}(\vec{r}) = \nabla \times A = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial}{\partial \phi} (A_\theta) \right] \hat{r} \\
+ \frac{1}{r} \left[ \frac{\partial}{\partial \theta} (A_r) - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial}{\partial \theta} (A_r) \right] \hat{\phi},
\]  
(6.74)
inside the sphere \((r < R)\), we find
\[
\vec{B}(\vec{r}) = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\mu_0 M}{3} r \sin (\theta) \right) \right] \hat{r} + \frac{1}{r} \left[ - \frac{\partial}{\partial r} \left( r \frac{\mu_0 M}{3} r \sin (\theta) \right) \right] \hat{\theta} \\
= \frac{2 \mu_0 M}{3} \left[ \cos (\theta) \hat{r} - \sin (\theta) \hat{\theta} \right].
\]  
(6.75)

Similarly, outside the sphere \((r > R)\)
\[
\vec{B}(\vec{r}) = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\mu_0 M R^3}{3 r^2} \sin (\theta) \right) \right] \hat{r} + \frac{1}{r} \left[ - \frac{\partial}{\partial r} \left( r \frac{\mu_0 M R^3}{3 r^2} \sin (\theta) \right) \right] \hat{\theta} \\
\Rightarrow \vec{B}(\vec{r}) = \frac{2 \mu_0 M R^3}{3 r^3} \cos (\theta) \hat{r} + \frac{\mu_0 M R^3}{3 r^3} \sin (\theta) \hat{\theta}.
\]  
(6.76)

In terms of the Magnetization vector
\[
\vec{M} = M \hat{z} = M \cos (\theta) \hat{r} - M \sin (\theta) \hat{\theta} \Rightarrow M \sin (\theta) \hat{\theta} = M \cos (\theta) \hat{r} - \vec{M},
\]  
(6.77)
one can rewrite the magnetic field, inside the sphere, as
\[
\vec{B}(\vec{r}) = \frac{2 \mu_0 \vec{M}}{3}
\]  
(6.78)
and outside the sphere
\[
\vec{B}(\vec{r}) = \frac{2 \mu_0 M R^3}{3 r^3} \cos (\theta) \hat{r} + \frac{\mu_0 R^3}{3 r^3} \left( M \cos (\theta) \hat{r} - \vec{M} \right) \\
= \frac{3 \mu_0 M R^3 \cos (\theta) \hat{r} - \mu_0 R^3 \vec{M}}{3 r^3} \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 R^3 \left( 3 \vec{M} \hat{r} \right) \hat{r} - \vec{M}}{3 r^3}.
\]  
(6.79)

Noting that for a sphere of radius \(R\) with a uniform magnetization \(\vec{M}\), in terms of the total magnetic dipole moment, \(\vec{m}_{\text{total}}\),
\[
M = \frac{\vec{m}_{\text{total}}}{\frac{4}{3} \pi R^3} \Rightarrow \vec{M} = \frac{\vec{m}_{\text{total}}}{\frac{4}{3} \pi R^3}.
\]  
(6.80)
one can write the magnetic field outside the sphere as

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3} \left[ (3\vec{m}_{\text{total}} \cdot \hat{r}) \hat{r} - \vec{m}_{\text{total}} \right].
\]

(6.81)

**Example 6.3** A long circular cylinder of radius \( R \) carries a magnetization \( \vec{M} = ks^2\hat{\varphi} \), where \( \hat{k} \) is a constant, \( s \) is the distance from the axis, and \( \hat{\varphi} \) is the usual azimuthal unit vector. Find the magnetic field due to \( \vec{M} \), for points inside and outside the cylinder.

[Figure 6.5: A long magnetized cylinder with magnetization, \( \vec{M} = ks^2\hat{\varphi} \).]

**Solution:** The magnetization is not constant and therefore unlike the previous example the volume current is not zero. It is given by

\[
\vec{J}_b(\vec{r}) = \nabla \times \vec{M}(\vec{r})
\]

(6.82)
in cylindrical coordinates, the curl is given by

\[
\nabla \times \vec{M} = \left[ \frac{1}{s} \frac{\partial}{\partial \varphi} M_z - \frac{\partial}{\partial z} M_s \right] \hat{s} + \left[ \frac{1}{s} \frac{\partial}{\partial z} M_z - \frac{\partial}{\partial s} M_s \right] \hat{\varphi} + \frac{1}{s} \frac{\partial}{\partial s} \left[ \frac{\partial}{\partial \varphi} M_s - \frac{\partial}{\partial \varphi} M_s \right] \hat{z}.
\]

(6.83)

Using

\[
\vec{M} = ks^2\hat{\varphi} \Rightarrow M_z = M_s = 0, M_\varphi = ks^2,
\]

(6.84)

we find

\[
\vec{J}_b(\vec{r}) = -\frac{\partial}{\partial z} \hat{s} + \frac{1}{s} \frac{\partial}{\partial s} \hat{\varphi} + \frac{1}{s} \frac{\partial}{\partial s} \hat{z} = \frac{1}{s} \frac{\partial}{\partial s} (ks^3) \hat{z}
\]

\[
\Rightarrow \vec{J}_b(\vec{r}) = 3ks \hat{z}
\]

(6.85)

(6.86)
and the bound surface current

\[ \vec{K}_b(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n} \quad (6.87) \]

becomes

\[ \vec{K}_b(\vec{r}) = kR^2 \hat{\phi} \times \hat{s} \Rightarrow \vec{K}_b(\vec{r}) = -kR^2 \hat{z}. \quad (6.88) \]

The magnetic field can then be easily obtained if we apply Ampere’s law (Fig.6.4)

\[ \oint \vec{B}.d\vec{l} = \mu_0 I_{\text{enc}}. \quad (6.89) \]

Inside the cylinder, the total current enclosed by an Amperian loop of radius \( s \) is given by

\[ I_{\text{enc}} = \int_0^s J_b(s) da = \int_0^s 3ks (2\pi s ds) = 2\pi ks^3. \quad (6.90) \]

Then one finds for the magnetic field

\[ \oint \vec{B}.d\vec{l} = B2\pi s = 2\pi ks^3 \Rightarrow \vec{B} = ks^2 \hat{\phi}. \quad (6.91) \]

On the other hand outside the cylinder the total current enclosed by an Amperian loop of radius \( s > R \) consists of the volume current and the surface current that flows in opposite directions. Thus for the total current,
we have

\[ I_{\text{enc}} = \int_0^R J_b(s) \, ds - \int_0^{2\pi R} K(R) \, dl = \int_0^R 3ks \, (2\pi s \, ds) - \int_0^{2\pi R} kR^2 \, dl \]

\[ \Rightarrow I_{\text{enc}} = 2\pi kR^3 - 2\pi kR^3 = 0 \]  

so that the magnetic field outside the cylinder becomes

\[ \oint \vec{B} \cdot d\vec{l} = B2\pi s = 0 \Rightarrow B = 0. \]  

6.5 Bound currents-physical interpretation

*Uniform Magnetization:* Consider a material with uniform magnetization, \( \vec{M} = M \hat{z} \), that has thickness \( t \) as shown in the figure. The magnetization is the sum of tiny magnetic dipole moments divided by the volume of the material. Each tiny magnetic dipole moments can be visualized as a small rectangular loops carrying a current \( I \) in the counter clockwise direction as shown in Fig. 6.5.

On adjacent sides of two loops the direction of the current is opposite, as we can see from the figure above, all internal currents cancel each other resulting in a surface current shown below.
Consider a portion of the material that has an area $a$ and thickness $t$, the magnetic dipole moment inside this volume can be expressed as $m_{total} = Mat$. We also know that the magnetic dipole moment for a rectangular loop of area $a$ carrying a current $I$ is, $m_{total} = Ia$. (Fig. 6.5) Then we can write

\[ Ia = Mat \Rightarrow I = Mt. \quad (6.94) \]

The magnitude of the surface current density will then be

\[ K = \frac{I}{t} = M \quad (6.95) \]

Taking into consideration the direction of the surface current, the magnetization, and the unit vector normal to the surface, we can write

\[ \vec{K} = \vec{M} \times \hat{n} \quad (6.96) \]

**Nonuniform Magnetization:** For a nonuniform magnetization the internal currents will no longer cancel out. For the two adjacent sides of the loop (see figure below) there exists a current along the positive x-direction. This current, applying the result in Eq. (6.94), can write

\[ L_{x+} = [M_z (y + dy) - M_z (y)] dz = \frac{\partial M_z}{\partial y} dy dz \Rightarrow (J_b)_{x+} = \frac{\partial M_z}{\partial y} \quad (6.97) \]
6.6.  THE AUXILIARY FIELD, $\vec{H}$

Similarly for the loops shown in the figure below there is current flowing in the negative x-direction. This current can be expressed as

$$ I_{x-} = - [M_y (z + dz) - M_y (z)] dy = - \frac{\partial M_y}{\partial z} dz dy \Rightarrow (J_b)_{x-} = - \frac{\partial M_y}{\partial z} \quad (6.98) $$

Therefore, the net volume current density along the x-direction will be

$$ (J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \quad (6.99) $$

Following a similar procedure one can easily show that

$$ (J_b)_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}, (J_b)_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \quad (6.100) $$

Using vector notation one can write

$$ \vec{J}_b = (J_b)_x \hat{x} + (J_b)_y \hat{y} + (J_b)_z \hat{z} = \left( \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \right) \hat{x} + \left( \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} \right) \hat{y} + \left( \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right) \hat{z} \Rightarrow \vec{J}_b = \nabla \times \vec{M} \quad (6.101) $$

6.6  The Auxiliary Field, $\vec{H}$

We recall from chapter 5 the differential form of Ampere’s law

$$ \nabla \times \vec{B} = \mu_0 \vec{J}, \quad (6.102) $$

where $\vec{J}$ was being taken as the free volume current density. However, in this chapter we have seen that whenever there is none uniformly magnetized material, there would be an additional volume current described by the bound volume current density, $\vec{J}_b$. Therefore, Ampere’s law in more general form should be written as

$$ \nabla \times \vec{B} = \mu_0 \left( \vec{J}_f + \vec{J}_b \right) \quad (6.103) $$
Using Eq. (6.101), we may write
\[ \nabla \times \vec{B} = \mu_0 \left( \vec{J}_f + \nabla \times \vec{M} \right) = \nabla \times \left( \frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_f \]  
(6.104)
or
\[ \nabla \times \vec{H} = \vec{J}_f, \]  
(6.105)
where
\[ \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}. \]  
(6.106)
is called the Auxiliary field. The integral form of Ampere’s law can then be expressed as
\[ \oint \nabla \times \vec{H} \cdot d\vec{a} = \oint \vec{J}_f \cdot d\vec{a} \Rightarrow \oint \vec{H} \cdot d\vec{l} = I_{\text{free}}. \]  
(6.107)
Note that \( \vec{H} \) plays a role in magnetostatic analogous to \( \vec{D} \) in electrostatic.

**Example 6.4** A long copper rod of radius, \( R \), carries a uniformly distributed (free) current \( I \) (Fig. below). Find \( \vec{H} \) inside and outside the rod.

**Solution:** Using Ampere’s law, for the Amperian loop-1, the magnetic field, \( \vec{H} \), inside the rod is given by
\[ \oint \vec{H} \cdot d\vec{l} = I_{\text{free}} \Rightarrow H 2\pi s = \frac{I}{\pi R^2} \pi s^2 \Rightarrow \vec{H} = \frac{Is}{2\pi R^2} \hat{\varphi} \]  
(6.108)
and outside the rod (using Amperian loop-2)
\[ \oint \vec{H} \cdot d\vec{l} = I_{\text{free}} \Rightarrow H 2\pi s = \frac{I}{\pi R^2} \pi R^2 \Rightarrow \vec{H} = \frac{I}{2\pi s} \hat{\varphi}. \]  
(6.109)
Example 6.5 An infinitely long cylinder, of radius $R$, carries a "frozen-in" magnetization parallel to the axis

$$\vec{M} = ks\hat{z},$$  \hspace{1cm} (6.110)

where $k$ is a constant and $s$ is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:

(a) Locate all the bound currents, and calculate the field they produce.

(b) Use Ampere's law (involving $\vec{H}$) to find $\vec{H}$, and then find $\vec{B}$ from Eq. (6.106)

Solution:

(a) The bound volume current density is given by

$$\vec{J}_b = \nabla \times \vec{M}$$  \hspace{1cm} (6.111)

$$\nabla \times \vec{M} = \left[ \frac{1}{s} \frac{\partial M_x}{\partial \varphi} \ - \ \frac{\partial M_x}{\partial z} \right] \hat{s} + \left[ \frac{1}{s} \frac{\partial M_x}{\partial r} \ - \ \frac{\partial M_z}{\partial s} \right] \hat{\varphi}$$

$$+ \frac{1}{s} \left[ \frac{\partial (sM_z)}{\partial s} \ - \ \frac{\partial M_s}{\partial \varphi} \right] \hat{z} = - \frac{\partial (ks)}{\partial s} \hat{\varphi} \Rightarrow \vec{J}_b = \nabla \times \vec{M} = -k\hat{\varphi}$$  \hspace{1cm} (6.112)

and the bound surface current density

$$\vec{K}_b = \vec{M} \times \hat{n} \Rightarrow \vec{K}_b = (kR\hat{z}) \times \hat{s} = kR\hat{\varphi}.$$  \hspace{1cm} (6.113)
First, let’s find the magnetic field outside. We consider a rectangular loop of width \(2s > 2R\) and length \(l\). The total current enclosed by this loop will then be

\[
I = \int J_b da + \int K_b dl \Rightarrow I = 0.
\]  

(6.114)

Therefore, since the current is zero the magnetic field outside becomes zero. However, inside the cylinder the magnetic field is not zero. To show this we consider a rectangular loop of width \(w\) and length \(l\) at a distance \(s < R\). The total enclosed current will be

\[
I = \int J_b da + \int K_b dl = -k(R-s)l + kRl = ksl
\]  

(6.115)

\[
\int \vec{B} \cdot d\vec{r} = \mu_0 I \Rightarrow Bl = \mu_0 ksl \Rightarrow \vec{B} = \mu_0 ksl \hat{z}
\]  

(6.116)

(b) Ampere’s law in terms of \(\vec{H}\) is given by

\[
\oint \vec{H} \cdot d\vec{r} = I_{free}.
\]  

(6.117)

Whether we are inside or outside the cylinder, we do not have any free current. Hence, we find

\[
\oint \vec{H} \cdot d\vec{r} = 0 \Rightarrow \vec{H} = 0
\]  

(6.118)
6.7. **BOUNDARY CONDITIONS**

in all regions, which means

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \Rightarrow \vec{B} = \mu_0 \vec{M}. \quad (6.119)$$

Inside the cylinder, the magnetization is $\vec{M} = ks\hat{z}$ and outside $\vec{M} = 0$. Then the magnetic field becomes

$$\vec{B} = \begin{cases} 
0 & \text{outside} \\
\mu_0 ks\hat{z} & \text{inside}
\end{cases} \quad (6.120)$$

### 6.7 Boundary conditions

Taking the divergence of Eq.(6.106), we have

$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \nabla \cdot \vec{B} - \nabla \cdot \vec{M} \quad (6.121)$$

and recalling that $\nabla \cdot \vec{B} = 0$, we find

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \Rightarrow \vec{H}_{\text{above}}^+ - \vec{H}_{\text{below}}^- = -(\vec{M}_{\text{above}} - \vec{M}_{\text{below}}) \quad (6.122)$$

From Eq. (6.105)) we can show that

$$\nabla \times \vec{H} = \vec{J}_f \Rightarrow \vec{H}_{\text{above}}^\parallel - \vec{H}_{\text{below}}^\parallel = \vec{K}_f \times \hat{n} \quad (6.123)$$

### 6.8 Linear and nonlinear Media

*Magnetic Susceptibility and Permeability:* When an external magnetic field is applied to a given material the magnetic dipole moments align themselves parallel (paramagnets) or antiparallel (diamagnet) to the external field. This would give rise to a net magnetization to the material. If the strength of the magnetization is proportional to the external magnetic field, the media is known as linear media. The proportionality is expressible as

$$\vec{M} = \chi_m \vec{H}, \quad (6.124)$$

where $\chi_m$ is known as the *magnetic susceptibility which is positive for paramagnets and negative for diamagnets*. Using Eq. (6.124), we may write

$$\vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right) = \mu_0 (1 + \chi_m) \vec{H} = \mu \vec{H}, \quad (6.125)$$

$\mu$ is known as the magnetic permeability of the material. For a vacuum there is nothing being magnetized and we have $\chi_m = 0$ and $\mu = \mu_0$.

*N. B.* it is important to note that even though $\vec{M}$ and $\vec{H}$ are both proportional to $\vec{B}$ and the divergence of the magnetic field always vanishes ($\nabla \cdot \vec{B} = 0$),
the divergence of the magnetization and also the auxiliary field do not vanish
\( \nabla \cdot \vec{M} \neq 0 \)

Consider the pill box shown in the figure above. One of its side belongs to
a vacuum while the other is inside a magnetized object with magnetization, \( \vec{M} \),
this means
\[
\int \vec{M} \cdot d\vec{a} = Ma \neq 0 \Rightarrow \int \vec{M} \cdot d\vec{a} = \int \left( \nabla \cdot \vec{M} \right) d\tau \neq 0 \Rightarrow \nabla \cdot \vec{M} \neq 0 \quad (6.126)
\]

In a \textit{homogenous linear material} the bound volume current density is propor-
tional to the free current density
\[
\vec{J}_b = \nabla \times \vec{M} = \chi_m \left( \nabla \times \vec{H} \right) = \chi_m \vec{J}_f. \quad (6.127)
\]
This means that inside a linear medium if there is no free current there
is no bound volume current.

\textbf{Example 6.6} An infinite solenoid (\( n \) turns per unit length, current \( I \)) is filled
with linear material of susceptibility \( \chi_m \). Find the magnetic field inside
the solenoid.

\textbf{Solution:} To find the magnetic field we use Ampere’s law involving \( \vec{H} \),
\[
\oint \vec{H} \cdot d\vec{l} = I_{\text{free}} \Rightarrow Hl = nIl \Rightarrow \vec{H} = nI \hat{z} \quad (6.128)
\]
Then using
\[
\vec{B} = \mu_0 (1 + \chi_m) \vec{H}, \quad (6.129)
\]
the magnetic field inside the solenoid is found to
\[
\vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 (1 + \chi_m) nI \hat{z}. \quad (6.130)
\]
The magnetic field outside is zero.
Example 6.7 A coaxial cable consists of two very long cylindrical tubes separated by linear insulating material of magnetic susceptibility, $\chi_m$. A current $I$ flows down the inner conductor and returns along the outer one, in each case the current distributes itself uniformly over the surface. Find the magnetic field in the region between tubes. As a check calculate, the magnetization and the bound currents, and confirm that (together, of course, with free currents) they generate the correct field.

Solution: We consider a circular Amperian loop of radius $s$ ($a < s < b$) and use Ampere's law involving $\vec{H}$.

$$\oint \vec{H} \cdot d\vec{l} = I_{\text{free}} \Rightarrow H2\pi s = I \Rightarrow \vec{H} = \frac{I}{2\pi s} \hat{\varphi}. \quad (6.131)$$
Then the magnetic field
\[ \vec{B} = \mu_0 (1 + \chi_m) \vec{H}, \] (6.132)
becomes
\[ \vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \frac{\mu_0 I}{2\pi s} (1 + \chi_m) \hat{\varphi} \] (6.133)
To find the bound currents we need the magnetization
\[ \vec{M} = \chi_m \vec{H} = \frac{\chi_m I}{2\pi s} \hat{\varphi}. \] (6.134)
The bound volume current
\[ \vec{J}_b = \nabla \times \vec{M} \]
\[ \nabla \times \vec{M} = \left[ \frac{1}{s} \frac{\partial M_z}{\partial \varphi} - \frac{\partial M_\varphi}{\partial z} \right] \hat{s} + \left[ \frac{1}{s} \frac{\partial M_z}{\partial z} - \frac{\partial M_z}{\partial s} \right] \hat{\varphi} \]
\[ + \frac{1}{s} \left[ \frac{\partial (sM_\varphi)}{\partial s} - \frac{\partial M_s}{\partial \varphi} \right] \hat{z} = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\chi_m I}{2\pi s} \right) \hat{z} = 0 \Rightarrow \vec{J}_b = 0. \] (6.135)
The bound surface current is given by
\[ \vec{K}_b = \vec{M} \times \hat{n} \] (6.136)
Noting that the unit vector, \( \hat{n} = -\hat{s} \), for the inner surface and \( \hat{n} = \hat{s} \) for the outer surface we find
\[ \vec{K}_b = \begin{cases} 
-\frac{\chi_m I}{2\pi a} (\hat{\varphi} \times \hat{s}) = \frac{\chi_m I}{2\pi a} \hat{z}, & \text{inner} \\
\frac{\chi_m I}{2\pi b} (\hat{\varphi} \times \hat{s}) = -\frac{\chi_m I}{2\pi b} \hat{z}, & \text{outer} 
\end{cases} \] (6.137)
The total enclosed current by the Amperian loop we considered earlier is the sum of the free and bound currents.
\[ I_{\text{total}} = I_f + I_b = I + K_b (2\pi a) = I + \left( \frac{\chi_m I}{2\pi a} \right) (2\pi a) \]
\[ \Rightarrow I_{\text{total}} = (1 + \chi_m) I. \] (6.138)
Therefore, the magnetic field \( \vec{B} \) is
\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{total}} \]
\[ \Rightarrow B 2\pi s = \mu_0 (1 + \chi_m) I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} (1 + \chi_m) \hat{\varphi}, \] (6.139)
which is the same as we found earlier.
6.9 Ferromagnetism and hysteresis

Ferromagnetic materials exhibit a long-range ordering phenomenon at the atomic level which causes the unpaired electron spins to line up parallel with each other in a region called a domain. Within the domain, the magnetic field is intense, but in a bulk sample the material will usually be unmagnetized because the many domains will themselves be randomly oriented with respect to one another (see Fig. 6.6). Ferromagnetism manifests itself in the fact that a small externally imposed magnetic field, for example from a solenoid, can cause the magnetic domains to line up with each other and the material is said to be magnetized. The driving magnetic field will then be increased by a large factor which is usually expressed as a relative permeability for the material.

![Figure 6.6: Magnetic domains in Ferromagnets.](image)

**Hysteresis**: A property of a system such that an output value is not a strict function of the corresponding input, but also incorporates some lag, delay, or history dependence, and in particular when the response for a decrease in the input variable is different from the response for an increase. Ferromagnetic materials are know to exhibit magnetic Hysteresis. To understand magnetic hysteresis in let’s consider a ferromagnetic rod (e.g. an iron rod) inserted inside a solenoid (see Fig. 6.7). The solenoid is connected to an ac current source. Consider an iron core inside a solenoid as shown in the figure below. The current in the solenoid is $I(t)$ and it can be varied both in magnitude and direction to a desired maximum or minimum values. Suppose the current produced by this source is a

![Figure 6.7: A ferromagnetic rod inside a solenoid with $n$ turns per unit length connected to an current source.](image)
sinusoidal current shown in Fig. 6.8. For a long solenoid with \( n \) number of turns per unit length, the axillary field inside, is given by

\[
\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}} \Rightarrow H = nI(t) \Rightarrow \mathbf{H} = nI(t) \hat{z},
\]  

(6.140)

with the \( z \) axis along the axis of the solenoid. Then the magnetic field strength, \( \mathbf{B} \), without the ferromagnetic rod is given by

\[
\mathbf{B} = \mu_0 nI(t) \hat{z}.
\]  

(6.141)

With the ferromagnetic rod, we have

\[
\mathbf{B} = \mu_0 [M(H) + H] \hat{z}.
\]  

(6.142)

where \( M(H) \) is the magnetization of the rod.

As the magnitude and direction of the current, \( I \), in the solenoid changes the magnetic field \( \mathbf{H} \) changes. This causes the magnetization of the iron core to undergo the history described below which is known as Hysteresis (see Fig. 6.9).

The Curie Temperature: For a given ferromagnetic material the long range order abruptly disappears at a certain temperature which is called the Curie temperature for the material. The Curie temperature of iron is about \( 1043 \, K \). The Curie temperature gives an idea of the amount of energy takes to break up the long-range ordering in the material. At \( 1043 \, K \) the thermal energy is about \( 0.135 \, eV \) compared to about \( 0.04 \, eV \) at room temperature. The Curie point is rather like the boiling point or the freezing point in that there is no gradual transition from ferro- to para-magnetic behavior, any more than there is between water and ice. These abrupt changes in the properties of a substance, occurring at sharply defined temperatures, are known as in statistical mechanics as phase transitions.
6.9. FERROMAGNETISM AND HYSTERESIS

Figure 6.9: Magnetic Hysteresis.

Antiferromagnet and Ferrimagnet: According Heisenberg theory of Ferromagnetism which is based on quantum mechanics, when the spins on neighboring atoms change from parallel alignment to antiparallel alignment, there must be simultaneous change in the electron charge distribution in the atoms. The change in charge distribution alters the electrostatic energy of the system. When this change in energy favors parallel alignment and is at the same time of sufficient magnitude, the material composed of these atoms is ferromagnetic. However, if the energy change favors antiparallel alignment, although it is still possible to obtain ordered spin structure, the material become either antiferromagnet or Ferrimagnet. An ordered spin structure with zero net magnetic moment is called antiferromagnet (Fig. 6.10a). The most general ordered spin structure contains both "spin-up" and "spin-down" components but has a net, nonzero magnetic moment in one of these directions. Such a material is called Ferrimagnet or simply ferrite (Fig. 6.10b). The classic example of a ferrite is the mineral magnetite (Fe₃O₄).
Figure 6.10: Sping ordering in Antiferromagnet (a) and Ferrimagnet (b).
Chapter 7

Electrodynamics

7.1 Electromotive force

Ohm’s Law: Current is a flow of charges. In order to make the charges flow in a conductor or any other material there has to be some kind of force that pushes the charges. In a conductor this the current is proportional to the pushing force \( I \propto f \), where \( I \) is the current and \( f \) is the pushing force per unit charge. But it is more convenient if we talk in terms of the current density \( \vec{I}/A \) which is a vector quantity since force is a vector quantity. In terms of the current density we can write

\[
\vec{J} \propto \vec{f} \text{ or } \vec{J} = \sigma \vec{f}, 
\]

(7.1)

where \( \sigma \) is the proportionality constant known as the conductivity of the material. The conductivity depends on the nature and geometry of the conductor. The conductivity is related to the resistivity, \( \rho \), of the conductor by

\[
\sigma = \frac{1}{\rho}. 
\]

(7.2)

Charges can experience gravitational, electrical, or magnetic forces. In electrodynamics the dominant forces are electrical and magnetic forces. The force on a charge can then be determined from

\[
\vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \Rightarrow \vec{f} = \frac{\vec{F}}{q} = \vec{E} + \vec{v} \times \vec{B}. 
\]

(7.3)

Since the magnitude of the velocity of the charge (electron) is small we usually neglect the magnetic force contribution.

\[
\vec{f} = \frac{\vec{E}}{q} = \vec{E} \Rightarrow \vec{J} = \sigma \vec{E}. 
\]

(7.4)

Eq. (7.4) is called Ohm’s law. Since you are familiar with \( V = IR \) as Ohm’s law, you may be a little bit confused. So let’s do some examples if we can some how come up with the formula, \( V = IR \).
Example 7.1 A cylindrical resistor of cross sectional area \( A \) and length \( L \) is made from material with conductivity, \( \sigma \). If the potential is constant over each end, and the potential difference between the ends is \( V \), what is the current flowing through the conductor?

Solution: The magnitude of the current flowing in the material is

\[
I = JA, \quad (7.5)
\]

and we know

\[
J = \sigma E, \quad (7.6)
\]

We recall the potential difference and the electric field are related by

\[
E = \frac{V}{L}. \quad (7.7)
\]

Then the current

\[
I = JA = \sigma EA = \sigma \frac{V}{L}A \Rightarrow I = \left( \frac{\sigma A}{L} \right) V. \quad (7.8)
\]

Let’s put the above expression in a little bit different form

\[
V = \left( \frac{L}{\sigma A} \right) I = IR, \quad (7.9)
\]

where

\[
R = \frac{L}{\sigma A}. \quad (7.10)
\]

7.2 Conductivity and density of the free electrons

Conductivity, \( \sigma \), is proportional to the density of the free electrons and decreases with increasing temperature. Due to thermal effect electrons move randomly
inside the material. Since their motion is random, the average speed is zero. But when there is an external force applied to the electron, it would move a mean displacement, \( \lambda \). Then the mean time it took to travel this distance is

\[
t = \frac{\lambda}{v_{\text{thermal}}}
\]  

(7.11)

During this time the average speed of the electron is

\[
v_{\text{av}} = \frac{1}{2} \alpha t = \frac{a \lambda}{2v_{\text{thermal}}},
\]  

(7.12)

where \( a \) is the acceleration of the electron that can be expressed in terms of the applied external force (electric field) as

\[
\vec{a} = \frac{\vec{F}}{m} = \frac{q \vec{E}}{m}.
\]  

(7.13)

Then the average speed becomes

\[
v_{\text{av}} = \frac{1}{2} \alpha t = \frac{qE\lambda}{2mv_{\text{thermal}}},
\]  

(7.14)

If there are \( n \) molecules per unit volume and \( f \) electrons per molecule then the total number of charge carriers (electrons) per unit volume would be \( nf \). If the cross sectional area of the material is \( A \), then the total number of electrons crossing this area per unit time is \( nf v_{\text{av}} A \). Then the current density which is defined as \( J = \frac{Q_{\text{total}}}{At} \) becomes

\[
\vec{J} = qnfv_{\text{av}} = \frac{nfq^2}{2mv_{\text{thermal}}} \vec{E} \Rightarrow \sigma = \frac{nq^2}{2mv_{\text{thermal}}},
\]  

(7.15)

**Example 7.2** Two long metallic cylinders (radii \( a \) and \( b \)) are separated by material of conductivity, \( \sigma \). If they are maintained at a potential difference, \( V \), what current flows from one to the other, in a length, \( l \)?

**Solution:** The current density

\[
\vec{J} = \sigma \vec{E}
\]  

(7.16)
If the potential difference between the two cylinders is $V$ which is a constant, then we can imagine that the inner cylinder carries a total charge $Q$ and the outer cylinder carries a total charge $-Q$ which is uniformly distributed on each surface of the cylinders over a length $L$. Then using Gauss's law the electric field in between the two surfaces at a distance $s$ from the axis of the inner cylinder, can be expressed as

$$\int E \cdot d\hat{s} = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{2\pi l \epsilon_0} \frac{1}{s} \Rightarrow \vec{E} = \frac{Q}{2\pi l \epsilon_0} \frac{1}{s} \hat{s}$$

(7.17)

Let’s assume the current is flowing from the inner cylinder to the outer cylinder. Then we may write

$$\vec{J} = \frac{dI}{da} \hat{s} = \sigma \vec{E} = \frac{Q}{2\pi l \epsilon_0} \frac{1}{s} \hat{s},$$

and noting that $d\hat{a} = d\alpha \hat{s}$

we find

$$I = \int \vec{J} \cdot d\hat{a} = \int \frac{\sigma Q}{2\pi l \epsilon_0} \frac{1}{s} da = \int_0^{2\pi} \frac{\sigma Q}{2\pi l \epsilon_0} \frac{1}{s} l ds \varphi = \frac{\sigma Q}{\epsilon_0}$$

We need to relate the charge $Q$ to the potential difference $V$ between the cylinders. This can be done as follows

$$V = -\int_b^a \vec{E} \cdot d\hat{s} = -\int_b^a \frac{Q}{2\pi l \epsilon_0} \frac{1}{s} \hat{s} \cdot d\hat{s} = -\frac{Q}{2\pi l \epsilon_0} \int_b^a l ds \varphi = -\frac{Q}{2\pi l \epsilon_0} \ln \left(\frac{b}{a}\right) \Rightarrow Q = \frac{2\pi \epsilon_0 l V}{\ln \left(\frac{b}{a}\right)}.$$  

(7.18)

Then the current and the potential difference are related by

$$I = \frac{\sigma Q}{\epsilon_0} = \frac{1}{\epsilon_0} \left(\frac{2\pi \epsilon_0 l V}{\ln \left(\frac{b}{a}\right)}\right) \Rightarrow I = \left(\frac{2\pi \sigma l}{\ln \left(\frac{b}{a}\right)}\right) V$$

(7.19)

We can rewrite the above expression in the form

$$V = \left(\frac{\ln \left(\frac{b}{a}\right)}{2\pi \sigma l}\right) I = IR,$$

(7.20)

where the constant, $R$, is given by

$$R = \frac{\ln \left(\frac{b}{a}\right)}{2\pi \sigma l}.$$  

(7.21)

It is the resistance of the material in between the cylinders. It is a function of the geometry of the arrangement and the conductivity of the medium. It is measured in Ohms ($\Omega = \frac{V}{I}$). We recall from the continuity equation,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

(7.22)
7.2. CONDUCTIVITY AND DENSITY OF THE FREE ELECTRONS

If there is a steady current (i.e. the density of the charge at any time must be a constant. Because there is an equal in and out flow of charge in the volume for a steady current. Thus for steady current

\[ \nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t} = 0 \]  

(7.23)

one finds

\[ \nabla \cdot \vec{E} = \frac{1}{\sigma} \nabla \cdot \vec{j} = 0. \]  

(7.24)

**Example 7.3** Two concentric metallic spherical shells of radius \( a \) and \( b \), respectively, are separated by weakly conducting material of conductivity, \( \sigma \).

(a) If they are maintained at a potential difference \( V \), what current flows from one to the other?

(b) What is the resistance between the shells?

(c) Notice that if \( b >> a \), the outer radius, \( b \), is irrelevant. How do you account for that? Explain this observation to determine the current flowing between two metal spheres, each of radius \( a \), immersed deep in the sea and held very far apart, if the potential difference between them is \( V \). (This arrangement can be used to measure the conductivity of sea water.)

**Solution:**

(a) First let’s find the electric field in the region between the spheres. Let’s assume the inner sphere carries a total charge \( Q \) that is uniformly distributed over the surface and the outer sphere carries a charge \(-Q\) that is also
uniformly distributed over the surface. Using Gauss’s law, we may write
\[ \oint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0} \Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}. \]  
(7.25)

Then the current density becomes
\[ \vec{J} = \sigma \vec{E} = \frac{\sigma Q}{4\pi\epsilon_0 r^2} \hat{r}. \]  
(7.26)

The magnitude of the total current will then be
\[ I = \int \vec{J} \cdot d\vec{a} = \int \frac{\sigma Q}{4\pi\epsilon_0 r^2} dr \Theta = \int_0^\pi \int_0^{2\pi} \frac{\sigma Q}{4\pi\epsilon_0 r^2} r^2 d\theta d\phi \]
\[ \Rightarrow I = \frac{\sigma Q}{\epsilon_0}. \]  
(7.27)

To express the total charge in terms of the potential, we use
\[ V = -\int_b^a \vec{E} \cdot d\vec{r} = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \cdot d\vec{r} = -\frac{Q}{4\pi\epsilon_0} \int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0} \left( \frac{b - a}{ab} \right) \]
\[ \Rightarrow Q = \frac{4\pi\epsilon_0 V ab}{b - a}. \]

Substituting this for \( Q \) the current can be expressed as
\[ I = \frac{\sigma Q}{\epsilon_0} = \frac{\sigma}{\epsilon_0} \left( \frac{4\pi\epsilon_0 V ab}{b - a} \right) = \left( \frac{4\pi\sigma ab}{b - a} \right) V. \]  
(7.28)

(b) Rewriting the above expression as
\[ V = I \left( \frac{b - a}{4\pi\sigma ab} \right) \]  
(7.29)

one finds for the resistance
\[ R = \frac{b - a}{4\pi\sigma ab} = \frac{1}{4\pi\sigma} \left( \frac{1}{a} - \frac{1}{b} \right). \]  
(7.30)

(c) Noting that for \( b \gg a \) the resistance of the material between the inner and outer sphere can be approximated as
\[ R = \frac{1}{4\pi\sigma a}. \]  
(7.31)
7.3. ELECTROMOTIVE FORCE

Now you have two spheres connected to a power source that generates a voltage $V$ and placed quite far apart deep in the sea, the resistance between the two spheres is

$$R_{eq} = 2R = \frac{1}{2\pi \sigma a} \quad (7.32)$$

and the current that can be measured is

$$I = \frac{V}{R_{eq}} = \frac{V}{2\pi \sigma a}. \quad (7.33)$$

7.3 Electromotive force

Electromotive force, or most commonly emf is "that which tends to cause current (actual electrons and ions) to flow. More formally, emf is the external work expended per unit of charge to produce an electric potential difference across two open-circuited terminals. Devices that can provide emf include voltaic cells, thermoelectric devices, solar cells, electrical generators, transformers. Two forces involve in driving current around a circuit: the source $f_s$ which is ordinarily confined to one portion of the loop and the electrostatic force which serve to smooth out the flow and communicate the influence of the source to distant parts of the circuit. Hence the net force per unit charge

$$\vec{f} = \vec{f}_s + \vec{E} \quad (7.34)$$

The electromotive force is defined as

$$\varepsilon = \oint \vec{f} \cdot d\vec{l} = \oint \left( \vec{f}_s + \vec{E} \right) \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} + \oint \vec{E} \cdot d\vec{l}$$

$$\Rightarrow \varepsilon = \oint \vec{f}_s \cdot d\vec{l} + \oint \left( \nabla \times \vec{E} \right) \cdot d\vec{a}. \quad (7.35)$$

But for electrostatic fields $\nabla \times \vec{E} = 0$, and we find

$$\varepsilon = \oint \vec{f}_s \cdot d\vec{l} \quad (7.36)$$
The potential difference between the two terminals \((a\) and \(b\)) of a source is given by

\[
V = - \int_{a}^{b} \vec{E} \cdot d\vec{l} \tag{7.37}
\]

For ideal source the resistance is zero and the net driving force per unit charge should be zero

\[
\vec{f} = \vec{f}_s + \vec{E} = 0 \Rightarrow \vec{f}_s = -\vec{E} \Rightarrow V = \int_{a}^{b} \vec{f}_s \cdot d\vec{l}. \tag{7.38}
\]

Since \(\vec{f}_s\) is zero outside the source, we may write

\[
V = \int_{a}^{b} \vec{f}_s \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} = \varepsilon \tag{7.39}
\]

Note: The function of a battery is to establish and maintain a voltage difference equal to the electromotive force.

Example 7.4 For the circuit shown in the figure below a rectangular conducting wire connected to a resistor of resistance \(R\) is pulled to the right with a constant speed \(v\). The loop is placed in a uniform magnetic field \(B\) directed into the page. If the width of the loop is \(h\) show that the emf (known as motional emf) is given by

\[
\varepsilon = \oint \vec{f}_{\text{mag}} \cdot d\vec{l} = vBh \tag{7.40}
\]

Solution: Consider a charge on the segment of the conducting wire which is inside the magnetic field. This charge experiences the two forces shown in the figure below.

The two forces are the external pulling force and the magnetic force. The pulling force per unit charge, \(f_p\), must balance the magnetic force per unit charge if the rectangular wire is moving with a constant velocity. It can be expressed as

\[
f_p = f_\mu = \mu B, \tag{7.41}
\]
where \( \mu \) is the component of the velocity of the charge. The work done by this force

\[
W = \int \vec{f}_p \cdot d\vec{r} = \int \mu B dl \cos (90 - \theta) = B \int \mu \sin(\theta) dl \quad (7.42)
\]

where \( l \) is the actual distance that the charge has traveled along the direction of the actual velocity \( \vec{w} \) and is given by

\[
l = \frac{h}{\cos \theta} \Rightarrow dl = \frac{dh}{\cos \theta}. \quad (7.43)
\]

Then for the work we have

\[
W = B \int_0^h \mu \tan(\theta) \, dh \quad (7.44)
\]

so that using

\[
\frac{\nu}{\mu} = \tan(\theta) \quad (7.45)
\]

we find

\[
W = B \nu \int_0^h dh = \nu Bh = f_m h = \varepsilon. \quad (7.46)
\]

This is motional emf.

**Example 7.5** A metal disk of radius, \( a \), rotates with angular velocity \( \omega \) about a vertical axis, through a uniform field \( B \), pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk. Find the current in the resistor.

**Solution:** In order to find the current we first need to determine the emf driving the charges in the circuit. This emf is *motional emf*. When the metallic disk is rotating with angular speed, \( \omega \), a positive charge located a distance \( s \) from the axis of rotation will also move with a velocity, \( v = s\omega \), normal
to the radial direction. When this velocity crossed with the magnetic field leads to a magnetic force which drives the charge radially out to the rim of the disk. This gives a current flowing in the clockwise direction as shown in the figure above. We recall that motional emf is

$$\varepsilon = \int \vec{f}_{mag} \cdot d\vec{l}$$

so that the motional emf becomes

$$\varepsilon = \int_0^a (B \omega s) \, ds = \frac{1}{2} B \omega a^2. \quad (7.48)$$

Then the current, $I$

$$I = \frac{B \omega a^2}{2R} \quad (7.49)$$

N.B. This kind of current is known as Eddy current. To see a demonstration on the effect of eddy currents you may go to MIT OPENCOURSEWARE [http://ocw.mit.edu/OcwWeb/Physics/8-02Spring-2007/Visualizations/detail/faraday.htm] and check out the levitating, the suspended, or the falling ring simulation.
7.4 Magnetic flux, motional emf, and Faraday’s Law

In the previous example we saw that a motional emf can be induced in a closed conducting loop moving inside a uniform magnetic field. This motional emf can also be expressed in terms of the magnetic flux through the area bounded by the conducting loop. In fact the motional emf in any shape of a closed conducting loop moving in a uniform magnetic field is given by

\[ \varepsilon = - \frac{d\Phi}{dt}, \]  

(7.50)

where

\[ \Phi = \int \vec{B} \cdot d\vec{a}. \]  

(7.51)

is the magnetic flux through the area bounded by the conducting loop. Next we want to proof this relation. To this end, we consider the surface bounded by a conducting loop inside a magnetic field, \( \vec{B}(\vec{r}) \), shown in Fig.7.1.

Figure 7.1: A conducting loop (shown by the red closed curve) moving with a velocity, \( \vec{v} \), in a none uniform magnetic field, \( \vec{B} \).

The infinitesimal flux change due to the movement of the loop (the region shown with a blue strips) is

\[ d\Phi = \Phi(t + dt) - \Phi(t) = \int \vec{B} \cdot d\vec{a} \]  

(7.52)

Let \( \vec{v} \) is the velocity of the loop with which the wire is moving and \( \vec{u} \) is the velocity of the charge along the wire caused by the magnetic force (see Example 7.4). Then the net velocity of the charge, \( \vec{v} \), is the vector sum of these two.
components, $\vec{w} = \vec{v} + \vec{u}$. The magnified image of the infinitesimal area, $d\vec{a}$, (in Fig. 7.1), is shown in Fig. 7.4. This area is given by

$$d\vec{a} = \vec{v}dt \times d\vec{l}$$

(7.53)

then the change in flux becomes

$$d\Phi = \Phi (t + dt) - \Phi (t) = \int_{\text{ribbon}} \vec{B} \cdot (\vec{v}dt \times d\vec{l})$$

$$d\Phi = \int_{\text{ribbon}} \vec{B} \cdot (\vec{v} \times d\vec{l}) dt \Rightarrow \frac{d\Phi}{dt} = \int_{\text{ribbon}} \frac{\vec{B} \cdot (\vec{v} \times d\vec{l})}{dt}.$$  (7.54)

Noting that

$$\vec{B} \cdot (\vec{v} \times d\vec{l}) = (\vec{v} \times d\vec{l}) \cdot \vec{B} = (\vec{B} \cdot \vec{v}) \cdot d\vec{l} = - (\vec{B} \times \vec{v}) \cdot d\vec{l}$$  (7.55)

we find rate of change of the magnetic flux

$$\frac{d\Phi}{dt} = -\int_{\text{ribbon}} (\vec{v} \times \vec{B}) \cdot d\vec{l} = -\int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\Rightarrow \varepsilon = \int f_{\text{mag}} \cdot d\vec{l} = -\frac{d\Phi}{dt}. $$  (7.56)

This motional emf was observed for the first time in 1831 by Michael Faraday. He observed a current in a closed conducting loop without any power source when the loop is dragged inside a uniform magnetic field. He realized that the current is an induced current due to an induced electromotive force. The induced electromotive force is proportional to the time change of the magnetic flux in the area bounded by the conducting loop,

$$\varepsilon = -\frac{d\Phi}{dt},$$  (7.57)

where the magnetic flux, $\Phi$, is

$$\Phi = \int_A \vec{B} \cdot d\vec{a},$$  (7.58)
7.4. MAGNETIC FLUX, MOTIONAL EMF, AND FARADAY’S LAW

in which $A$ is the area bounded by the conducting loop. This means an electric current (electromotive force) can be induced by changing the magnetic field inside a closed conducting wire by changing the area over which the magnetic field is acting and/or the magnetic field.

The direction of the induced current resulting from the induced voltage is determined by Lenz’s law. It states the induced current must have direction that gives an induced magnetic field which leads to an induced magnetic flux that opposes the cause for the change in the flux. That means if the cause for induced current is due to a decrease in a magnetic flux, the induced current must produce an induced magnetic field that increases the magnetic flux. For example, for the induced current (a moving positive charge, $e$) shown in Fig. 7.2, the induced magnetic field is out of the page which is opposite to the external magnetic field, $\vec{B}$. In this case the magnetic flux must have increased either by increasing the magnetic field or the area bounded by the conducting wire.

![Figure 7.2: A counter clockwise induced current.](image)

The induced electric field: The induced electromotive force (the current) is a result of some kind of force acting on the free charges in the conductor. So where this force is coming from? generally this force is a direct result of an induced electric field, $\vec{E}$ due to the change in magnetic flux

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}. \quad (7.59)$$

Applying Stoke’s theorem, we can write

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = \int_A \nabla \times \vec{E} \cdot d\vec{a} = -\frac{d\Phi}{dt} \quad (7.60)$$
and using Eq. (7.58)

\[ \int_A \nabla \times \vec{E} \cdot d\vec{a} = -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{a}. \]  

(7.61)

If the flux is a direct result of change in the magnetic field the above Eq. 7.61 can put in the form

\[ \int_A \nabla \times \vec{E} \cdot d\vec{a} = \int_A \left( -\frac{d\vec{B}}{dt} \right) \cdot d\vec{a} \]  

(7.62)

This results is the differential form of Faraday’s law

\[ \nabla \times \vec{E} = -\frac{d\vec{B}}{dt}. \]  

(7.63)

Example 7.6 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current, I, as shown in the figure below.

(a) Find the flux of \( \vec{B} \) though the loop.

(b) If someone now pulls the loop directly away from the wire, at speed \( v \), what emf is generated? In what direction (clockwise or counterclockwise does the current flow?

(c) What if the loop is pulled to the right at speed \( v \), instead of away?

Solution:

(a) For a very long straight wire carrying a current, I, at a distance, s, the magnetic field (assuming the positive-z direction is along the direction of the current flow) is given by

\[ \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{z}. \]  

(7.64)
For an infinitesimal area of thickness $ds$, we have

$$d\mathbf{a} = ads'\hat{\varphi}$$  \hfill (7.65)

so that the magnetic flux

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a}.$$  \hfill (7.66)

becomes

$$\Phi = \int_s^{s+a} \frac{\mu_0 I}{2\pi s} ads' = \frac{\mu_0 I a}{2\pi} \ln \left(\frac{s + a}{s}\right).$$  \hfill (7.67)

(b) Noting that the speed, $v$, is

$$v = \frac{ds}{dt}$$  \hfill (7.68)

the motional emf given by

$$\varepsilon = -\frac{d\Phi}{dt}$$  \hfill (7.69)

is found to be

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt} \left(\frac{\mu_0 I a}{2\pi} \ln \left(\frac{s + a}{s}\right)\right)$$

$$= -\frac{\mu_0 I a}{2\pi} \frac{s}{s + a} \ln \left(\frac{s + a}{s}\right)$$

$$= -\frac{\mu_0 I a}{2\pi} \frac{s}{s + a} \left(1 - \frac{s + a ds}{s^2 dt}\right) \Rightarrow \varepsilon = \frac{\mu_0 I v a^2}{2\pi s (s + a)}$$  \hfill (7.70)

(c) In this case the flux is constant and there is no emf.

**Example 7.7** A long cylindrical magnet of length, $L$, and radius, $a$, carries a uniform magnetization, $M$, parallel to its axis. It passes with a constant velocity $v$ through a circular conducting ring of slightly larger radius, $b$.

(a) Plot a qualitative graph for the magnetic flux as a function of time.

(b) Plot the emf induced as function of time.
(c) Determine the direction of the induced current.

**Solution:**

(a) We first need to find the magnetic field of the magnet. Using the auxiliary field, $\vec{H}$, and Ampere’s law

$$\oint \vec{H} \cdot d\vec{l} = (I_f)_{\text{enclosed}},$$

one can easily see that

$$\vec{H} = 0$$

both inside and outside the cylinder since there is no free current. Since the magnetization, $\vec{M} = 0$, outside and $\vec{M} = M \hat{z}$ inside the cylinder, from

$$\vec{H} = \frac{1}{\mu_0} (\vec{B} - \vec{M}),$$

we find

$$\vec{B} \simeq \left\{ \begin{array}{ll} \mu_0 M \hat{z}, & s < a \\ 0, & s > a \end{array} \right.$$

(7.74)

The magnetic flux

$$\Phi = \int_A \vec{B} \cdot d\vec{a},$$

(7.75)

when the magnet is far away from the ring, the flux is zero

$$\Phi = 0$$

(7.76)

and starts to increase as it moves closer and it reaches a maximum value

$$\Phi = \int_A \vec{B} \cdot d\vec{a} = \mu_0 M \pi a^2,$$

(7.77)

when its one end just enters into the ring. It then remain nearly constant until its other end begins move away from the ring (i.e. over a time period of $t = L/v$). It then starts to decrease back to zero as it moves further away from the ring. Qualitatively, the flux as function of time looks like as shown below.

(b) For the corresponding time intervals shown in the figure above the induced emf

$$\varepsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}.$$  

(7.78)

$$\varepsilon = \left\{ \begin{array}{ll} -\left| \frac{d\Phi}{dt} \right|, & \text{when flux is increasing} \\ 0, & \text{when flux is constant} \\ \left| \frac{d\Phi}{dt} \right|, & \text{when flux is decreasing} \end{array} \right.$$  

(7.79)

Therefore the emf as function of time looks as shown below
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(c) The negative voltage is a result of increase in flux and therefore the increase must opposed by the flux by the magnetic field of the induced current. In order to decrease this flux the induced current must flow in direction such that it gives a magnetic field opposite to the magnetization (i.e. in $-z$ direction). Therefore the induced current must flow in the clockwise direction as we look at it from the magnet side. For the positive voltage they must flow in the counterclockwise direction.

Example 7.8 If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on top, and plug it in, the ring will jump several feet in the air. Why?

Solution: As you plug it in the magnetic flux changes from zero to (assuming it is long solenoid)

$$\Phi = \mu_0 n I \pi a^2 \Rightarrow \Delta \Phi = \mu_0 n I \pi a^2,$$

(7.80)

where $n$ is the number of turns per unit length and $I$ is the current in the solenoid, $a$, is the radius of the ring. This shows the flux has suddenly increased and the induced current must oppose this increase. That means
it must produce a magnetic field in opposite direction (downward). For this to happen the induced current, as we look from the top, must flow in the clockwise direction. If we imagine the positive-\( z \) direction along the direction of the solenoid magnetic field, for the ring on the x-y plane we have what we see in the Fig. 7.5

![Figure 7.5: The induced current in the conducting ring.](image)

The component of the magnetic field of the solenoid on the x-y plane of the ring gives a net force along the positive \( z \)-direction causing the ring to jump.

**Example 7.9** A line charge is glued onto the rim of a wheel of radius, \( b \), which is then suspended horizontally, as shown in the figure below, so that it is free to rotate (the spokes are made of some nonconducting material). In the central region, out to radius \( a \), there is a uniform magnetic field, \( \vec{B}_0 \), pointing up. Now someone turns the field off.

(a) Explain what is going to happen?
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(b) Show that the angular momentum imparted to the wheel does not depend how fast or slow you turn off the field.

Solution:

(a) Originally before the magnetic field is turned off the magnetic flux is

\[ \Phi_i = \pi a^2 B_0 \]  

(7.81)

and suddenly the magnetic field is turned off. This causes the magnetic flux to drop to zero

\[ \Phi_f = 0, \]  

(7.82)

that leads to a decrease in magnetic flux

\[ \Delta \Phi = -\pi a^2 B_0. \]  

(7.83)

A decrease in magnetic flux results in induced current which must flow in a direction that gives a magnetic field opposing the decrease in magnetic flux. Which means the "induced magnetic field" must be reinforce the original magnetic field, \( B_0 \), in order to oppose the decrease. This means the induced magnetic field must be directed along the same direction of \( B_0 \) (up). Therefore, as shown in the figure the current flows in a counterclockwise direction. From Faraday’s law the induced current is a result of the induced electric field, \( \vec{E} \), resulting from the changing magnetic flux

\[ \oint \vec{E} \cdot d\vec{l} = -\pi a^2 \frac{dB}{dt}. \]  

(7.84)

Due to this electric field the charge experiences an electrical force, \( \vec{F} \). If we consider an infinitesimal charge, \( dq = \lambda dl \), the electrical force on this charge can be expressed as

\[ d\vec{F} = \lambda dl \vec{E}, \]  

(7.85)
and the torque, $d\tau$, on this charge

$$
d\tau = \vec{r} \times d\vec{F} = \lambda dl \vec{r} \times \vec{E}.
$$

(7.86)

Then the total torque on the line charge glued to the rim of the wheel becomes

$$
\tau = \lambda \int \left( \vec{r} \times \vec{E} \right) dl.
$$

(7.87)

Since the induced current is directed counterclockwise, the electric field is also in the counterclockwise direction. In cylindrical coordinates $(s, \varphi, z)$, we have

$$
\vec{r} = b\hat{s}, \vec{E} = E\hat{\varphi},
$$

(7.88)

so that the torque becomes

$$
\tau = \lambda b \int E (\hat{s} \times \hat{\varphi}) dl = \left( \lambda b \oint E dl \right) \hat{z}.
$$

(7.89)

Therefore the wheel will rotate in a counterclockwise direction.

(b) In order to show this, we must be able to prove that the change in angular momentum is a constant. The torque and the angular momentum are related by

$$
\tau = \frac{d\vec{L}}{dt}.
$$

(7.90)

The change in angular momentum (or the momentum imparted to the wheel) is given by

$$
\Delta \vec{L} = \int_{0}^{t} \tau dt = \lambda b \int_{0}^{t} \left( \oint E dl \right) \hat{z} dt.
$$

(7.91)

From Faraday’s law, we have

$$
\oint E \cdot d\vec{l} = \oint E dl = -\pi b^{2} \frac{dB}{dt}
$$

(7.92)

so that

$$
\Delta \vec{L} = \int_{0}^{t} \tau dt = -\lambda b^{3} \pi \int_{0}^{t} \frac{dB}{dt} dt \hat{z} = -\lambda b^{3} \pi \int_{b_{0}}^{0} dB \hat{z} = \lambda b^{3} \pi B_{0} \hat{z}.
$$

(7.93)

**Example 7.10** An infinitely long straight wire carries a slowly varying current, $I(t)$. Determine the induced electric field, as a function of the distance, $s$, from the wire.
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Solution: The magnetic field of a long straight wire carrying a current, \( I \), is given by

\[
\mathbf{B} = \frac{\mu_0 I (t)}{2\pi s} \hat{\varphi},
\]

(7.94)

where \( s \) is the distance from the center of the wire and we assumed the direction of the positive \( z \)-axis to be along the direction of the current. The induced electric field inside a rectangular conducting wire of length, \( l \), with its sides located at \( s_0 \) and \( s \) satisfies Faraday's law

\[
\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}.
\]

(7.95)

If we write the line integral for each side we have

\[
\oint \mathbf{E} \cdot d\mathbf{l} = \int_{\text{left}} \mathbf{E} \cdot d\mathbf{l} + \int_{\text{top}} \mathbf{E} \cdot d\mathbf{l} + \int_{\text{right}} \mathbf{E} \cdot d\mathbf{l} + \int_{\text{bottom}} \mathbf{E} \cdot d\mathbf{l};
\]

(7.96)

so that assuming at a given instant the induced current is flowing in a counterclockwise direction, we can write

\[
\oint \mathbf{E} \cdot d\mathbf{l} = \int_{-l/2}^{l/2} E(s) \, dz + \int_{s_0}^{s} E(s') \, ds' + \int_{-l/2}^{l/2} E(s_0) \, dz + \int_{s_0}^{s} E(s') \, ds' \\
= \int_{s_0}^{s} E(s') \, ds' - \int_{s_0}^{s} E(s') \, ds' + \int_{-l/2}^{l/2} E(s) \, dz - \int_{-l/2}^{l/2} E(s_0) \, dz \\
\Rightarrow \oint \mathbf{E} \cdot d\mathbf{l} = (E(s) - E(s_0)) l.
\]

(7.97)

If the current in the wire changes slowly the rate of change of the magnetic
flux in the rectangular loop becomes
\[ -\frac{d\Phi}{dt} = -\int dB = -\frac{\mu_0}{2\pi} \frac{dI}{dt} \int_{s_0}^{s} l ds = -\frac{\mu_0 l}{2\pi} \frac{dI}{dt} \ln \left( \frac{s}{s_0} \right) \] (7.98)
\[ \Rightarrow -\frac{d\Phi}{dt} = -\frac{\mu_0 l}{2\pi} \frac{dI}{dt} \ln \left( \frac{s}{s_0} \right) \] (7.99)

Then, the induced electric field
\[ \oint E \cdot dl = -\frac{d\Phi}{dt} \Rightarrow (E(s) - E(s_0)) l = -\frac{\mu_0 l}{2\pi} \frac{dI}{dt} \ln \left( \frac{s}{s_0} \right). \] (7.100)

that can be put in the form
\[ \vec{E}(s) = -\left( \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln (s) + K \right) \hat{z}, \] (7.101)

where the constant
\[ K = -\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln (s_0) + E(s_0). \] (7.102)

**Quasi-static approximation:** As we saw if the magnetic induction varies in time, an electric field is created, according to Faraday’s law; the situation is no longer purely magnetic in character. Nevertheless, if the time variation is not too rapid, the magnetic fields dominate and the behavior can be called quasi-static. "Quasi-static" refers to the regime for which the finite speed of light can be neglected and fields treated as if they propagated instantaneously (i.e. when \( s \ll c\tau; c \) is the speed of light and \( \tau \) is the time it takes the current \( I \) to change). Suppose the magnetic fields are not propagated instantaneously.

Then the magnetic flux at the position of the loop at time \( t \) must be due to a changing current current at earlier time \( t - s/c \). This means
\[ I(t) = I_0 \sin \left( \omega \left( t - \frac{s}{c} \right) \right) \] (7.103)

and the magnetic field becomes
\[ \vec{B}(s,t) = \frac{\mu_0 I_0}{2\pi s} \sin \left( \omega \left( t - \frac{s}{c} \right) \right) \hat{\phi} \]
\[ \Rightarrow \frac{d\vec{B}}{dt} = \frac{\mu_0 I_0 \omega}{2\pi s} \cos \left( \omega \left( t - \frac{s}{c} \right) \right) \hat{\phi}. \] (7.104)

The induced magnetic field we found becomes
\[ \vec{E}(s) = -\left( \frac{\mu_0}{2\pi} \frac{dI}{dt} \ln (s) + K \right) \hat{z} \]
\[ = \left\{ -\frac{\mu_0 I_0 \omega}{2\pi} \cos \left( \omega \left( t - \frac{s}{c} \right) \right) \ln \left( \frac{s}{s_0} \right) + E(s_0) \right\} \hat{z}, \] (7.105)
and using the curl in cylindrical coordinates, we find
\[
\nabla \times \vec{E}(s) = -\frac{\partial}{\partial s} \left\{ -\frac{\mu_0 I_0 \omega}{2\pi} \cos \left[ \omega \left( t - \frac{s}{c} \right) \right] \ln \left( \frac{s}{s_0} \right) + E(s_0) \right\} \hat{\phi} \\
= \frac{\mu_0 I_0 \omega}{2\pi s} \left\{ \cos \left[ \omega \left( t - \frac{s}{c} \right) \right] + \frac{\omega s}{c} \sin \left[ \omega \left( t - \frac{s}{c} \right) \right] \ln \left( \frac{s}{s_0} \right) \right\} \hat{\phi} 
\]
(7.106)

According to Faraday’s law
\[
\nabla \times \vec{E}(s) = -\frac{d\vec{B}}{dt}
\]
(7.107)
this means the results in Eqs. (7.104) and (7.106) must be the same
\[
\frac{\mu_0 I_0 \omega}{2\pi s} \cos \left[ \omega \left( t - \frac{s}{c} \right) \right] \hat{\phi} \\
= \frac{\mu_0 I_0 \omega}{2\pi s} \left\{ \cos \left[ \omega \left( t - \frac{s}{c} \right) \right] + \frac{\omega s}{c} \sin \left[ \omega \left( t - \frac{s}{c} \right) \right] \ln \left( \frac{s}{s_0} \right) \right\} \hat{\phi} 
\]
(7.108)
This equality holds only when
\[
\frac{\omega s}{c} \sin \left[ \omega \left( t - \frac{s}{c} \right) \right] \ln \left( \frac{s}{s_0} \right) \simeq 0 \Rightarrow \frac{\omega s}{c} << 1 \Rightarrow \omega s << c. 
\]
(7.109)
and this requires the current must not change fast.

**Example 7.11** A long solenoid with radius, \(a\), and \(n\) turns per unit length carries a time-dependent current, \(I(t)\), in the \(\hat{\phi}\) direction. Find the electric field (magnitude and direction) at a distance \(s\) from the axis (both inside and outside the solenoid), in the quasistatic approximation.

**Solution:** Inside the solenoid if we consider a circular conducting loop of radius \(s\) centered about the axis of the cylinder, a current will be induced inside the conductor. The induced electric field inside this conductor, using Faraday’s law, can be expressed as
\[
\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}.
\]
(7.110)

Inside a long solenoid carrying a current, \(I(t)\), in the \(\varphi\) direction, the magnetic field is given by
\[
\vec{B}(\vec{r}) = \mu_0 I(t) n\hat{z}
\]
(7.111)
so that the rate of change of the magnetic flux inside the circular loop of radius \(s\) is
\[
-\frac{d\Phi}{dt} = -\mu_0 n \pi s^2 \frac{dI}{dt}.
\]
(7.112)
Then the induced electric field becomes
\[
\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = E2\pi s = -\mu_0 n\pi s^2 \frac{dI}{dt} \Rightarrow \vec{E} = -\frac{\mu_0 ns}{2} \frac{dI}{dt}.
\] (7.113)

Outside the solenoid the rate of change of the magnetic flux inside the circular loop of radius \(s\), since the magnetic field outside is zero, becomes
\[
-\frac{d\Phi}{dt} = -\mu_0 n\pi a^2 \frac{dI}{dt}.
\] (7.114)

and the induced electric field becomes
\[
\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = E2\pi s = -\mu_0 n\pi a^2 \frac{dI}{dt} \Rightarrow \vec{E} = -\frac{\mu_0 na^2}{2s} \frac{dI}{dt}.
\] (7.115)

Again it is important to note note that this result is valid when the current is not changing rapidly (quasistatic approximation).

### 7.5 Inductance

Consider two conducting loops with arbitrary shape as shown in the figure below.

The first loop carries a current, \(I_1\), and the second loop carries a current, \(I_2\). Due to the current in the first loop, there is a magnetic field, \(\vec{B}_1\), at the position of the second loop and likewise due to the current in the second loop a magnetic field, \(\vec{B}_2\), exists at the position of the first loop. Then the magnetic flux though the area bounded by the second loop, \(\Phi_{21}\), can be expressed as
\[
\Phi_{21} = \int_{a_2} \vec{B}_1 \cdot d\vec{a}_2,
\] (7.116)

where \(a_2\) is the area bounded by the second loop. In terms of the vector potential, \(\vec{A}_1\), for the the magnetic field, \(\vec{B}_1\), we have
\[
\vec{B}_1 = \nabla \times \vec{A}_1,
\] (7.117)
and the magnetic flux can be put in the form

\[ \Phi_{21} = \int_{A_2} \nabla \times \vec{A}_1 \cdot d\vec{a}_2. \]  

(7.118)

Using Stokes theorem, one can write

\[ \Phi_{21} = \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2, \]  

(7.119)

where \( C_2 \) is the second closed wire. We recall the vector potential for the first loop can be expressed as

\[ \vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{|\vec{r} - \vec{r}'|}, \]  

(7.120)

so that the magnetic flux can be put in the form

\[ \Phi_{21} = \oint_{C_2} \left( \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{|\vec{r} - \vec{r}'|} \right) \cdot d\vec{l}_2 = \left( \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r} - \vec{r}'|} \right) I_1 \]  

(7.121)

or

\[ \Phi_{21} = M_{21} I_1, \]  

(7.122)

where

\[ M_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r} - \vec{r}'|}, \]  

(7.123)

the Neumann formula known as the mutual inductance of the two conducting wires. It is purely geometrical quantity depending on the shapes, sizes, and relative positions of the two loops. Following a similar procedure one can easily show that the flux in the area bounded by the first loop, \( \Phi_{12} \), due to the current in the second loop is given by

\[ \Phi_{12} = M_{12} I_2, \]  

(7.124)

where

\[ M_{12} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{|\vec{r} - \vec{r}'|}, \]  

(7.125)

which shows

\[ M_{21} = M_{12}. \]  

(7.126)

Therefore, we can conclude that if one places a conducting loop in the vicinity of a magnetic field by another loop carrying a current, \( I \), the flux in the conducting loop can be determined from

\[ \Phi = MI. \]  

(7.127)

where \( M \) is the mutual inductance of the two wires. If the source of the magnetic field is the current in the conducting loop itself then we have a self inductance given by

\[ L = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_1} \frac{d\vec{l} \cdot d\vec{l}}{|\vec{r} - \vec{r}'|}. \]  

(7.128)
and self magnetic flux

$$\Phi = LI.$$  \hspace{1cm} (7.129)

For a current changing with time, the induced emf in the conducting loop can be expressed as

$$\varepsilon = -\frac{d\Phi}{dt} = -M\frac{dI}{dt},$$  \hspace{1cm} (7.130)

and for a self induced emf,

$$\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$  \hspace{1cm} (7.131)

The SI unit of inductance is Henry (H).

**Example 7.12** A short solenoid (length $l$ and radius $a$, with $n_1$ turns per unit length) lies on the axis of a very long solenoid (radius, $b$, with $n_2$ turns per unit length) as shown in Fig. 7.6). Current $I$ flows in the short solenoid. What is the flux through the long solenoid?

![Figure 7.6: A short solenoid embedded in along solenoid.](image)

**Solution:** The side view of the two solenoid is shown in Fig.7.5. The short solenoid carries a current in a counterclockwise direction. What we normally would do to find the flux through the long solenoid, is to find the magnetic field produced by the first solenoid. However, since the length of the first solenoid is short it is not an easy task finding this magnetic field. The convenient way is to use the fact that the mutual inductance...
is the same, \( M_{12} = M_{21} = M \), and the flux through each solenoid can be determined using the mutual inductance

\[
\Phi_2 = MI_1 \quad \text{and} \quad \Phi_1 = MI_2, \quad (7.132)
\]

where \( \Phi_1 \) is the flux in the first (short) solenoid due to the second (long) solenoid and \( \Phi_2 \) is the flux in the second (long) solenoid due to the first (short) solenoid. Suppose the long solenoid carries a current, \( I_2 \), the magnetic field due to this current is given by

\[
\vec{B}_2 = \mu_0 n_2 I_2 \hat{z}, \quad (7.133)
\]

and the flux through the short solenoid due to this magnetic field,

\[
\Phi_1 = B_2 n_1 l \pi a^2 = \mu_0 n_2 I_2 n_1 l \pi a^2 = \left( \mu_0 n_1 n_2 l \pi a^2 \right) I_2 = M_{12} I_2, \quad (7.134)
\]

where the mutual inductance is

\[
M_{12} = M_{21} = \mu_0 n_1 n_2 l \pi a^2, \quad (7.135)
\]

Then the flux through the long solenoid due to the field in the short solenoid can be written as

\[
\Phi_2 = M_{21} I_1 = \left( \mu_0 n_1 n_2 l \pi a^2 \right) I. \quad (7.136)
\]

**Example 7.13** Find the self-inductance of a toroidal coil with rectangular cross section (inner radius \( a \), outer radius \( b \), height \( h \)), which carries a total of \( N \) turns.(Fig. 7.7)

![Figure 7.7: A toroid with total turns, \( N \), inner radius \( a \) and outer radius \( b \) and height \( h \).](image)

**Solution:** We recall the magnetic field of a toroid is given by

\[
\vec{B} = \begin{cases} 
\frac{\mu_0 NI}{2 \pi} \hat{\varphi}, & \text{inside the coil} \\
0, & \text{outside the coil}
\end{cases}. \quad (7.137)
\]

Then the self magnetic flux through a single rectangular loop (Fig. 7.5), can be expressed as

\[
\Phi_1 = \int_{\text{Area}} \vec{B} \cdot d\vec{a} = \int_{\text{Area}} \frac{\mu_0 NI}{2 \pi s} \hat{\varphi} \cdot \hat{\varphi} d\alpha = \Phi = \int_{\text{Area}} \frac{\mu_0 NI}{2 \pi s} d\alpha \quad (7.138)
\]
then the total flux through the \( N \) turns of the toroid

\[
\Phi = N \Phi_1 = \int_a^b \frac{\mu_0 NI}{2\pi s} Nhds = \frac{\mu_0 N^2 hI}{2\pi} \int_a^b \frac{ds}{s} ds
\]

\[
\Rightarrow \Phi = \frac{\mu_0 N^2 h}{2\pi} \ln \left( \frac{b}{a} \right) I = LI, 
\]

(7.139)

where the self inductance, \( L \), of the toroid is given by

\[
L = \frac{\mu_0 N^2 h}{2\pi} \ln \left( \frac{b}{a} \right). 
\]

(7.140)

**Example 7.14** Suppose there is a current, \( I \), around a loop when someone suddenly cuts the wire. The current drops "instantaneously" to zero. This generates a whopping back emf, for although, \( I \), may be small, \( \frac{dI}{dt} \), is enormous. That’s why you often draw a spark when you unplug an iron or toaster—electromagnetic induction is desperately trying to keep the current going even if it has to jump the gap in the circuit. The same thing happens when you plug in a toaster or iron. Model a circuit representing the toaster or the iron and find the current in the circuit as a function of time.

**Solution:** A toaster and an iron are made of a coil of wire. This coil of wire have a self inductance \( L \) and some resistance \( R \). When we plug it in to a power outlet generating a constant voltage \( \varepsilon_0 \) (7.8), a current \( I \) is being generated in the circuit.

Apply Kirchoff’s voltage rule for the circuit, one finds

\[
\varepsilon_0 - L \frac{dI}{dt} - RI = 0. 
\]

(7.141)
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From which follows that

\[ L \frac{dI}{dt} = \varepsilon_0 - RI \Rightarrow \int_{I_0}^{I(t)} \frac{dI}{\varepsilon_0 - RI} = \int_0^t \frac{dt}{L} \]

\[ \Rightarrow \left. - \frac{1}{R} \ln (\varepsilon_0 - RI) \right|_{I_0}^{I(t)} = \frac{t}{L} \Rightarrow \ln \left( \frac{\varepsilon_0 - RI(t)}{\varepsilon_0 - RI_0} \right) = -\frac{Rt}{L} \]

\[ \Rightarrow \frac{\varepsilon_0 - RI(t)}{\varepsilon_0 - RI_0} = \exp (-Rt/L) \Rightarrow \varepsilon_0 - RI(t) = (\varepsilon_0 - RI_0) \exp (-tR/L) \]

\[ \Rightarrow I(t) = \frac{\varepsilon_0}{R} - \left( \frac{\varepsilon_0}{R} - I_0 \right) \exp (-tR/L) . \quad (7.142) \]

One may write this current as

\[ I(t) = I_s + I_L \]

where

\[ I_s = \frac{\varepsilon_0}{R}, \quad I_L = -\left( \frac{\varepsilon_0}{R} - I_0 \right) \exp (-tR/L) \]

\[ I_s \] is the dc current that is continually flow out of the source in a clockwise direction and \( I_L \) is the induced current generated in the inductor and flows in a counterclockwise direction. If there was no current in the circuit just before we turn the switch on, \( I_0 = 0 \), then we find

\[ I_s (t) = \frac{\varepsilon_0}{R}, \quad I_L (t) = -\frac{\varepsilon_0}{R} \exp \left( -\frac{t}{\tau} \right) \]

where

\[ \tau = \frac{L}{R} \quad (7.143) \]

is know as the inductive time constant which you have studied in Intro physics II (PHYS 2020/2021). It tells you how long the current takes to reach a substantial fraction (roughly two third) of its final (maximum) value (see Fig. ??).

The voltage across the inductor

\[ \Delta V_L = -L \frac{dI}{dt} = -\varepsilon_0 \exp \left( -\frac{t}{\tau} \right) \quad (7.144) \]
At \( t = 0 \) although the current in the circuit is zero since \( \frac{dI}{dt} \neq 0 \), it generates an induced voltage

\[
\Delta V_L = -\varepsilon_0 \quad (7.145)
\]

in the inductor (coil) which causes the spark.

**Example 7.15** A square loop of wire, of side \( a \), lies midway between two long wires, \( 3a \) apart, and in the same plane. (Actually the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current \( I \) in the square loop is gradually increasing: \( dI/dt = k \) (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

**Solution:** The magnetic flux in the big loop is proportional to the current in the square loop and can be expressed as

\[
\Phi = MI \quad (7.146)
\]

where \( M \) is the mutual inductance. The induced emf can then be expressed as

\[
\varepsilon = -\frac{d\Phi}{dt} = -M \frac{dI}{dt} = -Mk \quad (7.147)
\]

The easy way to find the mutual inductance \( M \) is to find the flux in the area bounded by the square loop. The magnetic field inside the square loop is due to the magnetic field by the longest sides of the rectangular loop. If we imagine same magnitude of current \( I \) flowing in clockwise direction in this loop, the net magnetic field at a distance \( s \) from the left side of the wire can be written as

\[
\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\varphi} + \frac{\mu_0 I}{2\pi(3a - s)} \hat{\varphi}. \quad (7.148)
\]
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Then the magnetic flux in the square loop is

\[
\Phi = \int \vec{B} \cdot d\vec{a} = \int_a^{2a} \left[ \frac{\mu_0 I}{2\pi s} + \frac{\mu_0 I}{2\pi (3a - s)} \right] ads
\]

\[
= \frac{\mu_0 a I}{2\pi} \left[ \ln \left( \frac{2a}{a} \right) - \ln \left( \frac{3a - 2a}{3a - a} \right) \right] = \frac{\mu_0 a I}{2\pi} \left[ \ln 2 - \ln \left( \frac{1}{2} \right) \right]
\]

\[\Rightarrow \Phi = \left( \frac{\mu_0 a}{\pi} \ln 2 \right) I. \tag{7.149}\]

Now recalling that the flux is also expressible in terms of the mutual inductance, \( M \), of the two loops as

\[\Phi = MI \tag{7.150}\]

we find

\[M = \frac{\mu_0 a}{\pi} \ln 2. \tag{7.151}\]

Therefore the induced emf becomes

\[\varepsilon = -\frac{\mu_0 a I}{\pi} \ln 2. \tag{7.152}\]

Since the flux is increasing the induced current must oppose the increase. In order to oppose the increase, we need an induced magnetic field coming out of the page in the area bounded by the rectangular loop. This happens when the current is flowing in the counterclockwise direction.

**Example 7.16** A capacitor \( C \) is charged up to a potential \( V \) and connected to an inductor \( L \), as shown in the figure below. At time \( t = 0 \) the switch \( s \) is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor \( R \) is included in series with \( C \) and \( L \).

![Diagram of a capacitor and inductor circuit](image)

**Solution:** After the switch is closed the current in the inductor and the charge on the capacitor is going to be a function of time. Applying Kirchoff’s rule we may write

\[
\frac{Q(t)}{C} + L \frac{dI(t)}{dt} = 0 \tag{7.153}
\]
and using $I(t) = dQ/dt$, we find
\[
\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0
\]  
(7.154)

The solution of the above equation is given by
\[
Q(t) = A \cos \left( \frac{1}{\sqrt{LC}} t \right) + B \sin \left( \frac{1}{\sqrt{LC}} t \right)
\]  
(7.155)

At $t = 0$ the capacitor is fully charged and we can assume this charge to be $Q_0$. But the current in the circuit is zero at the initial time. Therefore
\[
Q(0) = Q_0 \Rightarrow A = Q_0
\]  
(7.156)
\[
I(t = 0) = \left. \frac{dQ}{dt} \right|_{t=0} = \frac{d}{dt} \left[ A \cos \left( \frac{1}{\sqrt{LC}} t \right) + B \sin \left( \frac{1}{\sqrt{LC}} t \right) \right]_{t=0} = 0
\]  
\[
\Rightarrow B = 0
\]  
(7.157)

Therefore the charge and the current at a given time $t$
\[
Q(t) = Q_0 \cos \left( \frac{1}{\sqrt{LC}} t \right) \to I(t) = \frac{dQ}{dt} = -\frac{Q_0}{\sqrt{LC}} \sin \left( \frac{1}{\sqrt{LC}} t \right)
\]  
(7.158)

Noting that $Q_0 = CV$, we may write
\[
Q(t) = CV \cos \left( \frac{1}{\sqrt{LC}} t \right), I(t) = V \sqrt{\frac{C}{L}} \sin \left( \frac{1}{\sqrt{LC}} t \right).
\]  
(7.159)

### 7.6 Energy in a magnetic field

Suppose we have only a single circuit with a constant current, $I$ (Fig. 7.9 a) at time $t$. If at a later time $t + dt$ the magnetic flux through the circuit changed (Fig. 7.9 b). Due to this change in magnetic flux there will be an induced current in the opposite direction. However, the source can maintain the current constant by doing a work that would balance the opposing induced current that depends on the induced electromotive force, $\varepsilon$. To determine the rate of this work done, we note that
\[
dW = \vec{F} \cdot d\vec{r} \Rightarrow \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{v} \cdot \vec{F}.
\]  
(7.160)

Then the added electric field, $\vec{E}'$, by the source to balance the induced electric field (due to the changing flux) on each electron of charge, $q_i$, and mean velocity $\vec{v}_i$ gives rise to a change in energy per unit time given by
\[
\frac{dW_i}{dt} = q_i \vec{v}_i \cdot \vec{E}'
\]  
(7.161)

Then if we sum this energy per unit time over all electrons in the closed con-
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![Figure 7.9: (a) A steady current $I$ in closed conducting wire at time $t$ when there is no external magnetic field, and (b) An external magnetic field $\vec{B}$ is turned on at time $t + dt$. The source voltage maintains the same current $I$.](image)

conducting loop,

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}' = \oint I dt \frac{d\vec{r}}{dt} \cdot \vec{E}' = \int \vec{E}' \cdot d\vec{r}; \quad (7.162)$$

we do find the rate at which the source must supply energy to maintain the current constant ($I$)

$$\frac{dW}{dt} = -I \varepsilon = -I \left( -\frac{d\Phi}{dt} \right) = I \frac{d\Phi}{dt}. \quad (7.163)$$

This is in addition to Ohmic losses in the circuit which are not to be included in the magnetic-energy content. Thus if the flux changes through a circuit carrying a current $I$ is $\delta \Phi$, the work done by the source is

$$\delta W = I \delta \Phi \quad (7.164)$$

We now consider the problem of the work done in establishing a general steady-state distribution of currents and fields. For steady current

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{J} = 0 \quad (7.165)$$

For the loop with cross-sectional area, $\Delta \sigma$, following a closed path, $C$, and spanned by a surface $S$ with normal $\hat{n}$, we may write the flux change as

$$\delta \Phi = \int_S \vec{B} \cdot \hat{n} da \quad (7.166)$$

and the current

$$I = J \Delta \sigma \quad (7.167)$$

so that the work done becomes

$$\delta W = I \delta \Phi \Rightarrow \delta W = J \Delta \sigma \int_S \delta \vec{B} \cdot \hat{n} da. \quad (7.168)$$
In terms of the vector potential, 
\[ \delta \vec{B} = \nabla \times \delta \vec{A} \Rightarrow (\delta W) = J \Delta \sigma \int_S \nabla \times \delta \vec{A} \cdot \hat{n} da \] (7.169)
and applying Stoke’s theorem,
\[ (\delta W) = J \Delta \sigma \oint_C \delta \vec{A} \cdot d\vec{l}, \] (7.170)
where \( C \) is the closed curve bounding the surface, \( S \), as shown in Fig. 7.6. Noting that
\[ J \Delta \sigma d\vec{l} = \vec{J} d\tau, \] (7.171)
where \( d\tau \) is an infinitessimal volume, we can write
\[ \delta W = \int \delta \vec{A} \cdot \vec{J} d\tau \] (7.172)
Applying Ampere’s law
\[ \nabla \times \vec{H} = \vec{J} \] (7.173)
we may write
\[ \delta W = \int \delta \vec{A} \cdot \left( \nabla \times \vec{H} \right) d\tau \] (7.174)
and using the relation
\[
\nabla \cdot (\vec{C} \times \vec{D}) = \vec{D} \cdot (\nabla \times \vec{C}) - \vec{C} \cdot (\nabla \times \vec{D})
\]
\[
\Rightarrow \vec{C} \cdot (\nabla \times \vec{D}) = \vec{D} \cdot (\nabla \times \vec{C}) - \nabla \cdot (\vec{C} \times \vec{D})
\]
(7.175)

we find
\[
\delta W = \int \vec{H} \cdot (\nabla \times \delta \vec{A}) \, d\tau - \int \nabla \cdot (\delta \vec{A} \times \vec{H}) \, d\tau
\]
(7.176)
or
\[
\delta W = \int \left( \vec{H} \cdot \delta \vec{B} \right) \, d\tau - \oint_S (\delta \vec{A} \times \vec{H}) \cdot \vec{n} \, d\sigma
\]
(7.177)

We recall the vector potential for the auxiliary field are related to the volume current density
\[
\vec{A} = \frac{\mu_0}{4\pi} \int_{vol} \frac{\vec{J}(r')}{|r-r'|} \, d\tau'.
\]
(7.178)
\[
\nabla \times \vec{H} = \vec{J}
\]
(7.179)

If the field distribution is localized, for infinitely large radius,
\[
\int \vec{J}(r') \, d\tau' \sim \text{constant}, \frac{1}{|r-r'|} \sim \frac{1}{r}, \vec{H} \sim 0
\]
the second integral vanishes
\[
\delta W = \int \left( \vec{H} \cdot \delta \vec{B} \right) \, d\tau.
\]
(7.180)

If we assume that the medium is para-or diamagnetic, so that a linear relation exists between \( \vec{H} \) and \( \vec{B} \), we may write
\[
\vec{H} \cdot \delta \vec{B} = \frac{1}{2} \delta \left( \vec{H} \cdot \vec{B} \right)
\]
(7.181)
so that
\[
W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \, d\tau.
\]
(7.182)

In free space
\[
W = \frac{1}{2\mu_0} \int_{\text{All space}} B^2 \, d\tau.
\]
(7.183)

For an inductor with self inductance \( L \), we have
\[
\varepsilon = -L \frac{dI}{dt}.
\]
(7.184)
and
\[ \frac{dW}{dt} = -I_\varepsilon = -I \left( -L \frac{dI}{dt} \right) = LI \frac{dI}{dt}. \]  
(7.185)
gives
\[ dw = LI dI \Rightarrow W = \frac{1}{2} LI^2 \]  
(7.186)

**Example 7.17** A long coaxial cable carries a current \( I \) (the current flows down the inner cylinder, radius \( a \), and back along the surface of the outer cylinder, radius \( b \)) as shown in the Figure below. The current is uniformly distributed. Find the magnetic energy stored in a section of length \( l \).

![Diagram of coaxial cable](image)

**Solution:** In order to find the magnetic energy, we first need to know the magnetic field in the regions \( s = 0 \to \infty \). We use Ampere’s law to find the field
\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \]  
(7.187)

For an Amperian loop of radius, \( s < a \)
\[ I_{\text{enc}} = \int_0^s \vec{J} \cdot d\vec{a} = \int_0^s \frac{I}{\pi a^2} 2\pi s ds = \frac{Is^2}{a^2}, \]  
(7.188)

for Amperian loop of radius \( a < s < b \)
\[ I_{\text{enc}} = \int_0^a \vec{J} \cdot d\vec{a} = \int_0^a \frac{I}{\pi a^2} 2\pi s ds = I, \]  
(7.189)

and for Amperian loop of radius \( s > b \)
\[ I_{\text{enc}} = 0. \]  
(7.190)

Therefore the magnetic field in the three regions is given by
\[ \vec{B} = \begin{cases} \frac{\mu_0 Is}{2\pi a^2} \hat{\phi} & s < a \\ \frac{\mu_0 I}{2\pi s} & a < s < b \\ 0 & s > b \end{cases} \]  
(7.191)

Then the magnetic energy
\[ W = \frac{1}{2\mu_0} \int_{\text{All space}} B^2 d\tau = \frac{1}{2\mu_0} \int_0^\infty B^2 l^2 2\pi s ds \]  
(7.192)
becomes

$$W = \frac{1}{2\mu_0} \left[ \int_0^a \left( \frac{\mu_0 I}{2\pi a^2} \right)^2 ds + \int_a^b \left( \frac{\mu_0 I}{2\pi s} \right)^2 ds \right]$$

$$= \frac{1}{2\mu_0} \left( \frac{\mu_0 I}{2\pi} \right)^2 \left[ \frac{1}{a^4} \int_0^a s^3 ds + \int_a^b \frac{ds}{s} \right]$$

$$= \frac{1}{2\mu_0} \left( \frac{\mu_0 I}{2\pi} \right)^2 \left[ \frac{1}{a^4} + \ln \left( \frac{b}{a} \right) \right] \Rightarrow W = \frac{1}{4\pi} \left[ \frac{1}{4} + \ln \left( \frac{b}{a} \right) \right]. \quad (7.193)$$

7.7 Maxwell’s Equations

Electrodynamics before Maxwell: So far we have studied the following laws:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss’s Law)}, \quad \nabla \cdot \vec{B} = 0 \quad \text{(No name)},$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday’s Law)}, \quad \nabla \times \vec{B} = \mu_0 \vec{J} \quad \text{(Ampere’s Law).} \quad (7.194)$$

We know the divergence of a curl of any vector is zero. This is always true for the curl of the electric field vector

$$\nabla \cdot \left( \nabla \times \vec{E} \right) = -\frac{\partial \left( \nabla \cdot \vec{B} \right)}{\partial t} = 0,$$  \quad (7.195)

since $\nabla \cdot \vec{B} = 0$. However, if we take the divergence for the curl of the magnetic field (Ampere’s law), we find

$$\nabla \cdot \left( \nabla \times \vec{B} \right) = -\frac{\partial \left( \nabla \cdot \vec{J} \right)}{\partial t}. \quad (7.196)$$

This is true only when the current is steady. Which means the current density is independent of position ($\nabla \cdot \vec{J} = 0$) which requires no accumulation of charges in some specific regions ($\frac{\partial \vec{J}}{\partial t} = 0$). So far we have considered steady currents.

What if the current is not steady ($\nabla \cdot \vec{J} \neq 0$) and there is accumulation of charge in some region? A good example is a parallel plate capacitor connected to a battery as shown in the figure below.

Consider the flat circular area for the Amperian loop shown in the figure the current enclosed is just $I_{enc}$

$$I_{enc} = \int_{\text{flat surface}} \vec{J} \cdot d\vec{a} = I. \quad (7.197)$$

On the other hand, if we consider the balloon shaped surface shown in Fig. 7.10 there is no current passing through this surface and therefore the current
enclosed is zero
\[ I_{\text{enc}} = \int_{\text{balloon surface}} \vec{J} \cdot d\vec{a} = 0. \]  
(7.198)

Does that mean Ampere’s law is wrong? It is not wrong as long as the current is steady. However, in the case considered above the current is not steady because of the build up of charge on the capacitor \( \frac{\partial \rho}{\partial t} \neq 0 \). This suggested that Ampere’s Law need to be fixed so that it can also be used when the current is not steady and it was fixed by Maxwell. We recall from the conservation of charges
\[ \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \]  
(7.199)

using
\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \left( \nabla \cdot \vec{E} \right) \]  
(7.200)

we may write
\[ \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\epsilon_0 \nabla \cdot \left( \frac{\partial \vec{E}}{\partial t} \right) \Rightarrow \nabla \cdot \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0. \]  
(7.201)

Maxwell defined what he call the displacement current density as
\[ \vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \]  
(7.202)

and fixed Ampere’s law by adding this displacement current density to Eq. (7.196)
\[ \nabla \times \vec{B} = \mu_0 \left( \vec{J} + \vec{J}_d \right) = \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right). \]  
(7.203)
This additional term shows that just like a changing magnetic field induces an electric field (Faraday’s law), a changing electric field induces a magnetic field. Now if we go back to the capacitor problem and assume that the plates are very close and large so that the electric field in between the plates can be expressed as

\[ E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0} \frac{Q}{A}, \quad (7.204) \]

we may write

\[ \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial Q}{\partial t} = \frac{I}{\varepsilon_0} = \frac{J_d}{\varepsilon_0}. \quad (7.205) \]

Now let’s use the modified Ampere’s law (Maxwell’s equation) Eq. (7.203)

\[ \oint B \cdot d\ell = \int \nabla \times \mathbf{B} \cdot d\mathbf{\alpha} = \mu_0 \int \mathbf{J} \cdot d\mathbf{\alpha} = \mu_0 \int \mathbf{J} \cdot d\mathbf{\alpha} = \mu_0 \int \mathbf{J} \cdot d\mathbf{\alpha} \]

\[ \Rightarrow \oint B \cdot d\ell = \mu_0 \int \mathbf{J} \cdot d\mathbf{\alpha} = \mu_0 \int \mathbf{J} \cdot d\mathbf{\alpha} = \mu_0 \int \mathbf{J} \cdot d\mathbf{\alpha} \]

\[ \Rightarrow \oint B \cdot d\ell = \mu_0 I. \quad (7.206) \]

Now if we use the balloon surface area, we still have

\[ \int_{\text{balloon surface}} \mathbf{J} \cdot d\mathbf{\alpha} = 0 \quad (7.207) \]

but for the displacement current

\[ \int_{\text{balloon surface}} \mathbf{J}_d \cdot d\mathbf{\alpha} = \int_{\text{balloon surface}} \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{\alpha} = \int_{\text{balloon surface}} \varepsilon_0 \frac{I}{\varepsilon_0} \, da = I \]

\[ \Rightarrow \oint B \cdot d\ell = \mu_0 I. \quad (7.208) \]

On the other hand if we use the flat surface shown in the figure, \( \int \mathbf{J}_d \cdot d\mathbf{\alpha} = 0 \), since the electric field is zero in this region. But

\[ \int \mathbf{J} \cdot d\mathbf{\alpha} = I \Rightarrow \oint B \cdot d\ell = \mu_0 I. \quad (7.209) \]

So the problem is fixed. Thanks to Maxwell!

**Example 7.19** A flat wire, radius \( a \), carries a constant current \( I \), uniformly distributed over its cross section. A narrow gap in the wire of width \( w << a \), forms a parallel-plate capacitor, as shown in the figure below. Find the magnetic field in the gap, at a distance \( s < a \) from the axis.

**Solution:** The magnetic field in the gap satisfies the Maxwell equation

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \mathbf{J}_d \right) = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad (7.210) \]
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In order to find this \( \vec{B} \) field let’s consider an Amperian loop of radius \( s < a \) inside the gap. The displacement current crossing the area bounded by this loop is

\[
\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}.
\] (7.211)

Assuming at a given instant of time the charge accumulated on the surfaces inside the gap is \( q(t) \), the surface charge density can be expressed as

\[
\sigma(t) = \frac{q(t)}{A} = \frac{q(t)}{\pi a^2}
\] (7.212)

and for \( w << a \) the electric field inside the gap is due to this positive charge on one side and the corresponding negative charge on the other side. It can be expressed as

\[
\vec{E} = \frac{\sigma}{\epsilon_0} = \frac{q(t)}{\epsilon_0 \pi a^2} \hat{z}.
\] (7.213)

Then the displacement current density

\[
\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \left( \frac{q(t)}{\epsilon_0 \pi a^2} \right) \hat{z} \Rightarrow \vec{J}_d = \frac{1}{\pi a^2} \frac{\partial q}{\partial t} \hat{z} = \frac{I}{\pi a^2} \hat{z}
\] (7.214)

Therefore, the magnetic field inside the gap

\[
\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int \vec{J}_d \cdot d\vec{a} \Rightarrow B2\pi s = \mu_0 \int \frac{I}{\pi a^2} 2\pi s ds
\]

\[
\Rightarrow \vec{B} = \frac{\mu_0 Is}{2\pi a^2} \hat{\varphi}
\] (7.215)

7.8 Maxwell’s Equation in Matter

Maxwell’s Equations in free space are

\[
\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \nabla \cdot \vec{B} = 0,
\]

\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}.
\] (7.216)

When the medium is a polarizable and magnetizable medium, we know that the charge and current density need to be replaced by

\[
\rho = \rho_{\text{free}} + \rho_{\text{bound}}, \vec{J} = \vec{J}_{\text{free}} + \vec{J}_{\text{bound}} + \vec{J}_{\text{polarization}},
\] (7.217)
where
\[ \rho_{\text{bound}} = -\nabla \cdot \vec{P}, \quad \vec{J}_{\text{bound}} = \nabla \times \vec{M}, \quad (7.218) \]
which we have derived when we study electric field and magnetic fields in matter in the previous chapters. The polarization current density, \( \vec{J}_{\text{polarization}} \), is what we have not seen so far. This current is a result of the change in polarization with time. It is important to note here that the bound current is a result of spin and orbital motion of the electron where as polarization current is a result of a small displacement of the bound charges and it is given by
\[ \vec{J}_{\text{polarization}} = \frac{\partial \vec{P}}{\partial t}. \quad (7.219) \]

Now replacing the charge and current densities, we can write
\[
\nabla \cdot \vec{E} = \frac{\rho_{\text{free}} + \rho_{\text{bound}}}{\varepsilon_0},
\]
\[
\nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left( \vec{J}_{\text{free}} + \vec{J}_{\text{bound}} + \vec{J}_{\text{polarization}} \right), \quad (7.220)
\]
and using the relations above
\[
\nabla \cdot \vec{E} = \frac{\rho_{\text{free}} - \nabla \cdot \vec{P}}{\varepsilon_0}, \quad \nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left( \vec{J}_{\text{free}} + \nabla \times \vec{M} + \frac{\partial \vec{B}}{\partial t} \right)
\]
\[
\Rightarrow \nabla \cdot \left( \varepsilon_0 \vec{E} + \vec{P} \right) = \rho_{\text{free}}, \nabla \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) - \frac{\partial \left( \varepsilon_0 \vec{E} + \vec{P} \right)}{\partial t} = (7.221)
\]
Recalling that
\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}. \quad (7.222) \]
Maxwell’s equations in matter can be expressed as
\[
\nabla \cdot \vec{D} = \frac{\rho_{\text{free}}}{\varepsilon_0}, \quad \nabla \cdot \vec{B} = 0
\]
\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_{\text{free}}. \quad (7.223)
\]

**Example 7.20** Consider a large parallel-plate capacitor immersed in a sea water and driven by a voltage \( V_0 \cos(2\pi vt) \). Sea water at frequency, \( \nu = 4 \times 10^8 \text{Hz} \), has permittivity \( \varepsilon = 81\varepsilon_0 \), permeability \( \mu = \mu_0 \), and resistivity, \( \rho = 0.23\Omega.m \).

(a) Find the free (conduction), polarization, and bound current densities.

(b) What is the ratio of the amplitude of the conduction current to displacement current?

**Solution:**
Consider a parallel-plate capacitor of area $A$ driven by $V_0 \cos(2\pi vt)$. If this capacitor is immersed in a water and the plates are separated by a distance $l$, the resistance $R$ of the water can be expressed as

$$R = \rho \frac{l}{A}. \quad (7.224)$$

Then the free current $I_f$ in the water is given by

$$I_f = \frac{V_0 \cos(2\pi vt)}{R} = \frac{V_0 A \cos(2\pi vt)}{\rho l} \Rightarrow J_f = \frac{I_f}{A} = \frac{V_0 \cos(2\pi vt)}{\rho l}. \quad (7.225)$$

We recall that the displacement current density $\vec{J}_d$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}. \quad (7.226)$$

Where $\vec{E}$ is the electric field inside the parallel-plate capacitor. If we assume the surface charge density is $\sigma$, the electric field inside the capacitor (if it is in a free space) is

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} = \frac{Q}{A \epsilon_0} \hat{z}. \quad (7.227)$$

Using Eq. (7.227), Eq. (7.226) can be written as

$$\vec{J}_d = \epsilon \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{V_0 \cos(2\pi vt)}{l} \hat{z} \right] = -\epsilon \frac{2\pi \nu V_0 \sin(2\pi vt)}{l} \hat{z}. \quad (7.228)$$

Since the electric field changes with time, the polarization $\vec{P}$ changes with time

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \frac{\epsilon - \epsilon_0}{\epsilon_0} \vec{E}. \quad (7.229)$$
and this gives rise to a polarization current. The polarization current density
\[
\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t} \left( \varepsilon_0 \varepsilon_r \mathbf{E} \right) = (\varepsilon - \varepsilon_0) \mathbf{E}_0 \frac{\partial \mathbf{E}}{\partial t} = - \left( \frac{\varepsilon - \varepsilon_0}{\varepsilon_0} \right) \frac{2\pi \nu V_0 \cos (2\pi \nu t)}{1} \mathbf{\hat{z}}.
\]
(7.230)
The bound current density is given by
\[
\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times \left( \chi_m \mathbf{H} \right) = \chi_m \nabla \times \left( \frac{\mathbf{B}}{\mu} \right) \Rightarrow \mathbf{J}_b = \chi_m \frac{\nabla \times \mathbf{B}}{\mu}. \quad (7.231)
\]
But we know that if the capacitor were in free space
\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_f,
\]
(7.232)
hence Eq. (7.231) can be expressed in terms of the free current density as
\[
\mathbf{J}_b = \chi_m \frac{\mu_0}{\mu} \mathbf{J}_f \quad (7.233)
\]
or using
\[
\mu = \mu_0 (1 + \chi_m) \Rightarrow \chi_m = \frac{\mu - \mu_0}{\mu_0} \quad (7.234)
\]
we find
\[
\mathbf{J}_b = \frac{\mu - \mu_0}{\mu} \mathbf{J}_f. \quad (7.235)
\]
(b) The ration of the amplitude of the conduction current to the displacement current will then be given by
\[
\frac{J_f}{J_d} = \frac{V_0}{\rho l} \frac{l}{\varepsilon 2\pi \nu V_0} = \frac{1}{2\pi \nu \rho} = 2.41 \quad (7.236)
\]
Example 7.21 A perfectly conducting spherical shell of radius \(a\) rotates about the \(z\) axis with angular velocity \(\omega\), in a uniform magnetic field, \(\mathbf{B} = B_0 \mathbf{\hat{z}}\). Calculate the emf developed between the "north pole" and the equator.

Solution: The motional emf induced is the result of the magnetic force exerted on the free charges inside the conductor. The magnetic force per unit charge responsible for the induced emf can be expressed as
\[
\mathbf{f}_m = \mathbf{v} \times \mathbf{B}.
\]
(7.237)
Noting that the shell is rotating about the \(z\) axis and the angular velocity is \(\omega\), the velocity of a charge on the surface of the shell is given by
\[
\mathbf{v} = \omega a \sin (\theta) \mathbf{\hat{r}}.
\]
(7.238)
Using the magnetic field in spherical coordinates

$$\vec{B} = B_0 \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right)$$  \hspace{1cm} (7.239)

the magnetic force per unit charge can be written as

$$\vec{f}_m = \omega a \sin (\theta) \hat{\varphi} \times \left( B_0 \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right) \right)$$

$$\Rightarrow \vec{f}_m = B_0 a \omega \left( \sin \theta \cos \theta \hat{\theta} + \sin^2 \theta \hat{r} \right). \hspace{1cm} (7.240)$$

We recall that induced emf is given by

$$\varepsilon = \int \vec{f}_m \cdot d\vec{l}. \hspace{1cm} (7.241)$$

Since we are interested in the emf developed between the north pole and the equator, we may write

$$d\vec{l} = a d\theta \hat{\theta} \hspace{1cm} (7.242)$$

and the emf developed can be written as

$$\varepsilon = \int \vec{f}_m \cdot d\vec{l} = B_0 a \omega \int_0^{\pi/2} \left( \sin \theta \cos \theta \hat{\theta} + \sin^2 \theta \hat{r} \right) \cdot a d\theta \hat{\theta}$$

$$\Rightarrow \varepsilon = B_0 a^2 \omega \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \frac{B_0 a^2 \omega}{2} \left. \sin^2 \theta \right|_0^{\pi/2} \Rightarrow \varepsilon = \frac{1}{2} B_0 a^2 \omega \hspace{1cm} (7.243)$$

**Example 7.22 Transformer**: Two coils are wound around a Ferromagnetic material as shown in Fig. 7.11. The Ferromagnetic material allows to
create same flux, $\Phi$, through every winding in both coils. The "primary" winding has $N_P$ turns and the secondary winding has $N_S$ turns. If the current, $I_P$, in the primary winding is changing, show that the emf in the secondary winding is given by

$$\varepsilon_S = \frac{N_S}{N_P} \varepsilon_P$$

where $\varepsilon_P$ is the (back) emf which is equal to the source voltage supplying the changing current, $I_P$, in the primary.

Solution: Due to the current in the primary coil we have magnetic field directed parallel to the axial direction. Let the flux due to this magnetic field on a single loop which could be in the primary or secondary is $\Phi$. Then when the current in the primary changes the magnetic field changes and so does the flux. This change in magnetic flux produces a back emf on the primary which can be expressed as

$$\varepsilon_P = N_P \frac{d\Phi}{dt}$$

On the other hand it induces a direct emf on the secondary given by

$$\varepsilon_S = N_S \frac{d\Phi}{dt}$$

Therefore the emf in the secondary can be related to the primary by

$$\frac{\varepsilon_S}{\varepsilon_P} = \frac{N_S}{N_P}$$
Example 7.23 A transformer in (Example 7.22) takes an input AC voltage of Amplitude, $V_P = \varepsilon_P$, and delivers an output voltage of Amplitude, $V_S$, which is determined by the turns ratio ($V_S/V_P = N_S/N_P$). If $N_S > N_P$ the output voltage is greater than the input voltage.

(a) Why does not this violate conservation of energy?

(b) In ideal transformer the same flux passes through all turns of the primary and of the secondary as shown in Fig. 7.11. Show that in this case, $M^2 = L_P L_S$, where $M$ is the mutual inductance of the coils, and $L_P$, $L_S$ are their individual self-inductances.

Solution:

(a) If we look at the power it is the product of voltage and current. Since when the voltage goes up, the current comes down the power remain the same and there will not be violation of conservation of energy.

(b) The flux in the primary can be expressed in terms of the self and mutual inductances

$$\Phi_P = L_P I_P - M I_S$$ (7.248)

Similarly the flux on the secondary

$$\Phi_S = M I_P - L_S I_S.$$ (7.249)

The flux per single turn can then be expressed as

$$\Phi = \frac{\Phi_P}{N_P} = \frac{\Phi_S}{N_S} \Rightarrow \frac{L_P}{N_P} I_P - \frac{M}{N_P} I_S = \frac{M}{N_S} I_P - \frac{L_S}{N_S} I_S.$$ (7.250)

If the current in the primary is zero ($I_P = 0$), we get

$$\frac{M}{N_P} I_S = \frac{L_S}{N_S} I_S \Rightarrow \frac{M}{N_P} = \frac{L_S}{N_S}.$$ (7.251)

and if it is zero in the secondary ($I_S = 0$)

$$\frac{L_P}{N_P} = \frac{M}{N_S}.$$ (7.252)

Taking the ratio of the two result, we find

$$\frac{M}{N_P} = \frac{L_S}{N_S} \Rightarrow \frac{M}{L_P} = \frac{L_S}{M} \Rightarrow M^2 = L_P L_S,$$ (7.253)

for the mutual inductance.
Chapter 8

Electromagnetic conservation laws

The motion of interacting particles with no charges is governed by three fundamental conservation laws. These are conservation of linear momentum, conservation of angular momentum, and conservation of energy.

(a) Conservation of energy: For an object subject to translational, rotational, or both motion, the work done by the net external force acting on the object is equal to the change in energy.

\[ dW = \vec{F}_{\text{net}} \cdot d\vec{r} \]

(b) Conservation of linear momentum: The net external force, \( \vec{F}_{\text{net}} \), acting on an object is equal to the change in linear momentum, \( \vec{P} \).

\[ \vec{F}_{\text{net}} = \frac{d\vec{P}}{dt} \]

(c) Conservation of angular momentum: For an object that is under a rotational motion, the net torque, \( \vec{\tau} \), on the object is equal to the change in angular momentum, \( \vec{L} \).

\[ \vec{\tau} = \frac{d\vec{L}}{dt} \]

What we studied as three conservation laws actually given by one conservation law—conservation of four-dimensional momentum in general theory of relativity where time and space form a four-dimensional coordinate manifold, (Minkowski spacetime manifold: \( x^1 = ct, x^2 = x, x^3 = y, x^3 = z \)). Each coordinate is parameterized by what is known as the proper time, \( \tau \). The three laws can beautifully derived from the conservation of 4D momentum.

In this chapter we shall see how these conservation laws works when the objects in motion carry a charge that leads to fields (Electric and Magnetic) and
create electrical and magnetic interaction which adds electrical and magnetic energy to the system. We already know that when there is a system of charged objects in a region described by a charge density, \( \rho (\vec{r}, t) \), and if there are charges moving into or out of this region with a velocity \( \vec{v}(\vec{r}, t) \), there is one other conservation law that we need to include, the conservation of charge. (see Fig. 8.1) This is given by the continuity equation,

\[
\frac{\partial \rho (\vec{r}, t)}{\partial t} = -\nabla \cdot \vec{J}(\vec{r}, t),
\]

where \( \rho \) is the charge density and

\( \vec{J} = \vec{v}(\vec{r}, t) \rho (\vec{r}, t) \)

is the current density

### 8.1 Conservation of energy

Suppose we have some charge distribution shown in Fig. 8.1 with charge density, \( \rho (\vec{r}, t) \). The electric field and magnetic field in this region are, \( \vec{E}(\vec{r}, t) \), and magnetic field, \( \vec{B}(\vec{r}, t) \), respectively, at time \( t \). Consider an infinitesimal charge, \( dq \), in an infinitesimal volume, \( d\vec{r} \), in this region. The work done required to move this charge to move a displacement, \( d\vec{r} \), in the time interval \( dt \), is given by,

\[
d (\delta W) = \delta \vec{F} \cdot d\vec{r},
\]

where \( \delta W \) is the change in energy, \( \delta \vec{F} \) is the applied force on the charge, and \( d\vec{r} \) is the displacement.
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where \( \delta \vec{F} \) is the Lorentz force given by

\[
\delta \vec{F} = dq \left( \vec{E}(\vec{r}, t) + \vec{v} \times \vec{B}(\vec{r}, t) \right),
\]

(8.3)

where \( \vec{v}(\vec{r}, t) = \frac{d\vec{r}}{dt} \).

Note that \( \vec{E}(\vec{r}, t) \) and \( \vec{B}(\vec{r}, t) \) are the electric and magnetic field of all the other charges at the position of charge \( dq \), which we described by the vector, \( \vec{r} \). The work done per unit time can then be expressed as

\[
\frac{d(\delta W)}{dt} = \delta \vec{F} \cdot \frac{d\vec{r}}{dt} = dq \left( \vec{E} + \vec{v} \times \vec{B} \right) \cdot \vec{v} = q\vec{E} \cdot \vec{v} + q \left( \vec{v} \times \vec{B} \right) \cdot \vec{v}.
\]

(8.4)

Since \( \vec{v} \times \vec{B} \) is normal to \( \vec{v} \), the second term in the above expression becomes zero

\[
\frac{d(\delta W)}{dt} = dq \vec{E} \cdot \vec{v}
\]

(8.5)

Noting that for infinitesimal charge, \( dq \)

\[
dq = \rho(\vec{r}, t) d\tau
\]

(8.6)

we may write

\[
\frac{d(\delta W)}{dt} = dq \vec{E} \cdot \vec{v} \Rightarrow \frac{dW}{dt} = \int \vec{E}(\vec{r}, t) \cdot (\rho(\vec{r}, t) \vec{v}(\vec{r}, t)) d\tau.
\]

(8.7)

For a volume charge moving with a velocity, \( \vec{v} \), we can use the current density

\[
\vec{J}(\vec{r}, t) = \rho(\vec{r}, t) \vec{v}(\vec{r}, t)
\]

(8.8)

which leads to

\[
\frac{dW}{dt} = \int \left[ \vec{E} \cdot \vec{J} \right] d\tau.
\]

(8.9)

Using one of Maxwell's equation

\[
\nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \Rightarrow \vec{J} = \frac{1}{\mu_0} \left( \nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)
\]

(8.10)

we may write

\[
\frac{dW}{dt} = \frac{1}{\mu_0} \int \vec{E} \left( \nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) d\tau = \frac{1}{\mu_0} \int \left[ \vec{E} \cdot \left( \nabla \times \vec{B} \right) - \mu_0 \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right] d\tau.
\]

(8.11)
Applying the product rule, we have
\[ \nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B}) \]
\[ \Rightarrow \vec{E} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{B}) \]  
(8.12)
and using Faraday’s law
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]  
(8.13)
\[ \vec{E} \cdot (\nabla \times \vec{B}) = -\vec{B} \cdot \left( \frac{\partial \vec{B}}{\partial t} \right) - \nabla \cdot (\vec{E} \times \vec{B}). \]  
(8.14)

The work done per unit time
\[ \frac{dW}{dt} = \frac{1}{\mu_0} \int_V \left( \vec{E} \cdot (\nabla \times \vec{B}) - \mu_0 \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right) \, d\tau \]  
(8.15)
becomes
\[ \frac{dW}{dt} = -\int_V \left( \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} + \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right) \, d\tau - \frac{1}{\mu_0} \int_V \nabla \cdot (\vec{E} \times \vec{B}) \, d\tau. \]  
(8.16)
Noting that
\[ \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{d}{dt} (\vec{B} \cdot \vec{B}) = \frac{1}{2} \frac{d}{dt} (B^2), \]
\[ \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{d}{dt} (E^2), \]
\[ \int_V \nabla \cdot (\vec{E} \times \vec{B}) \, d\tau = \oint_S (\vec{E} \times \vec{B}) \cdot d\vec{a} \]  
(8.17)
we can write
\[ \frac{dW}{dt} = -\frac{d}{dt} \int_V \left( \frac{1}{\mu_0} B^2 + \epsilon_0 \vec{E}^2 \right) \, d\tau - \frac{1}{\mu_0} \oint_S (\vec{E} \times \vec{B}) \cdot d\vec{a}. \]  
(8.19)
In terms of the Poynting vector, which is defined as energy per unit time per unit area transported by the fields (Intensity), given by
\[ \vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \]  
(8.20)
one can express the work done per unit time as
\[ \frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S} \cdot d\vec{a}. \]  
(8.21)
This is the Poynting theorem. It is the "work-energy theorem" of electrodynamics which usually expressed as in the form

$$\frac{dU_{em}}{dt} + \oint_S \mathbf{S} \cdot d\mathbf{a} = -\frac{dW}{dt}.$$  \hspace{1cm} (8.22)

It is the statement of conservation of energy. The physical meaning of this equation is that the time rate of change of the electromagnetic energy within a certain volume plus the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume.

**Example 8.1** A cylindrical conductor of length, \(L\), and radius, \(a\), is connected to a dc power supply which generates a voltage \(V\) and a current \(I\). (Fig.8.1). Find the power delivered by the source using the pointing vector.

![Diagram of a cylindrical conductor](image)

**Solution:** We already know the power delivered by the source is just

$$P = VI.$$  \hspace{1cm} (8.23)

Here we want to find relation using the pointing vector. Let the direction of the positive \(z\) axis be along the axis of the cylinder parallel to the direction of the current flow. Then using the electric and magnetic field vectors given by

$$\mathbf{E} = \frac{V}{L} \hat{z}, \quad \mathbf{B} = \frac{\mu_0 I s}{2\pi a^2} \hat{\varphi},$$  \hspace{1cm} (8.24)

the poynting vector can be expressed as

$$\mathbf{S} = \frac{1}{\mu_0} \left( \mathbf{E} \times \mathbf{B} \right) = \frac{1}{\mu_0} \left( \frac{V}{L} \hat{z} \right) \times \left( \frac{\mu_0 I s}{2\pi a^2} \hat{\varphi} \right) = \frac{V I s}{2\pi L a^2} \left( -\hat{z} \right).$$  \hspace{1cm} (8.25)

This show the amount of energy per unit area per unit time propagating radially inwards. Then the power flow (energy delivered by the power source each second) can be expresses as

$$P = \left| \oint \mathbf{S} \cdot d\mathbf{a} \right| = \frac{V I a}{2\pi L a^2} (2\pi a l) = VI.$$  \hspace{1cm} (8.26)
8.2 Conservation of momentum

We recall that momentum is conserved when the sum of all external forces is zero and the internal forces cancel each other. In magnetostatic and electrostatic we have seen that electrical forces have the same magnitude but opposite direction and the same is true in magnetostatic. In order the momentum to remain conserved in electrodynamics the internal forces must cancel out. Consider the following two positive charges. One is forced to move on a track along the x-axis with a velocity, $v_1$, and the other is forced to move along the y-axis with velocity, $v_2$, as shown in the Fig. 8.2.

![Figure 8.2: Two moving point charges.](image)

The electrostatic forces by one to another ($\vec{F}_{E1}$ and $\vec{F}_{E2}$) add up to zero since these forces are equal in magnitude and opposite in direction. However, the magnetic forces ($\vec{F}_{B1}$ and $\vec{F}_{B2}$) even though are equal in magnitude do not have the same direction and the net magnetic forces do not add up to zero. It seems the internal forces do not cancel each other which is in violation of Newton’s third law. Does this mean momentum is not conserved? Absolutely not!!! So how can the law of conservation of momentum is rescued in electrodynamics? Well if we realize that the fields themselves carry momentum, conservation of momentum will be rescued in electrodynamics. Momentum lost by the charges in the example above is gained by the fields.

Let’s re-consider the general case where we have a volume of charge with charge density, $\rho(\vec{r}, t)$, shown in Fig. 8.3. The force on an infinitesimal charge, $dq$ in an infinitesimal volume, $d\tau$,

$$dq = \rho(\vec{r}, t)$$

moving with a velocity,

$$\vec{v} = \frac{d\vec{r}}{dt}$$
8.2. CONSERVATION OF MOMENTUM

is determined using Lorentz force

\[ d\vec{F}(\vec{r}) = \left( \vec{E}(\vec{r}, t) + \vec{v} \times \vec{B}(\vec{r}, t) \right) \rho(\vec{r}, t) \, d\tau. \]  \hfill (8.27)

Note that \( \vec{E}(\vec{r}, t) \) and \( \vec{B}(\vec{r}, t) \) are the electric and magnetic field of all the other charges at the position of charge \( dq \), which we described by the vector, \( \vec{r} \).

The "force density" which we define as the force per unit volume in the region, can be expressed as

\[ \vec{f} = \frac{d\vec{F}}{d\tau} = \rho \vec{E} + \rho \vec{v} \times \vec{B} = \rho \vec{E} + \vec{J} \times \vec{B}. \]  \hfill (8.28)

Applying Maxwell’s equations

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0 \nabla \cdot \vec{E} \]
\[ \nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \Rightarrow \vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \]  \hfill (8.29)

we may write the force density as

\[ \vec{f} = \left( \varepsilon_0 \nabla \cdot \vec{E} \right) \vec{E} + \left( \frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B} \]
\[ \Rightarrow \vec{f} = \left( \varepsilon_0 \nabla \cdot \vec{E} \right) \vec{E} + \frac{1}{\mu_0} \left( \nabla \times \vec{B} \right) \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} \]  \hfill (8.30)
CHAPTER 8. ELECTROMAGNETIC CONSERVATION LAWS

Noting that
\[
\frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t}
\] (8.31)
and using Faraday’s law
\[
\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \Rightarrow \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \vec{E} \times (\nabla \times \vec{E})
\]
\[
\Rightarrow \frac{\partial \vec{E}}{\partial t} \times \vec{B} = \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\nabla \times \vec{E})
\] (8.32)
the force density can then be expressed as
\[
\vec{f} = (\varepsilon_0 \nabla \cdot \vec{E}) \vec{E} + \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} - \varepsilon_0 \left[ \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\nabla \times \vec{E}) \right] + \vec{E} \times (\nabla \times \vec{E})
\]
\[
\Rightarrow \vec{f} = \varepsilon_0 \left[ (\nabla \cdot \vec{E}) \vec{E} - \vec{E} \times (\nabla \times \vec{E}) \right] + \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} - \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) .
\] (8.33)
Since
\[
\nabla \cdot \vec{B} = 0 \Rightarrow (\nabla \cdot \vec{B}) \vec{B} = 0
\] (8.34)
we can rewrite the force density as
\[
\vec{f} = \varepsilon_0 \left[ (\nabla \cdot \vec{E}) \vec{E} - \vec{E} \times (\nabla \times \vec{E}) \right] + \frac{1}{\mu_0} \left[ (\nabla \cdot \vec{B}) \vec{B} - \vec{B} \times (\nabla \times \vec{B}) \right] - \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) .
\] (8.35)
Using the product rule
\[
\nabla (E^2) = 2 (\vec{E} \cdot \nabla) \vec{E} + 2 \vec{E} \times (\nabla \times \vec{E}) \Rightarrow \nabla \times (\nabla \times \vec{E}) = \frac{1}{2} \nabla (E^2) - (\vec{E} \cdot \nabla) \vec{E} .
\] (8.36)
and with similar expression for the magnetic field
\[
\vec{B} \times (\nabla \times \vec{B}) = \frac{1}{2} \nabla (B^2) - (\vec{B} \cdot \nabla) \vec{B}
\] (8.37)
the force density put in the form
\[
\vec{f} = \varepsilon_0 \left[ (\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla \cdot (E^2) \right] + \frac{1}{\mu_0} \left[ (\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla \cdot (B^2) \right] - \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) .
\] (8.38)
First let’s expand the first term that involves the electric field only

\[
\left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})
\]

\[
+ \left( \frac{E_x}{\partial x} \right) \left( \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})
\]

\[
= \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})
\]

\[
+ \left( \frac{\partial E_x}{\partial x} \right) (E_y \hat{y} + E_z \hat{z}) \hat{y} + \left( \frac{\partial E_y}{\partial y} \right) (E_x \hat{x} + E_z \hat{z}) \hat{x}
\]

\[
+ \left( \frac{\partial E_z}{\partial z} \right) (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \hat{z} \quad (8.39)
\]

\[
\Rightarrow \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \frac{\partial E_x}{\partial x} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})
\]

\[
+ \frac{\partial E_y}{\partial y} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \hat{y} + \frac{\partial E_z}{\partial z} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \hat{z} \quad (8.40)
\]

\[
\Rightarrow \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \frac{\partial E_x}{\partial x} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \hat{x}
\]

\[
+ \frac{\partial E_y}{\partial y} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \hat{y} + \frac{\partial E_z}{\partial z} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \hat{z} \quad (8.41)
\]

\[
\Rightarrow \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \frac{\partial E_x}{\partial x} (E_x \hat{x} \cdot \hat{x}) \hat{x} + \frac{\partial E_y}{\partial y} (E_x \hat{x} \cdot \hat{y}) \hat{y} + \frac{\partial E_z}{\partial z} (E_x \hat{x} \cdot \hat{z}) \hat{z} \quad (8.42)
\]

\[
\Rightarrow \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \frac{\partial E_x}{\partial x} (E_x \hat{x} \cdot \hat{x}) \hat{x} + \frac{\partial E_y}{\partial y} (E_x \hat{x} \cdot \hat{y}) \hat{y} + \frac{\partial E_z}{\partial z} (E_x \hat{x} \cdot \hat{z}) \hat{z} \quad (8.43)
\]
\[
\n\Rightarrow \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot \left( E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \right) \\
+ E_x E_z \hat{x} \hat{z} + \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot \left( E_y E_x \hat{x} \hat{y} + E_y E_y \hat{y} \hat{y} + E_z E_z \hat{z} \hat{z} \right) \\
+ \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot \left( E_z E_x \hat{x} \hat{z} + E_y E_y \hat{y} \hat{z} + E_x E_x \hat{x} \hat{x} \right) \\
= \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \cdot \left( E_x E_x \hat{x} + E_y E_y \hat{y} + E_z E_z \hat{z} \right) \\
\]

in a matrix form this can be written as
\[
\Rightarrow \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} E_x & E_y & E_z \\ E_x & E_y & E_z \\ E_x & E_y & E_z \end{pmatrix}
\]

Noting that
\[
\frac{1}{2} \nabla (E^2) = \frac{1}{2} \left( \frac{\partial E^2}{\partial x} \hat{x} + \frac{\partial E^2}{\partial y} \hat{y} + \frac{\partial E^2}{\partial z} \hat{z} \right) \\
\Rightarrow \frac{1}{2} \nabla (E^2) = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{1}{2}E^2 & 0 & 0 \\ 0 & \frac{1}{2}E^2 & 0 \\ 0 & 0 & \frac{1}{2}E^2 \end{pmatrix}
\]

we can write
\[
\left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \begin{pmatrix} E_x & E_y & E_z \\ E_x & E_y & E_z \\ E_x & E_y & E_z \end{pmatrix} - \begin{pmatrix} \frac{1}{2}E^2 & 0 & 0 \\ 0 & \frac{1}{2}E^2 & 0 \\ 0 & 0 & \frac{1}{2}E^2 \end{pmatrix}
\]

Following a similar procedure we can show
\[
\left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \begin{pmatrix} B_x & B_y & B_z \\ B_x & B_y & B_z \\ B_x & B_y & B_z \end{pmatrix} - \begin{pmatrix} \frac{1}{2}B^2 & 0 & 0 \\ 0 & \frac{1}{2}B^2 & 0 \\ 0 & 0 & \frac{1}{2}B^2 \end{pmatrix}
\]

so that
\[
ce_0 \left[ \left( \nabla \cdot \vec{E} \right) \vec{E} + \left( \vec{E} \cdot \nabla \right) \vec{E} - \frac{1}{2} \nabla (E^2) \right] \\
+ \frac{1}{\mu_0} \left[ \left( \nabla \cdot \vec{B} \right) \vec{B} + \left( \vec{B} \cdot \nabla \right) \vec{B} - \frac{1}{2} \nabla (B^2) \right] = \nabla \cdot \vec{T}
\]
where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}$$

$$\mathbf{T} = \epsilon_0 \begin{pmatrix} E_x E_x \hat{x} & E_x E_y \hat{y} & E_x E_z \hat{z} \\ E_x E_y \hat{x} & E_y E_y \hat{y} & E_y E_z \hat{z} \\ E_x E_z \hat{x} & E_y E_z \hat{y} & E_z E_z \hat{z} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} E^2 \hat{x} & 0 & 0 \\ 0 & \frac{1}{2} E^2 \hat{y} & 0 \\ 0 & 0 & \frac{1}{2} E^2 \hat{z} \end{pmatrix}$$

$$+ \frac{1}{\mu_0} \begin{pmatrix} B_x B_x \hat{x} & B_x B_y \hat{y} & B_x B_z \hat{z} \\ B_y B_x \hat{x} & B_y B_y \hat{y} & B_y B_z \hat{z} \\ B_z B_x \hat{x} & B_z B_y \hat{y} & B_z B_z \hat{z} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} B^2 \hat{x} & 0 & 0 \\ 0 & \frac{1}{2} B^2 \hat{y} & 0 \\ 0 & 0 & \frac{1}{2} B^2 \hat{z} \end{pmatrix} \quad (8.49)$$

The matrix, $\mathbf{T}$, defined by its elements, $T_{ij}$, given by

$$T_{ij} = \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

is called the Maxwell’s stress tensor. Using this tensor and the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \left( \mathbf{E} \times \mathbf{B} \right) \quad (8.50)$$

the force density can be written as

$$\mathbf{f} = \nabla \cdot \mathbf{T} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t} \quad (8.51)$$

The total force on a charge inside a volume $V$ can then be expressed as

$$\mathbf{F} = \int_V \mathbf{f} d\tau = \int_V \left( \nabla \cdot \mathbf{T} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t} \right) d\tau$$

$$= \int_V \left( \nabla \cdot \mathbf{T} \right) d\tau - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau \Rightarrow \mathbf{F} = \int_S \mathbf{T} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau. \quad (8.52)$$

In the static case since

$$\frac{d}{dt} \int_V \mathbf{S} d\tau = 0$$

the total electromagnetic force is given by

$$\mathbf{F} = \int_S \mathbf{T} \cdot d\mathbf{a}. \quad (8.53)$$

**Shorter approach:** using Einstein summation convention one can write

$$\left( \nabla \cdot \mathbf{E} \right) \mathbf{E} = \frac{\partial E_i}{\partial x_i} E_j, \quad \left( \mathbf{E} \cdot \nabla \right) \mathbf{E} = E_i \frac{\partial E_j}{\partial x_i} \nabla \left( E^2 \right) = \delta_{ij} \frac{\partial}{\partial x_i} \left( E_i E_j \right) \quad (8.54)$$
so that
\[
\left[ (\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla (E^2) \right] = \frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2} \delta_{ij} \frac{\partial}{\partial x_i} (E_i E_j)
\]
\[
= \frac{\partial}{\partial x_i} (E_i E_j) - \frac{1}{2} \delta_{ij} \frac{\partial}{\partial x_i} (E_i E_j) = \frac{\partial}{\partial x_i} \left[ E_i E_j - \frac{1}{2} \delta_{ij} E_i E_j \right]
\]
(8.55)
where repeated index refers to summation and \(i, j = 1, 2, 3\) which corresponds to (\(x, y,\) and \(z\)). Similarly,
\[
\left[ (\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla (B^2) \right] = \frac{\partial}{\partial x_i} \left[ B_i B_j - \frac{1}{2} \delta_{ij} B_i B_j \right]
\]
(8.56)
Then introducing a tensor quantity, \(\vec{T}\), defined by the elements
\[
T_{ij} = \epsilon_0 \left[ E_i E_j - \frac{1}{2} \delta_{ij} E_i E_j \right] + \frac{1}{\mu_0} \left[ B_i B_j - \frac{1}{2} \delta_{ij} B_i B_j \right]
\]
(8.57)
one can then write
\[
\epsilon_0 \left[ (\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla (E^2) \right] + \frac{1}{\mu_0} \left[ (\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla (B^2) \right] = \frac{\partial}{\partial x_i} T_{ij}
\]
(8.58)
and noting that the gradient
\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)
\]
(8.59)
using matrices one can write
\[
\epsilon_0 \left[ (\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla (E^2) \right] + \frac{1}{\mu_0} \left[ (\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B} - \frac{1}{2} \nabla (B^2) \right] = \left( \begin{array}{ccc}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{array} \right)
\]
(8.60)
Note that
\[
\left( \begin{array}{ccc}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}
\end{array} \right) = \left( \begin{array}{ccc}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{array} \right)
\]
\[
\left( \begin{array}{ccc}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{array} \right) = \epsilon_0 \left( \begin{array}{ccc}
E_x E_x & E_x E_y & E_x E_z \\
E_y E_x & E_y E_y & E_y E_z \\
E_z E_x & E_z E_y & E_z E_z
\end{array} \right) - \left( \begin{array}{ccc}
\frac{1}{2} E^2 & 0 & 0 \\
0 & \frac{1}{2} E^2 & 0 \\
0 & 0 & \frac{1}{2} E^2
\end{array} \right)
\]
\[
+ \frac{1}{\mu_0} \left( \begin{array}{ccc}
B_x B_x & B_x B_y & B_x B_z \\
B_y B_x & B_y B_y & B_y B_z \\
B_z B_x & B_z B_y & B_z B_z
\end{array} \right) - \left( \begin{array}{ccc}
\frac{1}{2} B^2 & 0 & 0 \\
0 & \frac{1}{2} B^2 & 0 \\
0 & 0 & \frac{1}{2} B^2
\end{array} \right)
\]
(8.61)
8.2. CONSERVATION OF MOMENTUM

(a) Conservation of linear Momentum: Newton’s second law states that

\[ \vec{F} = \frac{d \vec{P}_{\text{mech}}}{dt} \Rightarrow \oint_{S} \vec{T} \cdot d\vec{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_{V} \vec{S} d\tau = \frac{d \vec{P}_{\text{mech}}}{dt} \]

\[ \Rightarrow \oint_{S} \vec{T} \cdot d\vec{a} - \frac{d}{dt} \left( \varepsilon_0 \mu_0 \int_{V} \vec{S} d\tau \right) = \frac{d \vec{P}_{\text{mech}}}{dt}. \quad (8.62) \]

Since

\[ \vec{S} = \frac{1}{\mu_0} \left( \vec{E} \times \vec{B} \right) \quad (8.63) \]

which involves the electric and magnetic field, the integral

\[ \vec{p}_{\text{em}} = \varepsilon_0 \mu_0 \int_{V} \vec{S} d\tau \quad (8.64) \]

represents the momentum stored in the electromagnetic fields themselves. Then the conservation of momentum in electrodynamics can be expressed as

\[ \frac{d \vec{p}_{\text{mech}}}{dt} + \frac{d \vec{p}_{\text{em}}}{dt} = \oint_{S} \vec{T} \cdot d\vec{a}. \quad (8.65) \]

Or in terms of the electromagnetic momentum density defined as

\[ \vec{\rho}_{\text{em}} = \frac{d \vec{p}_{\text{em}}}{d\tau} \quad (8.66) \]

and mechanical momentum density

\[ \vec{\rho}_{\text{mech}} = \frac{d \vec{p}_{\text{mech}}}{d\tau} \quad (8.67) \]

and using

\[ \oint_{S} \vec{T} \cdot d\vec{a} = \int_{V} \left( \nabla \cdot \vec{T} \right) d\tau \quad (8.68) \]

The conservation of momentum in electrodynamics can be written as

\[ \vec{\rho}_{\text{em}} + \vec{\rho}_{\text{mech}} = \nabla \cdot \vec{T}. \quad (8.69) \]

(b) Conservation of angular momentum: Electromagnetic fields carry a linear momentum. The linear momentum density carried by the electromagnetic fields in terms of the pointing vector is given by

\[ \vec{\rho}_{\text{em}} = \mu_0 \varepsilon_0 \vec{S} = \varepsilon_0 \left( \vec{E} \times \vec{B} \right). \quad (8.70) \]

The angular momentum is defined as

\[ \vec{L} = \vec{r} \times \vec{p} \quad (8.71) \]
the angular momentum density of electromagnetic fields is given by
\[ \mathbf{l}_{em} = \mathbf{r} \times \mathbf{p}_{em} = \varepsilon_0 \left[ \mathbf{r} \times \left( \mathbf{E} \times \mathbf{B} \right) \right] \] (8.72)

**Problem:** Obtain an equation that describes conservation of angular momentum in electrodynamics...

**Example 8.2** Determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius, \( R \), and charge \( Q \).

![Figure 8.4: Uniformly charged solid sphere with a total charge, \( Q \).](image)

**Solution:** To find the force on the northern hemisphere (shown by the blue by the blue-white mesh in Fig. ??) of a uniformly charged solid sphere we use the electromagnetic force expressed in terms of the Maxwell’s stress tensor
\[ \mathbf{F} = \oint_S \mathbf{T} \cdot d\mathbf{a}, \] (8.73)

where
\[ T_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right), \] (8.74)

Since we have a stationary uniformly charged sphere the magnetic field is zero, for the elements of the Maxwell’s stress Tensor becomes
\[ T_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right). \] (8.75)
Since we are interested in the force on the northern hemisphere we consider a closed surface that is made of an equatorial circular plane and a hemispherical shell of radii \( R \). Using Gauss’s law the electric field on the surface of the hemispherical shell of radius \( R \) is found to be

\[
\vec{E} = \frac{Q}{4\pi \epsilon_0 R^2} \hat{r}.
\]  

(8.76)

But we need to use Cartesian coordinates to apply Maxwell’s stress tensor. Therefore, the electric field should be written as

\[
\vec{E} = \frac{Q}{4\pi \epsilon_0 R^2} \left[ \sin (\theta) \cos (\varphi) \hat{x} + \sin (\theta) \sin (\varphi) \hat{y} + \cos (\theta) \hat{z} \right].
\]  

(8.77)

which leads to

\[
E_x = \frac{Q}{4\pi \epsilon_0 R^2} \sin (\theta) \cos (\varphi),
\]

\[
E_y = \frac{Q}{4\pi \epsilon_0 R^2} \sin (\theta) \sin (\varphi),
\]

\[
E_z = \frac{Q}{4\pi \epsilon_0 R^2} \cos (\theta), \Rightarrow E^2 = \left( \frac{Q}{4\pi \epsilon_0 R^2} \right)^2
\]

(8.80)

Using \( i = 1, 2, 3 \) for \( x, y, z \), respectively, the diagonal elements for the Maxwell’s stress tensor, we find

\[
T_{xx} = \epsilon_0 \left( E_x E_x - \frac{1}{2} E^2 \right) = \frac{\epsilon_0}{2} \left( E_x^2 - E_y^2 - E_z^2 \right)
\]

\[
= \frac{1}{2\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \left[ 2 \sin^2 (\theta) \cos^2 (\varphi) - 1 \right],
\]

\[
T_{yy} = \epsilon_0 \left( E_y E_y - \frac{1}{2} E^2 \right) = \frac{\epsilon_0}{2} \left( E_y^2 - E_x^2 - E_z^2 \right)
\]

\[
= \frac{1}{2\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \left[ 2 \sin^2 (\theta) \sin^2 (\varphi) - 1 \right],
\]

\[
T_{zz} = \epsilon_0 \left( E_z E_z - \frac{1}{2} E^2 \right) = \frac{\epsilon_0}{2} \left( E_z^2 - E_x^2 - E_y^2 \right)
\]

\[
= \frac{1}{2\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \left[ 2 \cos^2 (\theta) - 1 \right],
\]

(8.81)

and for the none diagonal elements we find.

\[
T_{xy} = \epsilon_0 E_x E_y = T_{yx} = \frac{1}{\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \sin (\theta) \cos (\varphi),
\]

\[
T_{xz} = \epsilon_0 E_x E_z = T_{zx} = \frac{1}{\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \sin (\theta) \cos (\varphi),
\]

\[
T_{yz} = \epsilon_0 E_y E_z = T_{zy} = \frac{1}{\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \sin (\theta) \cos (\varphi).
\]

(8.82)
Now using
\[ \overrightarrow{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}, \quad (8.83) \]
and noting that in spherical coordinates the infinitesimal area over the surface of the sphere is expressible as
\[ d\mathbf{a} = R^2 \sin(\theta) \, d\theta d\phi \]
which can be expressed using matrix as
\[ d\mathbf{a} = \begin{bmatrix} R^2 \sin^2(\theta) \cos(\phi) \, d\theta d\phi \\ R^2 \sin^2(\theta) \sin(\phi) \, d\theta d\phi \\ R^2 \sin(\theta) \cos(\theta) \, d\theta d\phi \end{bmatrix}. \quad (8.85) \]
One can then write
\[ \overrightarrow{T} \cdot d\mathbf{a} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \begin{bmatrix} R^2 \sin^2(\theta) \cos(\phi) \, d\theta d\phi \\ R^2 \sin^2(\theta) \sin(\phi) \, d\theta d\phi \\ R^2 \sin(\theta) \cos(\theta) \, d\theta d\phi \end{bmatrix} = \begin{bmatrix} dF_x \\ dF_y \\ dF_z \end{bmatrix} \]
where
\[ dF_x = R^2 \sin(\theta) \{ \sin(\theta) \{ T_{xx} \cos(\phi) + T_{xy} \sin(\phi) \} + T_{xz} \cos(\phi) \} \, d\theta d\phi, \]
\[ dF_y = R^2 \sin(\theta) \{ \sin(\theta) \{ T_{yx} \cos(\phi) + T_{yy} \sin(\phi) \} + T_{yz} \cos(\phi) \} \, d\theta d\phi, \]
\[ dF_z = R^2 \sin(\theta) \{ \sin(\theta) \{ T_{zx} \cos(\phi) + T_{zy} \sin(\phi) \} + T_{zz} \cos(\phi) \} \, d\theta d\phi. \]
Upon substituting the expressions for the elements we determined for Maxwell’s stress tensor, we find
\[ dF_x = \frac{R^2}{2\varepsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \sin^2(\theta) \cos(\phi) \, d\theta d\phi, \]
\[ dF_y = \frac{R^2}{2\varepsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \sin^2(\theta) \sin(\phi) \, d\theta d\phi, \]
\[ dF_z = \frac{R^2}{2\varepsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 \sin(\theta) \cos(\theta) \, d\theta d\phi. \]
Therefore, the contribution of the integral
\[ \vec{F} = \oint_S \overrightarrow{T} \cdot d\mathbf{a}, \quad (8.86) \]
8.2. CONSERVATION OF MOMENTUM

for the semi-spherical shell (bowl) is found to be

\[
\vec{F}_1 = \frac{R^2}{2\varepsilon_0} \left( \frac{Q}{4\pi R^3} \right)^2 \left\{ \frac{\pi/2}{0} \sin^2(\theta) \left[ \int_0^{2\pi} \cos(\varphi) \, d\varphi \hat{\hat{x}} + \int_0^{2\pi} \sin(\varphi) \, d\varphi \hat{\hat{y}} \right] d\theta \\
+ 2\pi \int_0^{\pi/2} \sin(\theta) \cos(\theta) \, d\theta \hat{\hat{z}} \right\} 
\]

(8.87)

Noting that

\[
\int_0^{2\pi} \cos(\varphi) \, d\varphi = \int_0^{2\pi} \sin(\varphi) \, d\varphi = 0, \quad \int_0^{\pi/2} \sin(\varphi) \, d\varphi = \frac{\sin(\theta)}{2} \bigg|_0^{\pi/2} = \frac{1}{2}
\]

(8.88)

one finds

\[
\vec{F}_1 = \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{8R^2} \hat{\hat{z}} 
\]

(8.89)

Similarly, for the disk using

\[
d\vec{a} = \begin{bmatrix} 0 \\ 0 \\ -rdrd\varphi \end{bmatrix},
\]

(8.90)

we have

\[
\vec{T} \cdot d\vec{a} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -rdrd\varphi \end{bmatrix} = \begin{bmatrix} T_{xx} rdrd\varphi \\ T_{yx} rdrd\varphi \\ -T_{zz} rdrd\varphi \end{bmatrix}
\]

(8.91)

Noting that we are now inside the sphere the electric field is given by

\[
\vec{E} = \frac{Qr}{4\pi R^3} \hat{\hat{r}} = \frac{Qr}{4\pi R^3} [\sin(\theta) \cos(\varphi) \, \hat{\hat{x}} + \sin(\theta) \sin(\varphi) \, \hat{\hat{y}} + \cos(\theta) \, \hat{\hat{z}}]
\]

(8.92)

which leads to

\[
T_{xx} = \frac{1}{\varepsilon_0} \left( \frac{Qr}{4\pi R^3} \right)^2 \sin(\theta) \cos(\theta) \cos(\varphi),
\]

(8.93)

\[
T_{yz} = \frac{1}{\varepsilon_0} \left( \frac{Qr}{4\pi R^3} \right)^2 \sin(\theta) \cos(\theta) \sin(\varphi)
\]

(8.94)

\[
T_{zz} = \frac{1}{2\varepsilon_0} \left( \frac{Qr}{4\pi R^3} \right)^2 \left[ 2\cos^2(\theta) - 1 \right]
\]

(8.95)

so that on the circular plane where \( \theta = \pi/2 \), we find

\[
T_{xz} = 0, T_{yz} = 0, T_{zz} = -\frac{1}{2\varepsilon_0} \left( \frac{Qr}{4\pi R^3} \right)^2
\]

(8.96)
and
\[
\vec{T} \cdot \text{d}\vec{a} = \begin{bmatrix}
T_x r \text{d}r \text{d}\phi \\
T_y r \text{d}r \text{d}\phi \\
T_z r \text{d}r \text{d}\phi
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\frac{1}{2\pi\epsilon_0} \left( \frac{Q}{4\pi R^2} \right)^2 r \text{d}r \text{d}\phi
\end{bmatrix}.
\] (8.97)

Thus
\[
\vec{F}_2 = \int_V \vec{T} \cdot \text{d}\vec{a} = \frac{Q^2}{32\pi^2\epsilon_0 R^6} \int_0^R \int_0^{2\pi} r^3 \text{d}r \text{d}\phi \hat{\vec{z}} = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{16R^2} \hat{\vec{z}}.
\] (8.98)

Therefore, the net force will be
\[
\vec{F} = \vec{F}_1 + \vec{F}_2 = \frac{1}{4\pi\epsilon_0} \frac{3Q^2}{16R^2} \hat{\vec{z}}.
\] (8.99)

Example 8.3 A long coaxial cable, of length \( l \), consists of an inner conductor (radius \( a \)) and an outer conductor (radius \( b \)). It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length \( \lambda \), and a steady current \( I \) to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields? (See Fig. 8.2)

Solution: Because of the line charge on the inner conductor, \( \lambda \), there is an electric field, \( \vec{E} \), given by
\[
\vec{E} = \begin{cases}
0, & s < a \\
\frac{\lambda}{2\pi\epsilon_0} \hat{s}, & a < s < b \\
0, & s > b
\end{cases}
\] (8.100)
and due to the current we have a magnetic field, $\vec{B}$

$$\vec{B} = \begin{cases} \frac{\mu_0 I_s}{2\pi s^2} \hat{\phi} & s < a \\ \frac{\mu_0 I}{2\pi s} \hat{\phi} & a < s < b \\ 0 & s > b \end{cases} \quad (8.101)$$

The pointing vector

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \quad (8.102)$$

will then be

$$\vec{S} = \begin{cases} \frac{\lambda}{2\pi \epsilon_0 s} \hat{s} \times \frac{1}{2\pi} \hat{\phi} = \frac{\lambda I}{4\pi^2 \epsilon_0 s^2} \hat{z} & s < a \\ 0 & a < s < b \\ 0 & s > b \end{cases} \quad (8.103)$$

Then the electromagnetic momentum stored in the field given by

$$\vec{p}_{em} = \epsilon_0 \mu_0 \int_V \vec{S} d\tau \quad (8.104)$$

becomes

$$\vec{p}_{em} = \epsilon_0 \mu_0 \int_a^b \frac{\lambda I}{4\pi^2 \epsilon_0 s} 2\pi l s ds \hat{z} = \epsilon_0 \mu_0 \int_a^b \frac{\lambda I}{4\pi^2 \epsilon_0 s^2} 2\pi l s ds \hat{z}$$

$$\Rightarrow \vec{p}_{em} = \frac{\mu_0 \lambda I}{2\pi} \ln \left( \frac{b}{a} \right) \hat{z} \quad (8.105)$$

**Example 8.4** Imagine a very long solenoid with radius $R$, $n$ turns per unit length, and current $I$. Coaxial with the solenoid are two long cylindrical shells of length $l$—one, inside the solenoid at radius $a$, carries a charge $+Q$, uniformly distributed over its surface; the other, outside the solenoid at radius, $b$, carries charge $-Q$ (see Fig.8.5). The length the the cylindrical shells is much bigger than the radii (i.e. $l >> a, b$). The cylinders are free to rotate about the their axis ($z$-axis). The current in the solenoid is gradually reduced and the cylinders begin to rotate. Find the torque responsible for the rotation and explain where does it come from.

**Solution:** As the current in the solenoid decreases the magnetic field of the solenoid changes which leads to a change in magnetic flux in the region. Because of Faraday’s law for any closed conducting loop placed in a region of a changing magnetic field, $\vec{B}$, there will be an induced electric field, $\vec{E}$, tangent to the circle that is given by

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \quad (8.106)$$

where $\Phi$ is the magnetic flux
Figure 8.5: A cylindrical shell of radius $a$ (Inside) and another cylinder with radius $b$ (outside) along the solenoid with radius $R$. The two cylinders have the same length, $l$. The inner cylinder carries a charge $-Q$ and the outer a charge $+Q$ that is uniformly distributed over the surface of the cylinders.

\[
\Phi = \int \vec{B} \cdot d\vec{a}
\]

Recalling that the magnetic field of a solenoid (long) for a current in a counterclockwise direction (see the top view in Fig. ??) is given by

\[
\vec{B} = \begin{cases} 
\mu_0 I n \hat{z} & s < R \\
0 & s > R
\end{cases}
\]  

(8.107)

the magnetic flux for a conducting loop of radius, $s = a < R$ (on the inner cylinder), we find

\[
\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \Rightarrow E_2 \pi a = -\frac{d}{dt} \left[ \pi a^2 \mu_0 I n \right] \Rightarrow \vec{E}(a) = -\frac{\mu_0 n a}{2} \frac{dI}{dt} \hat{\varphi}
\]  

(8.108)

where as for a radius, $s = b > R$

\[
\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \Rightarrow E_2 \pi b = -\frac{d}{dt} \left[ \pi R^2 \mu_0 I n \right] \Rightarrow \vec{E}(b) = -\frac{\mu_0 n R^2}{2b} \frac{dI}{dt} \hat{\varphi}.
\]  

(8.109)

Due to this electric field the charges on the inner and outer shell of the cylinder experience different magnitude of force. The force on the inner shell of radius $a$ ($a < R$) which carries a total charge, $Q$, on its surface will experience a force

\[
\vec{F}_a = Q \vec{E}(a) = -\frac{Q \mu_0 n a}{2} \frac{dI}{dt} \hat{\varphi}
\]  

(8.110)
and the charges on the outer shell of radius $b$ ($b > R$) experiences a force

$$\vec{F}_b = -Q\vec{E}(b) = \frac{Q\mu_0 nR^2}{2b} \frac{dI}{dt} \hat{\phi}. \quad (8.111)$$

Then the corresponding torque resulting from these forces about the axes of the cylinder ($z$-axis) is given by

$$\vec{\tau}_a = a\hat{s} \times \vec{F}_a = -\frac{Q\mu_0 na^2}{2} \frac{dI}{dt} \left( \hat{s} \times \hat{\phi} \right) = -\frac{Q\mu_0 na^2}{2} \frac{dI}{dt} \hat{z}. \quad (8.112)$$

and the charges on the outer shell of radius, $b$ ($b > R$)

$$\vec{F}_b = b\hat{s} \times \vec{F}_b = \frac{Q\mu_0 nR^2}{2} \frac{dI}{dt} \left( \hat{s} \times \hat{\phi} \right) = \frac{Q\mu_0 nR^2}{2} \frac{dI}{dt} \hat{z}. \quad (8.113)$$

Then the net torque

$$\vec{\tau} = \vec{\tau}_a + \vec{\tau}_b = \frac{Q\mu_0 n (R^2 - a^2)}{2} \frac{dI}{dt} \hat{z}. \quad (8.114)$$

This torque come from the angular momentum lost by the electromagnetic fields as the current in the solenoid changes. One can show this by finding the rate of change of the angular moment of the electromagnetic fields as the current decreases. We recall that the angular momentum density is given by

$$\vec{I}_{em} = \epsilon_0 \left[ \vec{F} \times \left( \vec{E} \times \vec{B} \right) \right]$$

(8.115)
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Since the magnetic field exists only inside the solenoid \((s < R)\) and the electric field due to the charges on the inner \((Q)\) and outer shell \((-Q)\) gives an electric field in the region \((a < s < b)\) which is given by (Using Gauss's law)

\[
\oint E \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \Rightarrow E = \frac{Q}{2\pi \varepsilon_0 s l} \hat{s},
\]

the common region where both the electric and magnetic fields exist is, \(a < s < R\). The angular momentum density in this region is given by

\[
\vec{l}_{\text{em}} = \varepsilon_0 \left( (s\hat{s} + z\hat{z}) \times \left[ \left( \frac{Q}{2\pi \varepsilon_0 s l} \hat{s} \right) \times (\mu_0 I n \hat{z}) \right] \right) = \varepsilon_0 \left( (s\hat{s} + z\hat{z}) \times \left[ \frac{Q \mu_0 I n}{2\pi \varepsilon_0 s l} (-\hat{\varphi}) \right] \right)
\]

\[
\Rightarrow \vec{l}_{\text{em}} = \varepsilon_0 \left[ -\frac{Q \mu_0 I n}{2\pi \varepsilon_0 s l} \hat{z} + \frac{Q \mu_0 I n}{2\pi \varepsilon_0 s l} \hat{s} \right],
\]

(8.117)

In cylindrical coordinates, the unit vector \(\hat{s}\) is given by

\[
\hat{s} = \cos (\varphi) \hat{x} + \sin (\varphi) \hat{y}.
\]

(8.118)

Then for the angular momentum density one can write

\[
\vec{l}_{\text{em}} = \frac{Q \mu_0 I zn}{2\pi sl} (\cos (\varphi) \hat{x} + \sin (\varphi) \hat{y}) - \frac{Q \mu_0 I n}{2\pi l} \hat{z}
\]

(8.119)

so that the total electromagnetic angular momentum

\[
L_{\text{em}} = \int_V \vec{l}_{\text{em}} dv
\]

(8.120)

using

\[
dv = sdsd\varphi dz
\]

can be expressed as

\[
\vec{L}_{\text{em}} = \int_a^R \int_{-l/2}^{l/2} \int_0^{2\pi} \left[ \frac{Q \mu_0 I zn}{2\pi sl} (\cos (\varphi) \hat{x} + \sin (\varphi) \hat{y}) - \frac{Q \mu_0 I n}{2\pi l} \hat{z} \right] sdsd\varphi dz.
\]

(8.121)

Noting that

\[
\int_0^{2\pi} \cos (\varphi) d\varphi = \int_0^{2\pi} \sin (\varphi) d\varphi = 0
\]

the \(x\) and \(y\) components becomes zero and the angular momentum of the EM fields becomes

\[
\vec{L}_{\text{em}} = -\frac{Q \mu_0 I n}{l} \hat{z} \int_a^R \int_{-l/2}^{l/2} sdsdz = -\frac{Q \mu_0 I n (R^2 - a^2)}{2} \hat{z}.
\]

(8.122)
As the current decreases from $I$ to 0 we have seen the shells begin to rotate. That means the shells gains angular momentum from 0 to some value $\hat{L}$ which is related to the torque by

$$\tau = \frac{d\hat{L}}{dt} \Rightarrow \frac{Q \mu_0 n (R^2 - a^2)}{2} \frac{dI}{dt} \hat{z} = \frac{d\hat{L}}{dt} \Rightarrow d\hat{L} = \frac{Q \mu_0 n (R^2 - a^2)}{2} dI \hat{z}$$

$$\int_0^L d\hat{L} = \int_I^0 \frac{Q \mu_0 n (R^2 - a^2)}{2} dI \hat{z} \Rightarrow \hat{L} = -\frac{Q \mu_0 n (R^2 - a^2) I}{2} \hat{z} \quad (8.123)$$

This the angular moment lost by the EM fields and gained by the spherical shells. In other words it is the EM angular momentum transformed to mechanical angular momentum.
Chapter 9

Electromagnetic Waves

9.1 Review of waves

The wave equation: For a non-absorbing and non-dispersive medium a wave is defined as a disturbance of a continuous medium that propagates with a fixed shape at constant velocity. Let’s consider a disturbance in a non-absorbing and non-dispersive medium traveling with a speed, \( v \) along the \( z \)-direction. Suppose a point on this disturbance at a given time \( t \) can be described by a function, \( f(z, t) \). Then before a time interval, \( \Delta t \), this point on this disturbance is described by \( f(z - v \Delta t, t - \Delta t) \) and after the same time interval \( \Delta t \) can be described by \( f(z + v \Delta t, t + \Delta t) \). If this disturbance is a wave, it must keep the shape at all time. This means the following condition must be satisfied

\[
f(z - v \Delta t, t - \Delta t) = f(z, t) \tag{9.1}
\]

or

\[
f(z, t) = f(z + v \Delta t, t + \Delta t) \tag{9.2}
\]

Example 9.1

Show that disturbance defined by the functions

\[
f_1(z, t) = \exp[-b(z - vt)^2], \quad f_2(z, t) = \sin[b(z - vt)]
\]

describe a wave where as the disturbance defined by the functions

\[
f_4(z, t) = \exp[-b(bz^2 + vt)^2], \quad f_5(z, t) = \sin[bz] \cos[bvt]^3
\]

do not describe a wave.

Solution: For \( \Delta t = t \), we must show that

\[
f(z - vt, 0) = f(z, t) \tag{9.3}
\]

if the disturbance defined by the function is a wave. For the first two functions

\[
f_1(z - vt, 0) = \exp[-b(z - vt - v \times 0)^2] = \exp[-b(z - vt)^2] = f_1(z, t)
\]

\[
f_2(z - vt, 0) = \sin[b(z - vt - v \times 0)] = \sin[b(z - vt)] = f_2(z, t) \tag{9.4}
\]
on the other hand for the last two functions

\[ f_4(z - vt, 0) = \exp[-b(b(z - vt)^2 + v \times 0)^2] = \exp[-b(b^2(z - vt)^4)] \neq f_4(z, t) \]

\[ f_5(z - vt, 0) = \sin[b(z - vt)]\cos[bv \times 0]^3 = \sin[b(z - vt)] \neq f_5(z, t) \]

### 9.2 The wave equation on a string

Consider wave propagating on a string (Along the z-direction) under a tension force, \( T \), shown in Fig. 9.3.

Consider an infinitesimal segment of the string shown in Fig. 9.2. If the string is displaced from the equilibrium, the net transverse force on the segment between \( z \) and \( z + \Delta z \) can be expressed as

\[ \Delta F = T \left( \sin (\theta') - \sin (\theta) \right) \]  (9.5)
9.2. THE WAVE EQUATION ON A STRING

Figure 9.3: Wave propagating on a string with tension force, T.

For a wave of such kind the displacement from the equilibrium position is small, we can make the approximation \( \sin(\theta) \approx \tan(\theta) \),

\[
\Delta F = T \left( \tan(\theta') - \tan(\theta) \right). \tag{9.6}
\]

Noting that

\[
\tan(\theta') = \frac{\partial f}{\partial z} \bigg|_{z=z+\Delta z}, \quad \tan(\theta) = \frac{\partial f}{\partial z} \bigg|_{z=z} \tag{9.7}
\]

we may write

\[
\Delta F = T \left( \frac{\partial f}{\partial z} \bigg|_{z=z+\Delta z} - \frac{\partial f}{\partial z} \bigg|_{z=z} \right). \tag{9.8}
\]

One can write for the first derivative of a function \( f(z) \) at \( z \)

\[
\frac{\partial f}{\partial z} \bigg|_{z=z} = \lim_{\Delta z \to 0} \left[ \frac{f(z + \Delta z) - f(z)}{\Delta z} \right] \tag{9.9}
\]
and for the second derivative
\[
\frac{\partial^2 f}{\partial z^2} \bigg|_{z=z} = \lim_{\Delta z \to 0} \left[ \frac{\partial f}{\partial z} \bigg|_{z+\Delta z} - \frac{\partial f}{\partial z} \bigg|_z \right] \Rightarrow \frac{\partial^2 f}{\partial z^2} \bigg|_{z=z} \Delta z = \lim_{\Delta z \to 0} \left[ \frac{\partial f}{\partial z} \bigg|_{z+\Delta z} - \frac{\partial f}{\partial z} \bigg|_z \right]
\]

so that for small \( \Delta z \), we may write

\[
\Delta F \approx T \frac{\partial^2 f}{\partial z^2} \Delta z. \tag{9.10}
\]

For the transverse motion, Newton’s second law can be written as

\[
\Delta F = ma = m \frac{\partial^2 f}{\partial t^2} \tag{9.11}
\]

If the string has a linear mass density, \( \mu \), we can write \( m = \mu \Delta z \) so that

\[
\Delta F = \mu \Delta z \frac{\partial^2 f}{\partial t^2}. \tag{9.12}
\]

Combining Eqs. (9.10) and (9.12), we find

\[
T \frac{\partial^2 f}{\partial z^2} \Delta z = \mu \Delta z \frac{\partial^2 f}{\partial t^2}. \tag{9.13}
\]

Which leads to

\[
\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T} \frac{\partial^2 f}{\partial t^2} \Rightarrow \frac{\partial^2 f}{\partial z^2} f(z,t) = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} f(z,t) \tag{9.14}
\]

where

\[
v = \sqrt{\frac{T}{\mu}} \tag{9.15}
\]

is the speed of the wave. Eq. (9.14) is known as the wave equation. The general solution given by

\[
f(z,t) = g(z - vt) + g(z + vt) \tag{9.16}
\]

### 9.3 Important terminologies in a sinusoidal waves

The sinusoidal wave shown in Fig. 9.4 is described by the function (satisfying the wave equation)

\[
f(z,t) = A \cos(k(z - vt) + \delta) \tag{9.17}
\]

- **Amplitude**, \( A \): the maximum displacement from the equilibrium
- **Phase**, \( \delta \): the phase constant tells you by how much the central maxima is behind the origin.
9.3. IMPORTANT TERMINOLOGIES IN A SINUSOIDAL WAVES

9.3.1. Definitions of Basic Terminologies in a Sinusoidal Wave

**Wave number, \( k \):** It is related to the wave length by

\[
\lambda = \frac{2\pi}{k}
\]  
(9.18)

**Period, \( T \):** The time for one full cycle. If the wave is travelling with speed \( v \), then

\[
v = \frac{\lambda}{T} \Rightarrow T = \frac{\lambda}{v} = \frac{2\pi}{kv}
\]  
(9.19)

**Frequency, \( f \):** Number of oscillations per unit time

\[
u = \frac{1}{T} \Rightarrow \nu = \frac{kv}{2\pi} = \frac{v}{\lambda}
\]  
(9.20)

**Angular frequency, \( \omega \):**

\[
\omega = 2\pi \nu = kv
\]  
(9.21)

A sinusoidal wave travelling to the left: see Fig. 9.5

\[
f (z, t) = A \cos (k (z + vt) - \delta)
\]  
(9.22)

and travelling to the right, which is shown (Fig. 9.4) is given by

\[
f (z, t) = A \cos (k (z - vt) + \delta).
\]  
(9.23)

In terms of the angular frequency, we may write

\[
\begin{align*}
 f (z, t) &= A \cos (kz + \omega t - \delta) \text{ to the left} \\
f (z, t) &= A \cos (kz - \omega t + \delta) \text{ to the right}
\end{align*}
\]  
(9.24)

**Complex notation of a sinusoidal wave:** Using the Euler’s formula

\[
e^{i\theta} = \cos (\theta) + i \sin (\theta) \Rightarrow \cos (\theta) = \text{Re} [e^{i\theta}],
\]  
(9.25)
CHAPTER 9. ELECTROMAGNETIC WAVES

where $\text{Re}$ mean the real part of the complex number, we can express a sinusoidal wave as

$$f(z, t) = \text{Re}[A e^{i(kz - \omega t + \delta)}] = \text{Re}[A e^{i(kz - \omega t)]} = \text{Re} \left[ \tilde{A} e^{i(kz - \omega t)} \right] = \text{Re} \left[ \tilde{f}(z, t) \right],$$

where

$$\tilde{f}(z, t) = \tilde{A} e^{i(kz - \omega t)}. \tag{9.26}$$

Any wave function can be expressed as a linear combination of sinusoidal waves

$$\tilde{f}(z, t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz - \omega t)} dk \tag{9.27}$$

Note this is because of the fact that sinusoidal functions form a complete orthonormal set of functions, as you have studied in Theoretical physics I.

9.4 Wave Boundary conditions

Consider a wave propagation from one medium to another as shown in Fig. 9.6. The wave is traveling with a velocity $v_1$ along the $z$-direction from medium-1 to medium-2. Suppose the boundary separating these two media is defined by the plane, $z(x, y) = z_0$. At this boundary the incident wave partly reflects and partly transmitted as shown in Fig. 9.6. Suppose the function describing the waves in medium-1, that includes both the incident and reflected, $(z > z_0)$ be $\tilde{f}_1(z, t)$ and the transmitted waves in medium-2, $(z < z_0)$ be $\tilde{f}_2(z, t)$. At the boundary of these two media, at the interface, (on the plane defined by $z(x, y) = z_0$), the wave function must satisfy the following two conditions

$$\tilde{f}_1(z_0, t) = \tilde{f}_2(z_0, t), \quad \frac{\partial \tilde{f}_1(z, t)}{\partial z} \bigg|_{z=z_0} = \frac{\partial \tilde{f}_2(z, t)}{\partial z} \bigg|_{z=z_0} \tag{9.28}$$
Example 9.2 Consider two strings with different mass densities, $\mu_1$ and $\mu_2$ tied together at one end. The tension on the two strings is the same, $T$. A wave is propagating from left to right along the $z$ direction as shown in the figure above. Find the amplitude and phase constant of the reflected and transmitted waves in terms of the incident wave amplitude, $A_I$, and phase constant, $\delta_A$, for the case $\mu_2 < \mu_1$ and $\mu_2 > \mu_1$. What would happen when $\mu_2 >> \mu_1$?

Solution: The two strings have different mass densities. This means the velocities are different, $v_1 = \sqrt{T/\mu_1}$ and $v_2 = \sqrt{T/\mu_2}$. This leads to a different wave length and wave number. If we assume the knot is at $z = 0$ and the incident wave coming from the left as shown in Fig. ??, we can write the wave function for the incident wave

$$\tilde{f}_I(z,t) = \tilde{A}_Ie^{i(k_1z-\omega t)}, (z < 0), \quad (9.29)$$

the reflected wave

$$\tilde{f}_R(z,t) = \tilde{A}_Re^{i(-k_1z-\omega t)}, (z < 0) \quad (9.30)$$

and the transmitted wave

$$\tilde{f}_T(z,t) = \tilde{A}_Te^{i(k_2z-\omega t)}, (z > 0). \quad (9.31)$$

The wave function and its first derivative must be continuous at the bound-
ary:
\[
\tilde{f}_1(0, t) = \tilde{f}_2(0, t), \quad \left. \frac{\partial \tilde{f}_1(z, t)}{\partial z} \right|_0 = \left. \frac{\partial \tilde{f}_2(z, t)}{\partial z} \right|_0.
\] (9.32)

Using the incident, reflected, and transmitted waves the boundary conditions can be written as
\[
\tilde{f}_I(0, t) + \tilde{f}_R(0, t) = \tilde{f}_T(0, t), \quad \left. \frac{\partial \tilde{f}_I(z, t)}{\partial z} \right|_{z=0} = \left. \frac{\partial \tilde{f}_R(z, t)}{\partial z} \right|_{z=0}
\] (9.33)

which leads to
\[
\tilde{A}_I + \tilde{A}_R = \tilde{A}_T, \quad k_1 \left( \tilde{A}_I - \tilde{A}_R \right) = k_2 \tilde{A}_T.
\] (9.34)

Solving for the amplitudes of the reflected and transmitted waves, we find
\[
\tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2} \tilde{A}_I, \quad \tilde{A}_T = \frac{2k_1}{k_1 + k_2} \tilde{A}_I.
\] (9.35)

Using the relation \( \omega = kv \Rightarrow k = \omega/v \) and noting that the frequency is the same in the two strings, we can express the amplitudes as
\[
\tilde{A}_R = \frac{v_2 - v_1}{v_2 + v_1} \tilde{A}_I, \quad \tilde{A}_T = \frac{2v_2}{v_1 + v_2} \tilde{A}_I.
\] (9.36)

The real amplitudes can be expressed as
\[
A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_2 + v_1} A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \frac{2v_2}{v_1 + v_2} A_I e^{i\delta_I}.
\] (9.37)

where \( \delta_I, \delta_R, \) and \( \delta_T \) are the phase constant for the incident, reflected, and transmitted waves.

**Case 1: High density to low density \((\mu_1 > \mu_2)\)** for the corresponding speeds, we have
\[
v_1 = \sqrt{T/\mu_1} < v_2 = \sqrt{T/\mu_2},
\] (9.38)

(From low speed to high speed), which leads to
\[
A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_2 + v_1} A_I e^{i\delta_I} \Rightarrow A_R = \frac{v_2 - v_1}{v_2 + v_1} A_I, e^{i\delta_R} = e^{i\delta_I}
\]
\[
\Rightarrow A_R = \frac{v_2 - v_1}{v_2 + v_1} A_I, \delta_R = \delta_I
\] (9.39)
\[
A_T e^{i\delta_T} = \frac{2v_2}{v_1 + v_2} A_I e^{i\delta_I} \Rightarrow A_T = \frac{2v_2}{v_1 + v_2} A_I, \delta_T = \delta_I.
\]

The phase constant for both the incident and reflected wave would not change.
9.5. POLARIZATION

Case 2: Low density to high density \((\mu_1 < \mu_2)\), we have

\[
v_1 = \sqrt{\frac{T}{\mu_1}} > v_2 = \sqrt{\frac{T}{\mu_2}}
\]  \(9.40\)

which means from high speed to low speed. This leads to

\[
A_R e^{i\delta_R} = -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| A_I e^{i\delta_I} \Rightarrow A_R e^{i(\delta_R + \pi)} = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| A_I e^{i\delta_I}
\]

\[
\Rightarrow A_R = \frac{v_1 - v_2}{v_2 + v_1} A_I, e^{i(\delta_R + \pi)} = e^{i\delta_I} \Rightarrow A_R = \frac{v_1 - v_2}{v_2 + v_1} A_I, \delta_R = \delta_I - \pi
\]  \(9.41\)

The reflected wave phase will shift by \(\pi\). On the other hand for the transmitted wave we have

\[
A_T e^{i\delta_T} = \frac{2v_2}{v_1 + v_2} A_I e^{i\delta_I} \Rightarrow A_T = \frac{2v_2}{v_1 + v_2} A_I, \delta_T = \delta_I
\]  \(9.42\)

which means there is no phase shift. The transmitted wave phase constant would remain unchanged. For \(\mu_2 \gg \mu_1\) (if the second string is very massive)

\[
v_1 = \sqrt{\frac{T}{\mu_1}} > v_2 \simeq 0
\]  \(9.43\)

we find

\[
A_T = \frac{2v_2}{v_1 + v_2} \simeq 0, A_R = A_I
\]  \(9.44\)

The wave completely be reflected and there is no transmitted wave.

9.5 Polarization

Before we talk about polarization of waves it is important to classify waves into two types depending on the direction of the displacement from the equilibrium (vibration direction) with respect to the direction of propagation of the waves.

Longitudinal waves: the displacement is along the direction of propagation. Sound waves are longitudinal waves.

\[
\psi
\]

Figure 9.7: Longitudinal wave.

Transverse waves: the displacement is perpendicular to the direction of propagation. Electromagnetic waves are transverse waves. Transverse waves exist in two independent states of polarization.
Figure 9.8: x-polarized wave

**X-polarization (Horizontally polarized):** the waves oscillates parallel to the x-z plane as shown in the figure below.

This wave can be expressed as

\[ f(z, t) = A e^{i(kz - \omega t)} \hat{x}. \] (9.45)

**Y-polarization (Vertically polarized):** the wave oscillates parallel to the y-z plane as shown in the figure below.

It can be expressed as

\[ f(z, t) = A e^{i(kz - \omega t)} \hat{y}. \] (9.46)

**Application in quantum information:** x-polarized and y-polarized photons are used in quantum information. For example, a photon can be prepared in a state described by a 50/50 chance of being x-polarized or y-polarized. This state forms a qubit which is usually represented using bra-ket notation as

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \ 1\text{qubit} \] (9.47)

Using two photons one can also generate a two qubit which is given by

\[ |\psi\rangle = c_{00} |0, 0\rangle + c_{01} |0, 1\rangle + c_{10} |1, 0\rangle + c_{11} |1, 1\rangle, \ 2\text{qubit} \]

note that |0\rangle and |1\rangle represent horizontal and vertical polarization, respectively.

**Linearly polarized:** A wave propagating along the z-direction can also be polarized along arbitrary direction, \(\hat{n}\), in the x-y plane. This polarization direction for a wave propagating along the z-direction satisfies the condition

\[ \hat{n} \cdot \hat{z} = 0 \] (9.48)
### 9.5. POLARIZATION

In terms of the polarization direction, $\hat{n}$, this wave can be expressed as
\[
\tilde{f}(z, t) = A e^{i(kz - \omega t)} \hat{n}
\]

or in terms of the polarization angle, $\theta$, measured from the positive $x$ axis
\[
\tilde{f}(z, t) = A e^{i(kz - \omega t)} \hat{n} = A e^{i(kz - \omega t)} (\cos(\theta) \hat{x} + \sin(\theta) \hat{y}) \\
\Rightarrow \tilde{f}(z, t) = A \cos(\theta) e^{i(kz - \omega t)} \hat{x} + A \sin(\theta) e^{i(kz - \omega t)} \hat{y}
\]

---

**Circularly & Elliptically polarized: (Watch the demonstration online)**

\[
\tilde{f}(z, t) = A e^{i(kz - \omega t)} \hat{n} = A e^{i(kz - \omega t)} (n_x \hat{x} \pm i n_y \hat{y}) \\
= A e^{i(kz - \omega t)} (n_x \hat{x} + e^{\mp i \frac{\pi}{2}} n_y \hat{y}) \Rightarrow \tilde{f}(z, t) = A_x e^{i(kz - \omega t)} \hat{x} + A_y e^{i(kz - \omega t + \frac{\pi}{2})} \hat{y}
\]

If $A_x = A_y$ the wave is called circularly polarized otherwise it is elliptically polarized.
9.5.1 Electromagnetic waves in a vacuum

The wave equation for $\vec{E}$ and $\vec{B}$: We recall that Maxwell’s equations in free space with a charge density, $\rho$, and current density, $\vec{J}$, are given by

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \nabla \cdot \vec{B} = 0,$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}.$$  \hspace{1cm} (9.52)

If there is no charge and current in the space, $\rho = 0$ and $\vec{J} = 0$, Maxwell’s equations for the electric and magnetic fields in a vacuum become

$$\nabla \cdot \vec{B} = 0, \nabla \cdot \vec{E} = 0, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$ \hspace{1cm} (9.53)

Using the relation

$$\nabla \times \nabla \times \vec{A} = \nabla \left( \nabla \cdot \vec{A} \right) - \nabla^2 \vec{A}$$ \hspace{1cm} (9.54)

we may write

$$\nabla \times \nabla \times \vec{E} = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left( \nabla \times \vec{B} \right)$$ \hspace{1cm} (9.55)

and using

$$\nabla \cdot \vec{E} = 0, \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t},$$ \hspace{1cm} (9.56)

we find

$$\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left( \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$ \hspace{1cm} (9.57)

where

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$ \hspace{1cm} (9.58)

is the speed of light in a vacuum. Following a similar procedure we find for the magnetic field

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}.$$ \hspace{1cm} (9.59)

The electric and magnetic fields satisfy the three dimensional wave equation!

Monochromatic Plane waves: A monochromatic plane wave propagating along the $z$-direction has the following two properties:

- It is monochromatic: has a single frequency (or wave length)
- It is plane: The electric field and magnetic fields which are on the $x$ – $y$ plane does not depend on the $x$ and $y$ coordinates. The fields are constant over this plane at a given instant of time,

$$\vec{E}(z,t) = E_0 e^{i(kz-\omega t)}, \vec{B}(z,t) = B_0 e^{i(kz-\omega t)}.$$ \hspace{1cm} (9.60)

Here $E_0$ and $B_0$ are the complex amplitudes.
Let’s see that whether the solution to the electric and magnetic fields given above indeed satisfy Maxwell’s equation. We first begin by checking the divergence of $\vec{E}$ and $\vec{B}$ for a plane monochromatic waves,

$$\nabla \cdot \vec{E}(z,t) = \nabla \cdot \left( \vec{E}_0 e^{i(kz-\omega t)} \right)$$

$$= \frac{\partial}{\partial x} \left( E_{0x} e^{i(kz-\omega t)} \right) + \frac{\partial}{\partial y} \left( E_{0y} e^{i(kz-\omega t)} \right) + \frac{\partial}{\partial z} \left( E_{0z} e^{i(kz-\omega t)} \right)$$

$$= e^{i(kz-\omega t)} \left[ \frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} + ikE_{0z} + \frac{\partial E_{0z}}{\partial z} \right]. \quad (9.61)$$

For plane waves propagating along the $z$-direction, we know that the fields do not depend on $x$ and $y$

$$\frac{\partial E_{0x}}{\partial x} = \frac{\partial E_{0y}}{\partial y} = 0. \quad (9.62)$$

The waves are propagating in a vacuum (non absorptive medium) the amplitude is constant

$$\frac{\partial E_{0z}}{\partial z} = 0. \quad (9.63)$$

Thus

$$\nabla \cdot \vec{E}(z,t) = e^{i(kz-\omega t)} i k E_{0z}. \quad (9.64)$$

If this wave has to satisfy Maxwell’s equation, then

$$\nabla \cdot \vec{E}(z,t) = 0 \Rightarrow E_{0z} = 0. \quad (9.65)$$

This tells as the electric field is transverse. Which means it oscillates on a plane normal to the direction of propagation. Following a similar procedure we can show that

$$\nabla \cdot \vec{B}(z,t) = 0 \Rightarrow B_{0z} = 0. \quad (9.66)$$

Now let’s apply Faraday’s Law for the electric and magnetic fields in a vacuum

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (9.67)$$

Using what we already found, $E_{0z} = B_{0z} = 0$, we can write

$$\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
E_{0x} e^{i(kz-\omega t)} & E_{0y} e^{i(kz-\omega t)} & 0 \\
\end{vmatrix} = -\frac{\partial}{\partial t} \begin{bmatrix}
B_{0x} e^{i(kz-\omega t)} \\
B_{0y} e^{i(kz-\omega t)} \\
0
\end{bmatrix} \quad (9.68)$$

$$-\frac{\partial}{\partial z} \left( E_{0y} e^{i(kz-\omega t)} \right) \hat{x} + \frac{\partial}{\partial z} \left( E_{0x} e^{i(kz-\omega t)} \right) \hat{y} + \frac{\partial}{\partial x} \left( E_{0y} e^{i(kz-\omega t)} \right) \hat{z} + i \omega B_{0x} e^{i(kz-\omega t)} \hat{x} + i \omega B_{0y} e^{i(kz-\omega t)} \hat{y}. \quad (9.69)$$
Recalling that for a plane wave $E_{0x}$ and $E_{0y}$ are constant, we find
\[-i k E_{0y} e^{i(kz-\omega t)} \hat{x} + i k E_{0x} e^{i(kz-\omega t)} \hat{y} = i \omega B_{0x} e^{i(kz-\omega t)} \hat{x} + i \omega B_{0y} e^{i(kz-\omega t)} \hat{y},\] (9.70)

There follows that
\[-k E_{0y} = \omega B_{0x}, k E_{0x} = \omega B_{0y} \Rightarrow B_{0x} = -\frac{k}{\omega} E_{0y}, B_{0y} = \frac{k}{\omega} E_{0x} \] (9.71)

Noting that
\[\vec{k} = k \hat{z}, \vec{E}_0 = E_{0x} \hat{x} + E_{0y} \hat{y},\]

and the results we obtained for the magnetic field, we have
\[\vec{B}_0 = B_{0x} \hat{x} + B_{0y} \hat{y} = -\frac{k}{\omega} E_{0y} \hat{x} + \frac{k}{\omega} E_{0x} \hat{y}\]
\[= -\frac{k}{\omega} E_{0y} (\hat{y} \times \hat{z}) + \frac{k}{\omega} E_{0x} (\hat{z} \times \hat{x}) = \frac{k}{\omega} E_{0y} (\hat{z} \times \hat{y}) + \frac{k}{\omega} E_{0x} (\hat{z} \times \hat{x})\]

so that
\[\vec{B}_0 = \frac{1}{\omega} \vec{k} \times \vec{E}_0 = \frac{k}{\omega} \hat{k} \times \vec{E}_0.\] (9.72)

Using the relations
\[k = \frac{2 \pi}{\lambda}, c = \lambda f \Rightarrow \omega = 2 \pi f = \frac{2 \pi c}{\lambda} \Rightarrow \frac{k}{\omega} = \frac{1}{c}\] (9.74)

we find
\[\vec{B}_0 = \frac{1}{c} \vec{k} \times \vec{E}_0 \Rightarrow \vec{B}_0 = \frac{1}{c} \vec{E}_0.\] (9.75)

Generally, electromagnetic waves can propagate in any direction and we can define the wave vector $\vec{k}$ which has a magnitude equal to the wave number ($k = \frac{2 \pi}{\lambda}$) and its direction describes the direction of propagation of the wave. In terms of the wave vector and the polarization direction we can then describe the electric and magnetic fields for plane electromagnetic waves as
\[\vec{E} = E_0 \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \right] \hat{n},\] \[\vec{B} = \frac{E_0}{c} \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \right] \left( \vec{k} \times \hat{n} \right)\] (9.76) \hspace{1cm} (9.77)

The electric, magnetic, and the wave vector form an orthogonal set of vectors
\[\vec{k} \cdot \hat{n} = 0, \left( \vec{k} \times \hat{n} \right) \cdot \hat{n} = 0.\] (9.78)

Energy and momentum in electromagnetic waves: We recall that the energy carried by an electromagnetic waves per unit volume (energy density) is given by
\[u = \frac{1}{2} \left[ \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right].\] (9.79)
For a plane electromagnetic wave the real electric and magnetic fields are given by

\[
\vec{E} = E_0 \cos \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{n}
\]
\[
\vec{B} = \frac{E_0}{c} \cos \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \left( \hat{k} \times \hat{n} \right)
\]

which leads to

\[
E^2 = \vec{E} \cdot \vec{E} = E_0^2 \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right)
\]
\[
B^2 = \frac{E_0^2}{c^2} \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) = \epsilon_0 \mu_0 E_0^2 \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right).
\]

where we used

\[
c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.
\]

Then the total electromagnetic energy density becomes

\[
u = \frac{1}{2} \left[ \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right] = \epsilon_0 E_0^2 \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right).
\]

The energy flux density: the amount of energy incident on a unit area per unit time. It is given by the pointing vector

\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}
\]

\[
\vec{S} = \frac{1}{\mu_0} E_0 \cos \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{n} \times \frac{E_0}{c} \exp \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \left( \hat{k} \times \hat{n} \right)
\]

\[
= \frac{1}{c \mu_0} E_0^2 \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{n} \times \left( \hat{k} \times \hat{n} \right)
\]

Using the relation

\[
\vec{A} \times \left( \vec{B} \times \vec{C} \right) = \left( \vec{A} \cdot \vec{C} \right) \vec{B} - \left( \vec{A} \cdot \vec{B} \right) \vec{C}
\]

we may write

\[
\hat{n} \times \left( \hat{k} \times \hat{n} \right) = \left( \hat{n} \cdot \hat{n} \right) \hat{k} - \left( \hat{n} \cdot \hat{k} \right) \hat{n} = \hat{k}.
\]

Then the pointing vector can be rewritten as

\[
\vec{S} = \frac{1}{c \mu_0} E_0^2 \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{k} = \frac{c}{\mu_0} \frac{E_0^2}{c^2} \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{k}
\]
\[
= c \frac{E_0^2}{\mu_0 \sqrt{\epsilon_0 \mu_0}} \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{k} = c c_0 E_0^2 \cos^2 \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \hat{k}
\]
\[
\Rightarrow \vec{S} = c u \hat{k}.
\]
Electromagnetic waves momentum: We recall that the momentum density is given by
\[ \mathbf{p} = \epsilon_0 \mu_0 \mathbf{E} = \frac{1}{c^2} \mathbf{E} = \frac{u}{c} \mathbf{k}. \] (9.89)

We are interested in the time averages for the energy density, pointing vector, or the momentum density. To find the time averages we need to find the average for \( \cos^2 \left( k \cdot \mathbf{r} - \omega t + \delta \right) \) over a period of time.

\[ \left\langle \cos^2 \left( k \cdot \mathbf{r} - \omega t + \delta \right) \right\rangle = \frac{1}{T} \int_0^T \cos^2 \left( k \cdot \mathbf{r} - \omega t + \delta \right) dt \]
\[ = -\frac{1}{T \omega} \int_0^T \cos^2 \left( k \cdot \mathbf{r} - \omega t + \delta \right) d \left( k \cdot \mathbf{r} - \omega t + \delta \right) = -\frac{1}{T \omega} \int_{k \cdot \mathbf{r} + \delta + \omega T}^{k \cdot \mathbf{r} + \delta} \cos^2 \left( v \right) d \left( v \right), \]
\[ (9.90) \]

where \( v = k \cdot \mathbf{r} - \omega t + \delta \). Using
\[ T = \frac{1}{f} = \frac{2\pi}{\omega}, \cos^2 \left( v \right) = \frac{\cos(2v) + 1}{2} \]
we find
\[ \left\langle \cos^2 \left( k \cdot \mathbf{r} - \omega t + \delta \right) \right\rangle = -\frac{1}{2\pi} \int_{k \cdot \mathbf{r} + \delta}^{k \cdot \mathbf{r} + \delta + 2\pi} \left( \frac{\cos(2v) + 1}{2} \right) dv \]
\[ \left\langle \cos^2 \left( k \cdot \mathbf{r} - \omega t + \delta \right) \right\rangle = \frac{1}{2}, \]
\[ (9.92) \]

The time average energy density, pointing vector, and momentum become
\[ \langle u \rangle = \frac{1}{2} \epsilon_0 E_0^2, \]
\[ \langle S \rangle = c \langle u \rangle \hat{k} = \frac{1}{2} \epsilon_0 \mu_0 E_0^2 \hat{k}, \]
\[ \langle \mathbf{p} \rangle = \frac{\langle u \rangle}{c} \hat{k} = \frac{1}{2} \frac{\epsilon_0}{c} E_0^2 \mathbf{k}. \] (9.93)

Electromagnetic waves in a vacuum travels with a speed \( c \). In a time interval \( \Delta t \) the distance traveled can be expressed as, \( l = c \Delta t \). If the waves are normally incident on an area, \( A \), then the amount of average energy crossing this area in a time interval, \( \Delta t \), can be obtained by multiplying the energy density by the volume, \( V = A \Delta t \)
\[ \Delta w = \langle u \rangle V = \frac{1}{2} \epsilon_0 E_0^2 A c \Delta t \] (9.94)
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The intensity, $I$, defined as the amount of energy per unit area per unit time can then be given by

$$I = \frac{1}{A} \frac{\Delta w}{\Delta t} = \frac{1}{2} \varepsilon_0 E_0^2. \quad (9.95)$$

Suppose we have a perfect absorber. It can take all the momentum of the incident EM wave. Let’s assume that the wave loses its momentum to the perfect absorber in a time interval $\Delta t$. The average change in momentum for the wave, $\Delta \vec{p}$, can then be written as

$$\Delta \vec{p} = \vec{p}_{\text{final}} - \vec{p}_{\text{initial}} = - \langle \vec{p} \rangle V = - \langle \vec{p} \rangle A c \Delta t = - \langle u \rangle \frac{1}{c} A c \Delta t \hat{k}. \quad (9.96)$$

Then the force exerted on the absorber will be

$$\vec{F} = \frac{\Delta \vec{p}}{\Delta t} = \langle u \rangle A \hat{k}. \quad (9.97)$$

The radiation pressure which is the magnitude of the force exerted by the EM waves per unit area on the absorber

$$P = \frac{F}{A} = \langle u \rangle = \frac{1}{c} \left( \frac{1}{2} \varepsilon_0 E_0^2 \right) \Rightarrow P = \frac{I}{c}. \quad (9.98)$$

For it is a perfect reflector the radiation pressure will be twice since the change in momentum is

$$\Delta \vec{p} = \vec{p}_{\text{final}} - \vec{p}_{\text{initial}} = - \langle \vec{p} \rangle V - \langle \vec{p} \rangle V = - 2 \langle \vec{p} \rangle A c \Delta t$$

$$\Rightarrow \vec{F} = \frac{\Delta \vec{p}}{\Delta t} = - 2 \langle \vec{p} \rangle c A = - 2 \langle u \rangle A \hat{k}, \quad (9.99)$$

where we used, $\vec{p} = \frac{u}{c} \hat{k}$. The radiation pressure can then be expressed as

$$P = \frac{F}{A} = 2 \langle u \rangle = \varepsilon_0 E_0^2 \Rightarrow P = \frac{2I}{c}. \quad (9.100)$$

### Sun’s Radiation Pressure Table

<table>
<thead>
<tr>
<th>AU distance</th>
<th>$\mu$Pa ((\mu)N/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20 AU = close</td>
<td>227</td>
</tr>
<tr>
<td>0.39 AU = Mercury</td>
<td>60.6</td>
</tr>
<tr>
<td>0.72 AU = Venus</td>
<td>17.4</td>
</tr>
<tr>
<td>1.00 AU = Earth</td>
<td>9.08</td>
</tr>
<tr>
<td>1.52 AU = Mars</td>
<td>3.91</td>
</tr>
<tr>
<td>3.00 AU = asteroid</td>
<td>1.01</td>
</tr>
<tr>
<td>5.20 AU = Jupiter</td>
<td>0.34</td>
</tr>
</tbody>
</table>

The table shows that the accelerative forces very close to the Sun are very high, and almost of no comparative importance (for macroscopic particles) by the orbital distance of Jupiter. It is for this reason that most interplanetary radiation-pressure probe missions are sun grazers, whose orbital trajectory passes very close to the Sun so that at midpoint, the probe’s reflectors can be turned.
toward the Sun, adding considerable velocity to the craft. Because the ratio of surface area to volume (and thus mass) increases with decreasing particle size, dusty (micrometre-size) particles are susceptible to radiation pressure even in the outer solar system. For example, the evolution of the outer rings of Saturn is significantly influenced by radiation pressure.

9.5.2 Electromagnetic waves in matter

In the previous section we got introduced to the properties of EM waves in free space. In this section we shall study EM waves propagating in matter. We will consider linear and homogeneous media with no free charges ($\rho_{\text{free}} = 0$) and current ($J_{\text{free}} = 0$). Suppose the linear and homogeneous medium has an electrical permeability, $\epsilon$, and magnetic permeability, $\mu$. Then Maxwell’s equation in matter

$$\nabla \cdot \vec{D} = \frac{\rho_{\text{free}}}{\varepsilon_0}, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = J_{\text{free}}.$$  

(9.101)

becomes

$$\nabla \cdot \vec{D} = 0, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}.$$  

(9.102)
9.5. POLARIZATION

For linear media where the electric displacement vector field ($\vec{D}$) and auxiliary magnetic field ($\vec{H}$) are given by

$$\vec{D} = \varepsilon \vec{E}, \vec{H} = \frac{1}{\mu} \vec{B}. \quad (9.103)$$

we may write Maxwell’s equations as

$$\nabla \cdot \vec{E} = 0, \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \varepsilon \mu \frac{\partial \vec{E}}{\partial t}. \quad (9.104)$$

The speed of the EM waves in the medium can then be written as

$$v = \frac{1}{\sqrt{\varepsilon \mu}} = \frac{c}{n}, \quad (9.105)$$

where $n$ is the refractive index of the medium given by

$$n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} \quad (9.106)$$

and $c$ is the speed of light in vacuum. For most materials the response to external magnetic fields is negligible, one can make the approximation, $\mu \simeq \mu_0$, so that the refractive index can be expressed as

$$n \simeq \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r}, \quad (9.107)$$

$\varepsilon_r$ represent the dielectric constant of the medium. In a homogeneous and linear medium we can then write for the EM energy density

$$u = \frac{1}{2} \left( \varepsilon E^2 + \frac{1}{\mu} B^2 \right), \quad (9.108)$$

denoting the poynting vector

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B}, \quad (9.109)$$

the momentum density

$$\vec{p} = \varepsilon \mu S = \frac{1}{\varepsilon} \vec{S} = \frac{u}{v} \hat{k}, \quad (9.110)$$

and the intensity

$$I = \frac{1}{2} \varepsilon v E_0^2. \quad (9.111)$$

In order to study the properties of EM waves at the boundaries of two linear media it is important to know the electromagnetic boundary conditions for linear media that we got introduced to last semester. We recall these boundary conditions are given by

$$\varepsilon_1 E_1^\perp = \varepsilon_2 E_2^\perp, E_1^\parallel = E_2^\parallel, B_1^\perp = B_2^\perp, \frac{1}{\mu_1} B_1^\parallel = \frac{1}{\mu_2} B_2^\parallel \quad (9.112)$$
9.5.3 Reflection and Transmission at Normal incidence

Consider a monochromatic plane EM wave of frequency $\omega$ (see Fig. 9.12) propagating along the $z$-direction and incident at a normal angle at the interface of medium 1 and medium 2. We consider the EM wave is $x$-polarized so that one can express the incident electric field as

$$\vec{E}_I(z,t) = E_0 I \exp \left[ i (k_1 z - \omega t) \right] \hat{x}. \quad (9.113)$$

Since the wave is propagating along the $z$-direction ($\hat{k} = \hat{z}$), using the relation

$$\vec{B}_I(z,t) = \frac{E_I(z,t)}{v_1} \left( \hat{k} \times \hat{n} \right) \quad (9.114)$$

for the magnetic field, we can write

$$\vec{B}_I(z,t) = \frac{E_0 I}{v_1} \exp \left[ i (k_1 z - \omega t) \right] \hat{y}. \quad (9.115)$$

Similarly, for the reflected wave

$$\vec{E}_R(z,t) = E_0 R \exp \left[ i (-k_1 z - \omega t) \right] \hat{x},$$

$$\vec{B}_R(z,t) = \frac{E_R}{v_1} \left( \hat{k} \times \hat{n} \right) = \frac{E_R}{v_1} \left( -\hat{z} \times \hat{x} \right) = -\frac{E_0 R}{v_1} \exp \left[ i (-k_1 z - \omega t) \right] \hat{y}, \quad (9.116)$$

and for the transmitted wave

$$\vec{E}_T(z,t) = E_0 T \exp \left[ i (k_2 z - \omega t) \right] \hat{x}, \quad \vec{B}_T(z,t) = \frac{E_0 T}{v_2} \exp \left[ i (k_2 z - \omega t) \right] \hat{y} \quad (9.117)$$
For both electric and magnetic field there is no normal component to the interface. So we shall focus only on the boundary conditions for the tangential components
\[ E_1^\| = E_2^\|, \quad \frac{1}{\mu_1} B_1^\| = \frac{1}{\mu_2} B_2^\|. \]

Using
\[ E_1^\| = E_{0I} \exp[i (k_1 z - \omega t)] + E_{0R} \exp[i (-k_1 z - \omega t)] \]
\[ E_2^\| = E_{0T} \exp[i (k_2 z - \omega t)] \]
\[ B_1^\| = \frac{E_{0I}}{v_1} \exp[i (k_1 z - \omega t)] - \frac{E_{0R}}{v_1} \exp[i (-k_1 z - \omega t)] \]
\[ B_2^\| = \frac{E_{0T}}{v_2} \exp[i (k_2 z - \omega t)]. \]

at the boundary, \( z = 0 \), we find
\[ E_{0I} + E_{0R} = E_{0T}, \quad \frac{1}{v_1 \mu_1} (E_{0I} - E_{0R}) = \frac{1}{v_2 \mu_2} E_{0T}. \]

In terms of the incident electric field, the reflected and transmitted electric field amplitudes can be expressed as
\[ E_{0R} = \left( \frac{1 - \beta}{1 + \beta} \right) E_{0I}, \quad E_{0T} = \left( \frac{2}{1 + \beta} \right) E_{0I} \]

where
\[ \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \]

For most materials \( \mu_1 \approx \mu_2 = \mu_0 \) and this would give us
\[ \beta = \frac{v_1}{v_2} = \frac{n_2}{n_1} \]

and
\[ E_{0R} = \left( \frac{v_2 - v_1}{v_2 + v_1} \right) E_{0I}, \quad E_{0T} = \left( \frac{2v_2}{v_2 + v_1} \right) E_{0I}. \]

Which is similar to what we obtained for a wave on a string. In terms of the refractive indices it can be expressed as
\[ E_{0R} = \left( \frac{n_1 - n_2}{n_1 + n_2} \right) E_{0I}, \quad E_{0T} = \left( \frac{2n_1}{n_1 + n_2} \right) E_{0I}. \]

Reflection coefficient is determined by the ratio of the intensity of the reflected to incident EM wave. We recall intensity of an EM wave in a linear and homogeneous media is given by
\[ I = \frac{1}{2} \epsilon_0 E_0^2. \]
so that we may write
\[ I_R = \frac{1}{2} \varepsilon_1 v_1 E_R^2, \quad I_I = \frac{1}{2} \varepsilon_1 v_1 E_I^2 \] (9.125)
which leads to
\[ R = \frac{I_R}{I_I} = \left( \frac{E_{0R}}{E_{0I}} \right)^2 = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2. \] (9.126)
The transmission coefficient which is defined in a similar way becomes
\[ T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \left( \frac{E_{0T}}{E_{0I}} \right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2} \] (9.127)
where we used
\[ I_T = \frac{1}{2} \varepsilon_2 v_2 E_T^2. \] (9.128)
The reflection and transmission coefficient satisfy the relation
\[ R + T = 1. \] (9.129)
This proves that energy is conserved.

### 9.5.4 Reflection and transmission at oblique incidence

Now let’s consider an EM wave incident at some angle, \( \theta_I \), and this wave is partly reflected at some angle, \( \theta_R \), and partly transmitted at some angle, \( \theta_T \). (see Fig. 9.13)

The state of polarization of the EM wave is unknown. So we may write for the incident EM wave
\[ \vec{E}_I(\vec{r},t) = E_{0I} \exp \left[ i \left( \vec{k}_I \cdot \vec{r} - \omega t \right) \right], \quad \vec{B}_I(\vec{r},t) = \frac{1}{v_1} \vec{k}_I \times \vec{E}_I, \]
for the reflected EM wave
\[ \vec{E}_R(\vec{r},t) = E_{0R} \exp \left[ i \left( \vec{k}_R \cdot \vec{r} - \omega t \right) \right], \quad \vec{B}_R(\vec{r},t) = \frac{1}{v_1} \vec{k}_R \times \vec{E}_R, \]
and the transmitted wave
\[ \vec{E}_T(\vec{r},t) = E_{0T} \exp \left[ i \left( \vec{k}_T \cdot \vec{r} - \omega t \right) \right], \quad \vec{B}_T(\vec{r},t) = \frac{1}{v_2} \vec{k}_T \times \vec{E}_T, \]

At the boundary \((z = 0)\), we have
\[ \varepsilon_1 \left[ \vec{E}_I^\perp(\vec{r},t) + \vec{E}_R^\perp(\vec{r},t) \right] = \varepsilon_2 \vec{E}_T^\perp(\vec{r},t) \bigg|_{z=0}, \quad \vec{E}_I^\parallel(\vec{r},t) + \vec{E}_R^\parallel(\vec{r},t) = \vec{E}_T^\parallel(\vec{r},t) \bigg|_{z=0}, \]
\[ \frac{\mu_1}{\mu_2} \left[ \vec{B}_I^\perp(\vec{r},t) + \vec{B}_R^\perp(\vec{r},t) \right] = \frac{\vec{B}_T^\perp(\vec{r},t)}{\mu_2} \bigg|_{z=0}, \]
(9.130)
Figure 9.13: EM waves incident at an oblique angle. Note that the polarization is not specified. The EM wave can oscillate at any plane normal to the direction of propagation.

so that

\[
\epsilon_1 \tilde{E}_{0I}^+ \exp \left[ i \left( \vec{k}_I \cdot \vec{r} - \omega t \right) \right] + \epsilon_1 \tilde{E}_{0R}^+ \exp \left[ i \left( \vec{k}_R \cdot \vec{r} - \omega t \right) \right] = \epsilon_2 \tilde{E}_{0T}^+ \exp \left[ i \left( \vec{k}_T \cdot \vec{r} - \omega t \right) \right] \bigg|_{z=0}. \tag{9.131}
\]

We used the same frequency, \( \omega \), for the incident, reflected, and transmitted waves since we are considering a none dispersive and also a none dissipative media (for now). Thus the above equation becomes

\[
\epsilon_1 \left[ \tilde{E}_{0I}^+ \exp \left( i\vec{k}_I \cdot \vec{r} \right) + \tilde{E}_{0R}^+ \exp \left( i\vec{k}_R \cdot \vec{r} \right) \right] = \epsilon_2 \tilde{E}_{0T}^+ \exp \left( i\vec{k}_T \cdot \vec{r} \right) \bigg|_{z=0}. \tag{9.132}
\]

Suppose we have three two-dimensional vector functions \( \vec{A}, \vec{B} \) and \( \vec{C} \) that involve an exponential functions

\[
\vec{A} = \vec{A}_0 \exp \left( \vec{k}_a \cdot \vec{r} \right) = \left( A_{0x} \hat{x} + A_{0y} \hat{y} \right) \exp \left( i\vec{k}_a \cdot \vec{r} \right), \tag{9.133}
\]

\[
\vec{B} = \vec{B}_0 \exp \left( \vec{k}_b \cdot \vec{r} \right) = \left( B_{0x} \hat{x} + B_{0y} \hat{y} \right) \exp \left( i\vec{k}_b \cdot \vec{r} \right), \tag{9.134}
\]

\[
\vec{C} = \vec{C}_0 \exp \left( \vec{k}_c \cdot \vec{r} \right) = \left( C_{0x} \hat{x} + C_{0y} \hat{y} \right) \exp \left( i\vec{k}_c \cdot \vec{r} \right), \tag{9.135}
\]
and if these vector functions are related by

\[ \vec{A} + \vec{B} = \vec{C} \Rightarrow (A_{0x} \hat{x} + A_{0y} \hat{y}) \exp \left( i \vec{k}_a \cdot \vec{r} \right) + (B_{0x} \hat{x} + B_{0y} \hat{y}) \exp \left( i \vec{k}_b \cdot \vec{r} \right) = (C_{0x} \hat{x} + C_{0y} \hat{y}) \exp \left( i \vec{k}_c \cdot \vec{r} \right) \] (9.136)

one can write

\[ A_{0x} \exp \left( i \vec{k}_a \cdot \vec{r} \right) + B_{0x} \exp \left( i \vec{k}_b \cdot \vec{r} \right) = C_{0x} \exp \left( i \vec{k}_c \cdot \vec{r} \right), \] (9.137)

\[ A_{0y} \exp \left( i \vec{k}_a \cdot \vec{r} \right) + B_{0y} \exp \left( i \vec{k}_b \cdot \vec{r} \right) = C_{0y} \exp \left( i \vec{k}_c \cdot \vec{r} \right). \] (9.138)

Multiplying the first by \( A_{0y} \) and the second by \( A_{0x} \), we have

\[ A_{0x} A_{0y} \exp \left( i \vec{k}_a \cdot \vec{r} \right) + B_{0x} A_{0y} \exp \left( i \vec{k}_b \cdot \vec{r} \right) = C_{0x} A_{0y} \exp \left( i \vec{k}_c \cdot \vec{r} \right) \] (9.139)

\[ A_{0x} A_{0y} \exp \left( i \vec{k}_a \cdot \vec{r} \right) + A_{0x} B_{0y} \exp \left( i \vec{k}_b \cdot \vec{r} \right) = A_{0x} C_{0y} \exp \left( i \vec{k}_c \cdot \vec{r} \right). \] (9.140)

so that upon subtracting these two equations, one finds

\[ (B_{0x} A_{0y} - A_{0x} B_{0y}) \exp \left( i \vec{k}_b \cdot \vec{r} \right) = (C_{0x} A_{0y} - A_{0x} C_{0y}) \exp \left( i \vec{k}_c \cdot \vec{r} \right). \] (9.141)

There follows that

\[ B_{0x} A_{0y} - A_{0x} B_{0y} = C_{0x} A_{0y} - A_{0x} C_{0y} \]

\[ \exp \left( i \vec{k}_b \cdot \vec{r} \right) = \exp \left( i \vec{k}_c \cdot \vec{r} \right) \Rightarrow \vec{k}_b \cdot \vec{r} = \vec{k}_c \cdot \vec{r}. \] (9.142)

Similarly by \( B_{0y} \) and \( B_{0x} \), respectively, one finds

\[ A_{0x} B_{0y} \exp \left( i \vec{k}_a \cdot \vec{r} \right) + B_{0x} B_{0y} \exp \left( i \vec{k}_b \cdot \vec{r} \right) = C_{0x} B_{0y} \exp \left( i \vec{k}_c \cdot \vec{r} \right) \] (9.143)

\[ B_{0x} A_{0y} \exp \left( i \vec{k}_a \cdot \vec{r} \right) + B_{0x} B_{0y} \exp \left( i \vec{k}_b \cdot \vec{r} \right) = B_{0x} C_{0y} \exp \left( i \vec{k}_c \cdot \vec{r} \right) \] (9.144)

that leads to

\[ (A_{0x} B_{0y} - B_{0x} A_{0y}) \exp \left( i \vec{k}_b \cdot \vec{r} \right) = (C_{0x} B_{0y} - B_{0x} C_{0y}) \exp \left( i \vec{k}_c \cdot \vec{r} \right). \] (9.145)

There follows that

\[ A_{0x} B_{0y} - B_{0x} A_{0y} = C_{0x} B_{0y} - B_{0x} C_{0y} \]

\[ \exp \left( i \vec{k}_a \cdot \vec{r} \right) = \exp \left( i \vec{k}_c \cdot \vec{r} \right) \Rightarrow \vec{k}_a \cdot \vec{r} = \vec{k}_c \cdot \vec{r} \] (9.146)

Therefore, from Eq. (9.142) and (9.145), for the three vector functions one can establish the relations

\[ \vec{k}_a \cdot \vec{r} = \vec{k}_b \cdot \vec{r} = \vec{k}_c \cdot \vec{r} \] (9.146)
and

\[ B_{0x}A_{0y} - A_{0x}B_{0y} = C_{0x}A_{0y} - A_{0x}C_{0y} \]

\[ A_{0x}B_{0y} - B_{0x}A_{0y} = C_{0x}B_{0y} - B_{0x}C_{0y} \]

In view of the relation in Eq. (9.146), for the boundary conditions at \( z = 0 \), in Eq. (9.132), one can write

\[ \tilde{k}_I \cdot \vec{r} = \tilde{k}_R \cdot \vec{r} = \tilde{k}_T \cdot \vec{r} \left| _{z=0} \right. \quad (9.147) \]

From this result we will find three fundamental laws in optics that you were introduced to in introductory physics.

**First Law:** The incident, reflected, and transmitted wave vectors lie on the same plane. We know this from

\[ \tilde{k}_I \cdot \vec{r} = \tilde{k}_R \cdot \vec{r} = \tilde{k}_T \cdot \vec{r} \left| _{z=0} \right. \Rightarrow (\tilde{k}_I - \tilde{k}_R) \cdot \vec{r} = 0, (\tilde{k}_R - \tilde{k}_T) \cdot \vec{r} = 0 \]

\[ \Rightarrow \tilde{k}_I - \tilde{k}_R \perp \vec{r} \quad \text{and} \quad \tilde{k}_R - \tilde{k}_T \perp \vec{r} \]

Since \( \vec{r} \) is on the x-y plane \( \tilde{k}_I, \tilde{k}_R, \) and \( \tilde{k}_T \) must on a plane normal to the x-y plane. This plane is known as plane of incidence. It also contains the normal vector to the interface.

**Second Law:** In view of the result in Eq. (9.147), at \( z = 0 \), one can write

\[ \tilde{k}_I \cdot \vec{r} = \tilde{k}_R \cdot \vec{r} = \tilde{k}_T \cdot \vec{r} \left| _{z=0} \right. \Rightarrow k_{Ix}x + k_{Iy}y = k_{Rx}x + k_{ Ry}y = k_{Tx}x + k_{Ty}y \]

Since \( k_{Ix}, k_{Iy}, k_{Rx}, k_{Ry}, k_{Tx}, \) and \( k_{Ty} \) are single valued constants the above equation must hold true for all \( x \) and \( y \). For \( x = 0 \), we get

\[ k_{Iy}y = k_{Ry}y = k_{Ty}y, \Rightarrow k_{Iy} = k_{Ry} = k_{Ty}. \quad (9.149) \]

Similarly for \( y = 0 \),

\[ k_{Ix}x = k_{Rx}x = k_{Tx}x \Rightarrow k_{Ix} = k_{Rx} = k_{Tx}. \quad (9.150) \]

Now let’s consider the result in Eq. (9.150) and put it in terms of the angle of incidence (\( \theta_I \)), reflection (\( \theta_R \)), and transmission (refraction) (\( \theta_T \))

\[ k_{Ix} = k_{Rx} = k_{Tx} \Rightarrow k_I \sin(\theta_I) = k_R \sin(\theta_R) = k_T \sin(\theta_T) \quad (9.151) \]

We recall the angular frequency, the wave vector, and the speed of the wave are related by, \( kv = \omega \). Noting that the frequency is the same in the two media (none dispersive) one can write

\[ \omega = v_1 k_I = v_1 k_R = v_2 k_T \quad (9.152) \]
which leads to

\[ k_R = k_I, k_T = \frac{v_1}{v_2} k_I. \]  \hspace{1cm} (9.153)

Substituting Eq. (9.153) into Eq. (9.151)

\[ k_I \sin (\theta_I) = k_R \sin (\theta_R) \Rightarrow \sin (\theta_I) = \sin (\theta_R) \]  \hspace{1cm} (9.154)

There follows the second law that states the angle of incidence is the same as the angle of reflection

\[ \theta_I = \theta_R. \]  \hspace{1cm} (9.155)

Third Law (Snell’s law): For the incident and transmitted waves using

\[ k_I \sin (\theta_I) = k_T \sin (\theta_T) \Rightarrow \sin (\theta_I) = \frac{v_1}{v_2} \sin (\theta_T) \]  \hspace{1cm} (9.156)

recalling that the speed of an EM wave is related to the refractive index of the medium, \( n \), and the speed in a vacuum, \( c \), by

\[ v = \frac{c}{n} \]  \hspace{1cm} (9.157)

we have

\[ v_1 = \frac{c}{n_I}, v_2 = \frac{c}{n_T} \]  \hspace{1cm} (9.158)

so that

\[ n_I \sin (\theta_I) = n_T \sin (\theta_T) \]  \hspace{1cm} (9.159)

the angle of incidence and angle of transmission are related by

\[ n_I \sin (\theta_I) = n_T \sin (\theta_T) \]  \hspace{1cm} (9.160)

Now using the electromagnetic boundary conditions in Eq. (??) and the result in Eq. (9.148), we have

\[ \varepsilon_1 (E_{0I} + E_{0R})_z = \varepsilon_2 (E_{0T})_z, (E_{0I} + E_{0R})_{x,y} = (E_{0T})_{x,y} \]

\[ (B_{0I} + B_{0R})_z = (B_{0T})_z, \frac{1}{\mu_1} (B_{0I} + B_{0R})_{x,y} = \frac{1}{\mu_2} B_{T x,y} \]  \hspace{1cm} (9.161)

Suppose the polarization of the incident light is parallel to the plane of incidence that is the \( x - z \) plane as shown in Fig. 9.5.4

then the electric field has a normal component (\( z \)-component) and a tangential component (\( x \)-component) which can be expressed as

\[ (E_{0I})_x = E_{0I} \cos (\theta_I), (E_{0R})_x = E_{0R} \cos (\theta_R), (E_{0T})_x = E_{0T} \cos (\theta_T) \]  \hspace{1cm} (9.162)

\[ (E_{0I})_z = -E_{0I} \sin (\theta_I), (E_{0R})_z = E_{0R} \sin (\theta_R), (E_{0T})_z = -E_{0T} \sin (\theta_T). \]

Using the boundary conditions for the electric field in Eqs. (9.161), we find

\[ \varepsilon_1 (E_{0I} + E_{0R})_z = \varepsilon_2 (E_{0T})_z \]

\[ \Rightarrow \varepsilon_1 [E_{0R} \sin (\theta_R) - E_{0I} \sin (\theta_I)] = -\varepsilon_2 E_{0T} \sin (\theta_T) \]  \hspace{1cm} (9.163)
\[ (E_{0I} + E_{0R})_x = (E_{0T})_x, \]
\[ \Rightarrow E_{0I} \cos (\theta_I) + E_{0R} \cos (\theta_R) = E_{0T} \cos (\theta_T). \quad (9.164) \]

Since the electric field is on the x-z plane and the magnetic field must be normal to the electric field and the wave vectors we can conclude that the magnetic field has only tangential component (i.e. component along the y-direction only). Hence the third boundary condition

\[ (B_{0I} + B_{0R})_z = (B_{0T})_z \Rightarrow 0 = 0 \quad (9.165) \]

However the last boundary condition

\[ \frac{1}{\mu_1} (\vec{B}_{0I} + \vec{B}_{0R})_{x,y} = \frac{1}{\mu_2} \vec{B}_{0T} x,y \quad (9.166) \]

becomes

\[ \frac{1}{v_1 \mu_1} (E_{0I} - E_{0R}) = \frac{1}{v_2 \mu_2} (k_T \times \vec{E}_{0T}) \Rightarrow \frac{1}{v_1 \mu_1} (E_{0I} - E_{0R}) = \frac{1}{v_2 \mu_2} (E_{0T}) \]

Using the law of reflection and law of refraction we obtained earlier, we may write

\[ E_{0I} - E_{0R} = \beta E_{0T}, E_{0I} + E_{0R} = \alpha E_{0T} \quad (9.167) \]

where

\[ \alpha = \frac{\cos (\theta_T)}{\cos (\theta_I)}, \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \quad (9.168) \]
Figure 9.14: The reflected and transmitted wave amplitude vs the angle of incidence for $n_1 = 1.0$ and $n_2 = 1.5$

The reflected and transmitted amplitudes becomes

$$E_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0I}, \quad E_{0T} = \left(\frac{2}{\alpha + \beta}\right) E_{0I},$$  \hspace{1cm} (9.169)

These are known as Fresnel’s equations. The amplitude of the transmitted and reflected waves depends on the angle of incidence.

$$\alpha = \frac{\cos(\theta_T)}{\cos(\theta_I)} = \sqrt{1 - \sin^2(\theta_T)}$$ \hspace{1cm} (9.170)

Using Snell’s law

$$n_1 \sin(\theta_I) = n_2 \sin(\theta_T) \Rightarrow \sin(\theta_T) = \frac{n_1}{n_2} \sin(\theta_I)$$ \hspace{1cm} (9.171)

we can write

$$\alpha = \frac{1 - \left(\frac{n_1}{n_2} \sin(\theta_I)\right)^2}{\cos(\theta_I)}$$ \hspace{1cm} (9.172)
9.5. POLARIZATION

Case 1: $\theta_I = 0$ (Normal incidence)
\[
\alpha = 1 \Rightarrow E_{0R} = \left(\frac{1 - \beta}{1 + \beta}\right) E_{0I}, E_{0T} = \left(\frac{2}{1 + \beta}\right) E_{0I} \tag{9.173}
\]

Case 2: $\theta_I = \pi/2$ (Total reflection)
\[
\alpha \rightarrow \infty \Rightarrow E_{0R} = -E_{0I}, E_{0T} = 0 \tag{9.174}
\]

Case 3: $\theta_I = \theta_B$ (Brewster’s angle): it is the angle at which the reflected wave disappears (i.e. $E_{0R} = 0$) and $\theta_I = \theta_B$. This leads to
\[
E_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) E_{0I} = 0 \Rightarrow \alpha = \beta \Rightarrow \frac{1 - \left(\frac{n_1}{n_2}\sin(\theta_B)\right)^2}{\cos(\theta_B)} = \beta
\]
\[
\Rightarrow 1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2(\theta_B) = \beta^2 \Rightarrow \frac{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2(\theta_B)}{1 - \sin^2(\theta_B)} = \beta^2
\]
\[
\Rightarrow 1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2(\theta_B) = \beta^2 \left(1 - \sin^2(\theta_B)\right) \Rightarrow \sin^2(\theta_B) = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2} \tag{9.175}
\]

Since for most media $\mu_1 \approx \mu_2$, we have
\[
\beta = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{n_2}{n_1} \Rightarrow \sin^2(\theta_B) = \frac{1 - \left(\frac{n_2}{n_1}\right)^2}{\left(\frac{n_1}{n_2}\right)^2 - \left(\frac{n_2}{n_1}\right)^2} \tag{9.176}
\]

Figure 9.15: Reflection at the Brewster (polarization) angle.
CHAPTER 9. ELECTROMAGNETIC WAVES

Noting that
\[ \tan^2 \theta_B = \frac{\sin^2(\theta_B)}{1 - \sin^2(\theta_B)} = \left[ 1 - \left( \frac{n_2}{n_1} \right)^2 \right] / \left[ 1 - \left( \frac{n_2}{n_1} \right)^2 - \left( \frac{n_2}{n_1} \right)^2 - 1 \right] \]
\[ = 1 - \left( \frac{n_2}{n_1} \right)^2 - \left( \frac{n_2}{n_1} \right)^2 - 1 = 1 - \left( \frac{n_2}{n_1} \right)^2 \]
\[ = \left( \frac{n_2}{n_1} \right)^2. \]  
(9.177)

There follows that
\[ \tan(\theta_B) = \frac{n_2}{n_1}. \]  
(9.178)

From Snell’s law we have
\[ n_1 \sin(\theta_B) = n_2 \sin(\theta_B) \]  
(9.179)
and combining with
\[ \tan(\theta_B) = \frac{n_2}{n_1} \Rightarrow n_1 \sin(\theta_B) = n_2 \cos(\theta_B) \Rightarrow n_1 \sin(\theta_B) = n_2 \sin(90^\circ - \theta_B) \]  
(9.180)
we find
\[ \sin(\theta_i) = \sin(90^\circ - \theta_B) \Rightarrow \theta_B + \theta_i = 90 \]  
(9.181)
This is what you learned in introductory physics where we said the reflected light polarized along the plane parallel to the interface when the sum of the incident angle and the transmitted angle is 90°.

9.6 Absorption

Electromagnetic waves in conductors: What we have studied in the previous sections is the behavior of EM waves propagating in a vacuum or through insulating materials such as glass, water. In such media the Maxwell’s equations are given by
\[ \nabla \cdot \vec{E} = 0, \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}, \]
where we have set \( \rho_f = 0 \) and \( J_f = 0 \). And these equations has lead us to the wave equation and the solution of which found to be a plane electromagnetic monochromatic wave function given by
\[ \vec{E} = E_0 \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \right] \hat{n}, \vec{B} = \frac{E_0}{v} \exp \left[ i \left( \vec{k} \cdot \vec{r} - \omega t + \delta \right) \right] \left( \vec{k} \times \hat{n} \right), \]  
(9.182)
where
\[ v = \frac{1}{\sqrt{\varepsilon \mu}} \] (9.183)
is the speed of the wave in the medium. In this section we will study the behavior
of EM waves in a conductive linear media. Maxwell’s equation in such media
should be written as
\[ \nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon}, \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \mu \vec{J}_f + \epsilon \mu \frac{\partial \vec{E}}{\partial t}. \] (9.184)
Using the continuity equation
\[ \frac{\partial \rho_f}{\partial t} = -\nabla \cdot \vec{J}_f \] (9.185)
for Ohmic materials where
\[ \vec{J}_f = \sigma \vec{E} \Rightarrow \frac{\partial \rho_f}{\partial t} = -\sigma \nabla \cdot \vec{E}, \Rightarrow \nabla \cdot \vec{E} = -\frac{1}{\sigma} \frac{\partial \rho_f}{\partial t} \] (9.186)
one can write
\[ \nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \Rightarrow -\frac{1}{\sigma} \frac{\partial \rho_f}{\partial t} = \frac{\rho_f}{\epsilon} \Rightarrow \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon} \rho_f \Rightarrow \rho_f (t) = \rho_f (0) e^{-t/\tau}, \] (9.187)
where
\[ \tau = \frac{\epsilon}{\sigma} \] (9.188)
is called the characteristic time. For metals the real electrical permittivity is one (i.e. \( \text{Re}(\varepsilon) \approx 1 \)).

<table>
<thead>
<tr>
<th>Metals</th>
<th>Conductivity at 20°C, ( \sigma )</th>
<th>Characteristic time, ( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>6.30 \times 10^7</td>
<td>1.58 \times 10^{-8}</td>
</tr>
<tr>
<td>Copper</td>
<td>5.96 \times 10^7</td>
<td>1.68 \times 10^{-8}</td>
</tr>
<tr>
<td>Aluminium</td>
<td>3.77 \times 10^7</td>
<td>2.65 \times 10^{-8}</td>
</tr>
<tr>
<td>Calcium</td>
<td>2.98 \times 10^7</td>
<td>3.35 \times 10^{-8}</td>
</tr>
<tr>
<td>Tungsten</td>
<td>1.79 \times 10^7</td>
<td>5.59 \times 10^{-8}</td>
</tr>
<tr>
<td>Zinc</td>
<td>1.69 \times 10^7</td>
<td>5.92 \times 10^{-8}</td>
</tr>
<tr>
<td>Nickel</td>
<td>1.43 \times 10^7</td>
<td>6.99 \times 10^{-8}</td>
</tr>
<tr>
<td>Lithium</td>
<td>1.08 \times 10^7</td>
<td>9.25 \times 10^{-8}</td>
</tr>
<tr>
<td>Iron</td>
<td>1.00 \times 10^7</td>
<td>10.0 \times 10^{-8}</td>
</tr>
</tbody>
</table>

It means if there is a free charge density \( \rho_f (0) \) at \( t = 0 \), this charge will
dissipate (move to the surface) in a characteristic time, \( \tau = \frac{\epsilon}{\sigma} \).

\[
\begin{align*}
\tau = 0, & \quad \text{For a perfect conductor (} \sigma = \infty \text{)} \\
\tau << \frac{1}{\tau}, & \quad \text{For a good conductor (} \sigma \neq 0 \text{)} \\
\tau >> \frac{1}{\tau}, & \quad \text{For a poor conductor (} \sigma \approx 0 \text{)}
\end{align*}
\] (9.189)
For any time \( t >> \tau \), we may drop the free volume charge density in a conductor
and rewrite the Maxwells equations as
\[ \nabla \cdot \vec{E} = 0, \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \sigma \mu \vec{E} + \epsilon \mu \frac{\partial \vec{E}}{\partial t}. \]
where we used \( \vec{J}_f = \sigma \vec{E} \).

Using the relation
\[
\nabla \times \nabla \times \vec{A} = \nabla \left( \nabla \cdot \vec{A} \right) - \nabla^2 \vec{A}
\] (9.190)
we may write
\[
\nabla \times \nabla \times \vec{E} = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left( \nabla \times \vec{B} \right)
\] (9.191)
and using
\[
\nabla \cdot \vec{E} = 0, \nabla \times \vec{B} = \sigma \mu \vec{E} + \epsilon \mu \frac{\partial \vec{E}}{\partial t}
\] (9.192)
we find
\[
\nabla^2 \vec{E} = \frac{\partial}{\partial t} \left( \sigma \mu \vec{E} + \epsilon \mu \frac{\partial \vec{E}}{\partial t} \right) \Rightarrow \nabla^2 \vec{E} = \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} + \sigma \mu \frac{\partial \vec{E}}{\partial t}.
\] (9.193)

Following the same procedure, one can also find
\[
\nabla^2 \vec{B} = \epsilon \mu \frac{\partial^2 \vec{B}}{\partial t^2} + \sigma \mu \frac{\partial \vec{B}}{\partial t}.
\] (9.194)
If we assume a plane monochromatic wave propagating along the z-direction described by
\[
\vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)}
\] (9.195)
the above equation leads to
\[
-k^2 \vec{E} = -\epsilon \mu \omega^2 \vec{E} - i\sigma \mu \omega \vec{E} \Rightarrow -k^2 = -\epsilon \mu \omega^2 - i\sigma \mu \omega
\Rightarrow k^2 = \epsilon \mu \omega^2 + i\sigma \mu \omega,
\] (9.196)
which means inside a conductor the wave vector, \( k \), is a complex quantity and can be expressed as
\[
k = k_{Re} + ik_{Im}.
\] (9.197)
Using this we may write
\[
k^2 = \epsilon \mu \omega^2 + i\sigma \mu \omega
\] (9.198)
as
\[
k_{Re}^2 - k_{Im}^2 + 2ik_{Re}k_{Im} = \epsilon \mu \omega^2 + i\sigma \mu \omega
\] (9.199)
which leads to
\[
k_{Re}^2 - k_{Im}^2 = \epsilon \mu \omega^2 \Rightarrow k_{Re}^2 = \epsilon \mu \omega^2 + k_{Im}^2,
\]
\[
k_{Re}k_{Im} = \frac{\sigma \mu \omega}{2}.
\] (9.200)
9.6. ABSORPTION

Noting that

\[ k_{Re}^2 k_{Im}^2 = \left( \frac{\sigma \mu \omega}{2} \right)^2 \Rightarrow (\epsilon \mu \omega^2 + k_{Im}^2) k_{Im}^2 = \left( \frac{\sigma \mu \omega}{2} \right)^2 \]  \hspace{1cm} (9.201)

\[ (k_{Im}^2)^2 + \epsilon \mu \omega^2 (k_{Im}^2) - \left( \frac{\sigma \mu \omega}{2} \right)^2 = 0 \]  \hspace{1cm} (9.202)

we find

\[ k_{Im} = \omega \sqrt{\frac{\epsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} - 1 \right]^{1/2}} \]  \hspace{1cm} (9.203)

and

\[ k_{Re} = \omega \sqrt{\frac{\epsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} + 1 \right]^{1/2}} \]  \hspace{1cm} (9.204)

Substituting \( k \) in the expression for the electric and magnetic field vectors we find

\[ \vec{E}(z, t) = \vec{E}_0 e^{i[(k_{Re} + ik_{Im})z - \omega t]}, \vec{B}(z, t) = \vec{B}_0 e^{i[(k_{Re} + ik_{Im})z - \omega t]} \]  \hspace{1cm} (9.205)

which leads to

\[ \vec{E}(z, t) = \vec{E}_0 e^{i(-k_{Im}^2 e^{i[k_{Re} z - \omega t]})}, \vec{B}(z, t) = \vec{B}_0 e^{i(-k_{Im}^2 e^{i[k_{Re} z - \omega t]})} \]  \hspace{1cm} (9.206)

These results show that both the electric and magnetic fields are damping as the wave propagates inside the conductor. The distance over which the amplitude of the wave damped by a factor of \( 1/e \) is known as the skin depth, \( d \). It measures how far the wave penetrates in the medium.

\[ |\vec{E}(d, t)| = \frac{1}{e} |\vec{E}(0, t)| \Rightarrow |\vec{E}_0 e^{i(-k_{Im} d e^{i[k_{Re} d - \omega t]})}| = \frac{1}{e} |\vec{E}_0 e^{-i\omega t}| \Rightarrow e^{-k_{Im} d} = \frac{1}{e} \Rightarrow d = \frac{1}{k_{Im}}. \]  \hspace{1cm} (9.207)

For good conductors, where \( \sigma \rightarrow \infty \), we may approximate the imaginary part of the magnitude of the wavevector

\[ k_{Im} = \omega \sqrt{\frac{\epsilon \mu}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} - 1 \right]^{1/2}} \]  \hspace{1cm} (9.208)

as

\[ k_{Im} \approx \omega \sqrt{\frac{\epsilon \mu}{2} \left( \frac{\sigma}{\epsilon \omega} - 1 \right)^{1/2}} \approx \omega \sqrt{\frac{\epsilon \mu}{2} \frac{\sigma}{\epsilon \omega}} = \sqrt{\frac{\omega \mu \sigma}{2 \rho}} \]  \hspace{1cm} (9.209)

where \( \rho = 1/\sigma \) is the resistivity of the conductor. The skin-depth becomes

\[ d = \frac{1}{k_{Im}} \approx \sqrt{\frac{\rho}{\pi \mu_0 f}} = \frac{1}{\sqrt{\frac{\rho}{\mu_0 f}}} = \frac{1}{\sqrt{\mu_0 f}} \Rightarrow d \approx 503 \sqrt{\frac{\rho}{\mu_0 f}}, \]  \hspace{1cm} (9.210)
where $\mu_r$ is the relative magnetic permeability of the metals.

The graph shown below displays the skin depth for some metals as a function of the frequency of the incident EM waves. At very high frequencies the skin depth for good conductors becomes tiny. For instance, the skin depths of some common metals at a frequency of $10\,GHz$ (microwave region) are less than a micron:

<table>
<thead>
<tr>
<th>Conductor</th>
<th>Skin depth ($\mu$m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>0.80</td>
</tr>
<tr>
<td>Copper</td>
<td>0.65</td>
</tr>
<tr>
<td>Gold</td>
<td>0.79</td>
</tr>
<tr>
<td>Silver</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Comparing $E_0e^{-k_{im}z}e^{ik_{Re}z-\omega t}$ with $E_0e^{ik_{Re}z-\omega t}$, we can see that the real part, $k_{Re}$, can be related to the wave length of the wave by

$$k_{Re} = \frac{2\pi}{\lambda} \quad (9.211)$$

and to the speed

$$v = \frac{\omega}{k_{Re}} \quad (9.212)$$
and the refractive index
\[ n = \frac{c}{v} = \frac{ck_{\text{Re}}}{\omega}. \tag{9.213} \]

Let’s consider an x-polarized EM wave. The electric field can then be expressed as
\[ \vec{E}(z, t) = E_0 e^{-k_{\text{Im}}z} e^{i(k_{\text{Re}}z - \omega t)} \hat{x}. \tag{9.214} \]

Recalling that for a plane electromagnetic wave the electric field, the magnetic field, and the wave vector satisfy the condition
\[ \vec{B}(z, t) = \frac{1}{\omega} \vec{k} \times \vec{E}(z, t), \tag{9.215} \]

for a wave propagating along the z-direction, \( \vec{k} = k \hat{z} \), the magnetic field can be expressed as
\[ \vec{B}(z, t) = \frac{k}{\omega} E_0 e^{-k_{\text{Im}}z} e^{i(k_{\text{Re}}z - \omega t)} (\hat{z} \times \hat{x}) = \frac{k}{\omega} E_0 e^{-k_{\text{Im}}z} e^{i(k_{\text{Re}}z - \omega t)} \hat{y} \tag{9.216} \]

But we have already shown that \( k \) is complex in a conductor. Thus one may express the \( k \), using Euler’s notation, as
\[ k = k_{\text{Re}} + ik_{\text{Im}} = Ke^{i\theta} = K \cos(\theta) + iK \sin(\theta) \tag{9.217} \]

where
\[ K = \sqrt{k_{\text{Re}}^2 + k_{\text{Im}}^2} = \omega \sqrt{e\mu \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2}}, \theta = \tan^{-1}\left(\frac{k_{\text{Im}}}{k_{\text{Re}}}\right). \tag{9.218} \]

If we write the complex electric and magnetic field amplitudes as
\[ E_0 = |E_0| e^{i\delta_E}, B_0 = |B_0| e^{i\delta_B} \tag{9.219} \]
we may express
\[ |B_0| e^{i\delta_B} = \frac{k}{\omega} |E_0| e^{i\delta_E} = \frac{K |E_0|}{|E_0|} e^{i(\delta_E + \delta_E)} \]
\[ \delta_B = \theta + \delta_E \Rightarrow \delta_B - \delta_E = \theta. \tag{9.220} \]

This shows that the amplitudes of the magnetic and electric fields are related by
\[ |B_0| = \frac{|E_0|}{\omega} = |E_0| \sqrt{e\mu \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2}} \tag{9.221} \]

with a phase shift of \( \theta \) (the magnetic field lags behind the electric field by \( \theta \)).

The real electric and magnetic fields can then be expressed as
\[ \vec{E} = |E_0| e^{-k_{\text{Im}}z} \cos(k_{\text{Re}}z - \omega t + \delta_E) \hat{x} \]
\[ \vec{B} = |B_0| e^{-k_{\text{Im}}z} \cos(k_{\text{Re}}z - \omega t + \delta_E + \theta) \hat{y} \tag{9.222} \]
Figure 9.16: The electric and magnetic field propagating in conducting media.

These the expressions for the electric and magnetic fields show that as the waves propagates the amplitudes dampens inside a conductor as shown in Fig. 9.16.

Alternative way to show the field dampening in metals: For the partial differential equation

$$\nabla^2 \vec{E} = \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \sigma \frac{\partial \vec{E}}{\partial t}$$

using separation of variables

$$\vec{E} = R(\vec{r}) T(t) \hat{n}$$

one can write

$$T(t) \nabla^2 R(\vec{r}) = R(\vec{r}) \left[ \epsilon \frac{d^2 T(t)}{dt^2} + \sigma \frac{dT(t)}{dt} \right]$$

$$\Rightarrow \frac{1}{R(\vec{r})} \nabla^2 R(\vec{r}) = \frac{1}{T(t)} \left[ \epsilon \frac{d^2 T(t)}{dt^2} + \sigma \frac{dT(t)}{dt} \right] = -k^2.$$  

where $k$ is a constant that must be determined. There follows that

$$\frac{d^2 T(t)}{dt^2} + \frac{\sigma}{\epsilon} \frac{dT(t)}{dt} + \frac{k^2}{\epsilon} T(t) = 0$$

and

$$\nabla^2 R(\vec{r}) - k^2 R(\vec{r}) = 0.$$  

We note that the equation for the time dependent part is that of damped harmonic oscillator the solution of which can be expressed as

$$T(t) = \exp \left( -\frac{\sigma}{2\epsilon} t \right) \left[ A \cos \left( \sqrt{\frac{k^2}{\epsilon} - \left( \frac{\sigma}{2\epsilon} \right)^2} t \right) + B \cos \left( \sqrt{\frac{k^2}{\epsilon} - \left( \frac{\sigma}{2\epsilon} \right)^2} t \right) \right]$$
Considering a plane wave propagating along the z-direction one can write
\[
\frac{d^2}{dz^2} R(z) + k^2 R(z) = 0.
\] (9.223)
The solution of which can be expressed as
\[
R(z) = C \cos(kz) + D \cos(kz).
\] (9.224)
Therefore the electric field is found to be
\[
\mathbf{E}(z,t) = \sum_k \exp \left(-\frac{k}{2e}t\right) [C \cos(kz) + D \cos(kz)]
\times \left[ A \cos \left(\sqrt{\frac{k^2}{\epsilon} - \left(\frac{\sigma}{2e}\right)^2} t\right) + B \cos \left(\sqrt{\frac{k^2}{\epsilon} - \left(\frac{\sigma}{2e}\right)^2} t\right) \right] \hat{n}.
\]
Challenge: show that this expression can be expressed as
\[
\mathbf{E}(z,t) = E_0 e^{-k_{im}\cdot}\mathbf{i} \mathbf{e}^{i[k_{Re}z-\omega t]} \hat{n}.
\] (9.225)
where
\[
k_{im} = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}.
\] (9.226)
and
\[
k_{Re} = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2}.
\] (9.227)

9.7 Reflection at a conducting surface

In general whenever EM waves propagates from one medium to another the boundary conditions at the interface of the two media are given by
\[
\epsilon_1 \mathbf{E}_1^\perp - \epsilon_2 \mathbf{E}_2^\perp = \sigma_f, \mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0
\]
\[
\mathbf{B}_1^\perp - \mathbf{B}_2^\perp = 0, \frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel = \mathbf{K}_f \times \hat{n}.
\] (9.228)
Assume that medium 1 is nonconductive and medium two is conductive. An EM wave (x-polarized) is propagating from medium 1 to medium 2 along the direction as shown in the figure above. Then the incident, reflected, and transmitted electric and magnetic waves are given by
\[
\mathbf{E}_I(z,t) = E_0 e^{i[k_1 z - \omega t]} \hat{x}, \mathbf{B}_I(z,t) = \frac{E_{0T}}{\nu_1} e^{i[k_1 z - \omega t]} \hat{y},
\]
\[
\mathbf{E}_R(z,t) = E_0 e^{i[k_1 z - \omega t]} \hat{x}, \mathbf{B}_R(z,t) = -\frac{E_{0R}}{\nu_1} e^{i[k_1 z - \omega t]} \hat{y},
\]
\[
\mathbf{E}_T(z,t) = \hat{E}_0 T e^{i[k_2 z - \omega t]} \hat{x}, \mathbf{B}_T(z,t) = \frac{k_2 E_{0T}}{\omega} e^{i[k_2 z - \omega t]} \hat{y}.
\] (9.229)
The transmitted wave will be attenuated since it is traveling in a conducting medium. Noting that the free surface current density (Ohm’s law) is given by

\[ \vec{J}_f = \sigma \vec{E} \]

for ohmic conductors and this costs infinite electric field to generate a surface current which is impossible, we have \( \vec{k}_f = 0 \) for most conductors. Thus the boundary conditions given above can be written as

\[
\begin{align*}
\epsilon_1 \vec{E}_1^\perp - \epsilon_2 \vec{E}_2^\perp &= \sigma_f, \quad \vec{E}_1^\parallel - \vec{E}_2^\parallel = 0, \\
\vec{B}_1^\perp - \vec{B}_2^\perp &= 0, \quad \frac{1}{\mu_1} \vec{B}_1^\parallel - \frac{1}{\mu_2} \vec{B}_2^\parallel = 0.
\end{align*}
\]

(9.231)

For an x-polarized EM wave propagating along the z-direction, the normal component of the electric field to the interface (x-y plane) is zero and same is true for the magnetic field normal components. Hence, the above boundary conditions reduces to the tangential components only

\[
\vec{E}_1^\parallel = \vec{E}_2^\parallel, \quad \frac{1}{\mu_1} \vec{B}_1^\parallel = \frac{1}{\mu_2} \vec{B}_2^\parallel.
\]

(9.232)

Applying these conditions at \( z = 0 \) we find

\[
E_{0I} + E_{0R} = E_{0T}, \quad \frac{1}{\mu_1 v_1} (E_{0I} - E_{0R}) = \frac{k_2 E_{0T}}{\mu_2 \omega}
\]

(9.233)

The second equation can be put in the form

\[
E_{0I} - E_{0R} = \beta E_{0T}
\]

(9.234)
where
\[ \beta = \frac{\mu_1 v_1}{\mu_2 \omega} k_2 \]  
(9.235)

The complex amplitude of the reflected and transmitted electric fields are given by
\[ E_{0R} = \left( \frac{1 - \beta}{1 + \beta} \right) E_{0I}, E_{0T} = \left( \frac{2}{1 + \beta} \right) E_{0I}. \]  
(9.236)

For a perfect conductor we have \( \sigma = \infty \) and the real and imaginary part of the wave vector for the conducting medium becomes
\[ k_{\text{Re}} = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} + 1 \right]^{1/2} \rightarrow \infty, \]
\[ k_{\text{Im}} = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon \omega} \right)^2} - 1 \right]^{1/2} \rightarrow \infty \]  
(9.237)

and so does \( k \). Therefore the complex amplitudes of the reflected and transmitted electric fields becomes
\[ E_{0R} = -E_{0I}, E_{0T} = 0 \]  
(9.238)

Which means all the incident waves will be reflected with a phase shift of \( \pi \). That is why excellent conductors are used in making mirrors.

## 9.8 Dispersion

The Frequency Dependence of permittivity: The electrons in a nonconducting medium are bound to specific molecules. These electrons oscillates about the equilibrium position with a small amplitude. The Calcite (CaCO3) StructureThe electrons experiencing this kind of motion can be modeled as a harmonic oscillator. Then the electron experiences a spring force \( \vec{F}_s \) given by
\[ \vec{F}_s = -k \ddot{x}. \]  
(9.239)

In addition to this force, the electron can also experience some damping force \( \vec{F}_d \) which is proportional to the speed of the electron
\[ \vec{F}_d = -\gamma m \frac{dx}{dt} \dot{x}. \]  
(9.240)

When the electron is exposed to an EM wave with frequency, \( \omega \), polarized in the x-direction
\[ \vec{E} = E_0 \cos (\omega t) \hat{x} \]  
(9.241)

it experiences a driving force
\[ \vec{F}_d = qE \ddot{x} = qE_0 \cos (\omega t) \dot{x} \]  
(9.242)
Then using Newton’s second law, the net force acting on the electron can be written as

\[
m \frac{d^2x}{dt^2} = qE_0 \cos(\omega t) - \gamma \frac{dx}{dt} - kx \Rightarrow m \frac{d^2x}{dt^2} + \gamma m \frac{dx}{dt} + kx = qE_0 \cos(\omega t)
\]

or

\[
\frac{d^2\hat{x}}{dt^2} + \gamma \frac{d\hat{x}}{dt} + \omega_0^2 \hat{x} = \frac{qE_0}{m} \cos(\omega t)
\]

(9.243)

where

\[
\omega_0 = \sqrt{\frac{k}{m}}
\]

(9.245)

is the natural frequency of the electron. In terms of the complex variables, \( \hat{x} \) and \( \hat{E} \), we may write this equation as

\[
\frac{d^2\hat{x}}{dt^2} + \gamma \frac{d\hat{x}}{dt} + \omega_0^2 \hat{x} = \frac{qE_0}{m} \exp(-i\omega t).
\]

(9.246)
The homogenous solution to this differential equation is given by

\[ \ddot{x}_H(t) = e^{-\gamma t/2} (A \cos(\beta t) + B \sin(\beta t)). \]  

(9.247)

where

\[ \beta = \sqrt{\omega_0^2 - \gamma^2}. \]  

(9.248)

Substituting a particular solution of the form

\[ \ddot{x}_p(t) = \dot{x}_0 \exp(-i\omega t) \]  

(9.249)

into the differential equation, we find

\[ -m\omega^2 \ddot{x}_0 - i\gamma \omega m \dot{x}_0 + k \ddot{x}_0 = qE_0 \]  

(9.250)

which gives

\[ -m\omega^2 \ddot{x}_0 - im\gamma \omega \dot{x}_0 + m\omega_0^2 \ddot{x}_0 = qE_0 \Rightarrow \ddot{x}_0 = \frac{qE_0/m}{\omega_0^2 - \omega^2 - i\gamma \omega} \]  

(9.251)

and the particular solution becomes

\[ \ddot{x}_p(t) = \frac{qE_0/m}{\omega_0^2 - \omega^2 - i\gamma \omega} \exp(-i\omega t) \]  

(9.252)

Therefore the complex displacement of the electron is given by

\[ \ddot{x}(t) = \ddot{x}_H(t) + \ddot{x}_p(t) = e^{-\gamma t/2} (A \cos(\beta t) + B \sin(\beta t)) + \frac{qE_0/m}{\omega_0^2 - \omega^2 - i\gamma \omega} \exp(-i\omega t) \]  

(9.253)

But we are interested in the steady state of the electron which happen if we waited long enough (i.e. \( t \to \infty \)). Thus for steady state the complex displacement of the electron becomes

\[ \ddot{x}(t) \approx \frac{qE_0/m}{\omega_0^2 - \omega^2 - i\gamma \omega} \exp(-i\omega t), \]  

(9.254)

which shows at steady state the electron begins to oscillate with frequency of the EM field as we would expect. The complex dipole moment of the electron can then be expressed as

\[ \ddot{p}(t) = q\ddot{x}(t) = \frac{q^2E_0/m}{\omega_0^2 - \omega^2 - i\gamma \omega} \exp(-i\omega t). \]  

(9.255)

This can also be put in the form

\[ \ddot{p}(t) = \frac{q^2E_0/m}{(\omega_0^2 - \omega^2 + i\gamma \omega)^2} \exp(-i\omega t) = |\ddot{p}_0| \exp[-(\omega t - \varphi)], \]  

(9.256)
where
\[ |\vec{p}_0| = \frac{q^2 E_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma \omega)^2}} \]  \hspace{1cm} (9.257)
and
\[ \varphi = \tan^{-1} \left( \frac{\gamma \omega}{\omega_0^2 - \omega^2} \right). \]  \hspace{1cm} (9.258)

This means that \( p \) is out of phase by \( \varphi \) with respect to the electric field \( \vec{E} \).
Lagging behind by \( \varphi \) which is very small when \( \omega \ll \omega_0 \) and rises to \( \pi \) when \( \omega \gg \omega_0 \). If there are \( N \) molecules per unit volume and \( f_j \) electrons per molecule. If these electrons in the \( j \)th molecule is oscillating with the natural frequency \( \omega_j \) and damped by \( \gamma_j \), then the total dipole moment of the electrons per unit volume (the polarization \( \vec{P} \)) can be expressed as
\[ \vec{P} = \frac{Nq^2}{m} \sum_{j=1}^{N} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} E_0 \exp(-i\omega t) = \frac{Nq^2}{m} \sum_{j=1}^{N} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \vec{E}. \]  \hspace{1cm} (9.259)

This can also be put in the form
\[ \vec{P} = \varepsilon_0 \chi_e (\omega) \vec{E} \]  \hspace{1cm} (9.260)
where
\[ \chi_e (\omega) = \frac{Nq^2}{\varepsilon_0 m} \sum_{j=1}^{N} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}. \]  \hspace{1cm} (9.261)

the complex dielectric susceptibility. Then one can write for the complex permittivity
\[ \varepsilon (\omega) = \varepsilon_0 (1 + \chi_e (\omega)) \]  \hspace{1cm} (9.262)
and for the complex dielectric constant
\[ \varepsilon_r (\omega) = \frac{\varepsilon (\omega)}{\varepsilon_0} = 1 + \chi_e (\omega) = 1 + \frac{Nq^2}{\varepsilon_0 m} \sum_{j=1}^{N} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}. \]  \hspace{1cm} (9.263)

Eq. (9.263) shows the dielectric constant (the permittivity of the medium) depends on the frequency of the EM wave-the medium is dispersive. Still the wave equation is satisfied by the electric and magnetic fields. For a dispersive medium takes the form
\[ \nabla^2 \vec{E} = \varepsilon (\omega) \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]  \hspace{1cm} (9.264)
with a plane wave function
\[ \vec{E} = \vec{E}_0 \exp \left[ i \left( \tilde{k} z - \omega t \right) \right]. \]  \hspace{1cm} (9.265)

The wave number \( \tilde{k} \) is complex and is given by
\[ \tilde{k} = \frac{\omega}{\tilde{v}} \Rightarrow \tilde{k} = \sqrt{\varepsilon \mu_0} \omega \]  \hspace{1cm} (9.266)
which can be put in the form

\[ \tilde{k} = k_{\text{Re}} + ik_{\text{Im}}. \]  

(9.267)

Then the electric field may be put in the form

\[ \tilde{E} = \tilde{E}_0 \exp (-ik_{\text{Im}} z) \exp [i(k_{\text{Re}} z - \omega t)]. \]  

(9.268)

and the intensity which is proportional to the square of the amplitude of the electric field becomes

\[ I \propto \left| \tilde{E}_0 \right| \exp (-2ik_{\text{Im}} z) \]  

(9.269)

which shows a damping in the field amplitude due to the absorption by the medium. For that reason the absorption coefficient of the medium is given by

\[ \alpha = 2k_{\text{Im}}. \]  

(9.270)

The real wave velocity, \( v \), is given by

\[ v = \frac{\omega}{k_{\text{Re}}}. \]  

(9.271)

so that the refractive index of the medium becomes

\[ n = \frac{c}{v} = \frac{ck_{\text{Re}}}{\omega}. \]  

(9.272)

The complex wave number

\[ \tilde{k} = \sqrt{\varepsilon \mu_0 \omega} = \sqrt{\frac{\varepsilon c_0 \mu_0}{\varepsilon_0 \varepsilon_0 \mu_0}} = \frac{\omega}{c} \sqrt{1 + \frac{Nq^2}{\epsilon_0 \mu_0} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}} \]  

(9.273)

For gases the second term is very small and one can make a binomial expansion for the square root which gives

\[ \tilde{k} \approx \frac{\omega}{c} \left[ 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right] \]  

\[ = \frac{\omega}{c} \left[ 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_{j=1}^{\infty} f_j \left( \omega_j^2 - \omega^2 + i\gamma_j \omega \right) \right]. \]  

(9.274)

Then we can find the real and imaginary part of the wave number to be

\[ k_{\text{Re}} \approx \frac{\omega}{c} \left[ 1 + \frac{Nq^2}{2\epsilon_0 m} \sum_{j=1}^{\infty} \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2} \right], \]  

\[ k_{\text{Im}} \approx \frac{Nq^2 \omega^2}{2\epsilon_0 m} \sum_{j=1}^{\infty} \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2}. \]  

(9.275)
In terms of these expressions the absorption coefficient and the refractive index of the medium become

$$\alpha = 2k_{Im} = \frac{Nq^2\omega^2}{\varepsilon_0m} \sum_{j=1} f_j \gamma_j \frac{(\omega_j^2 - \omega^2)^2 + (\gamma_j\omega)^2}{(\omega^2 - \omega^2)}$$

(9.276)

$$n = \frac{c}{\omega} = \frac{ck_{Re}}{\omega} = 1 + \frac{Nq^2}{2\varepsilon_0m} \sum_{j=1} f_j (\omega_j^2 - \omega^2)$$

(9.277)

Let’s examine these for a very simplified case where we shall consider one particular molecule ($j^{th}$ molecule). To this end we note that

$$\alpha = \frac{Nq^2\omega^2}{\varepsilon_0m} \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + (\gamma_j\omega)^2} = \frac{Nq^2f_j}{\varepsilon_0m\omega_j} \frac{\omega_j^2 \gamma_j}{(1 - \frac{\omega^2}{\omega_j^2})^2 + \left(\frac{\gamma_j\omega}{\omega_j}\right)^2}$$

(9.278)

$$n = 1 + \frac{Nq^2}{2\varepsilon_0m} \frac{f_j \omega_j^2}{(\omega_j^2 - \omega^2)^2 + (\gamma_j\omega)^2} = 1 + \frac{Nq^2f_j}{2\varepsilon_0m\omega_j^2} \frac{(1 - \frac{\omega^2}{\omega_j^2})^2 + \left(\frac{\gamma_j\omega}{\omega_j}\right)^2}{(1 - \frac{\omega^2}{\omega_j^2})^2}$$

(9.279)

Introducing the transformation defined by the dimensionless variables

$$\nu = \frac{\omega}{\omega_j}, \eta = \frac{\gamma_j}{\omega_j}$$

(9.280)

we can write

$$\alpha = \frac{Nq^2f_j}{\varepsilon_0m\omega_j} \frac{\nu^2\eta}{(1 - \nu^2)^2 + (\eta\nu)^2} = \Gamma_1 \frac{\nu^2\eta}{(1 - \nu^2)^2 + (\eta\nu)^2},$$

$$n = 1 + \frac{Nq^2f_j}{2\varepsilon_0m\omega_j^2} \frac{1 - \nu^2}{(1 - \nu^2)^2 + (\eta\nu)^2} = 1 + \frac{Nq^2f_j}{2\varepsilon_0m\omega_j^2} \frac{1 - \nu^2}{(1 - \nu^2)^2 + (\eta\nu)^2}$$

(9.281)

where

$$\Gamma_1 = \frac{Nq^2f_j}{\varepsilon_0m\omega_j^2}, \Gamma_2 = \frac{Nq^2f_j}{2\varepsilon_0m\omega_j^2} = \frac{\varepsilon_1}{2\omega_j}$$

(9.282)

The absorption coefficient, $\alpha$, as function of $\nu = \omega/\omega_j$ for different values of $\eta = \gamma_j/\omega_j$ is displayed in Fig. 9.18. It shows that as the damping rate, $\gamma_j$, increases the range of the spectrum that would be absorbed by the medium increases with. The peak absorption occurs at resonance when the frequency of the EM wave is the same as the natural frequency of the electrons (i.e. $\lambda = 1 \Rightarrow \omega = \omega_j$).

On the other hand the refractive index, $n$, vs $\nu = \omega/\omega_j$ displayed in Fig. 9.19 shows unusual characteristics near the resonant region. In the near resonant region the refractive index decreases in contrary to what we know in optics. In optics you have been taught that $n = 1$ for vacuum and for any other material, $n > 1$. So we should expect that $n - 1 \geq 0$. However, the results in Fig. 9.19 in fact can be negative, $n - 1 < 0$, near the resonant region. Figure 9.19 also shows
that as the damping effect, $\gamma_j$ increases this unusual characteristics becomes significant for a wide range of the spectrum. Comparison of the absorption coefficient and the refractive index is shown in the figure below. The curves shown by the dotted lines are for the absorption and the solid line are for the refractive index. In the immediate neighborhood of a resonance the index of refraction drops sharply.

Refractive index increases for $\omega < \omega_1$ and $\omega > \omega_2$ which is consistent with our experience from optics (dispersion). The most familiar example of dispersion is probably a rainbow (Fig. 9.21), in which dispersion causes the spatial separation of a white light into components of different wavelengths (different colors). We see the spectacle of a rainbow glimmering against a dark stormy sky (Fig. 9.21.a) due to the dispersion of a white light passing through droplets of water. The same process causes white light to be broken into colors by a clear glass prism or a diamond (Fig. 9.21.b).

Refractive index decreases for $\omega_1 < \omega < \omega_2$, in the immediate neighborhood of a resonance the index of refraction drops sharply. The material may be practically opaque in this frequency range since it coincides with the region of maximum absorption. This is because of the electron are forced to oscillate with their favorite frequency and the amplitude of oscillation is maximum and correspondingly high amount of energy is dissipated by the damping mechanism. Because this behavior is atypical, it is called anomalous dispersion (the region $\omega_1 < \omega < \omega_2$).

Refractive index less than one for $\omega > \omega_0$ (above resonance) which means
Figure 9.19: The refractive index, $n - 1$ vs $\lambda = \omega/\omega_j$. For different values of $\eta = \gamma_j/\omega_j$, $\eta = 0.2$ (Red), $\eta = 0.7$ (blue), and $\eta = 1.4$ (Cyan), the wave speed exceeds the speed of light, $c$.

$$v = \frac{c}{n} > 1.$$ (9.283)

This is not an alarm since energy does not travel at the wave velocity but rather at a group velocity. Moreover we considered only one molecule.

Far away from resonance ($\omega >> \omega_j$): For the case where we are far away from resonance we can ignore the damping term and we can write $(\omega_j^2 - \omega^2)^2 + (\gamma_j\omega)^2 \approx (\omega_j^2 - \omega^2)^2$. Then index of refraction can be expressed as

$$n = 1 + \frac{Nq^2}{2\varepsilon_0m} \sum_{j=1}^{\infty} \frac{f_j}{(\omega_j^2 - \omega^2)}.$$ (9.284)

For transparent materials, the nearest significant resonance typically lie in the ultraviolet, then $\omega < \omega_j$. Taking this into account and noting that we are very far from resonance we can make the approximation

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2 (1 - \omega_j^2)} \approx \frac{1}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2}\right),$$ (9.285)

so that

$$n = 1 + \frac{Nq^2}{2\varepsilon_0m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2}\right) = 1 + \left(\frac{Nq^2}{2\varepsilon_0m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2}\right) + \omega^2 \left(\frac{Nq^2}{2\varepsilon_0m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2}\right)$$
9.8. DISPERSION

Figure 9.20: The absorption coefficient, $\alpha$, and the index of refraction, $n - 1$, vs $\lambda = \omega/\omega_j$. For different values of $\eta = \gamma_j/\omega_j$: $\eta = 0.2$ (Red), $\eta = 0.7$ (blue), and $\eta = 1.4$ (Cyan).

If we express the frequency in terms of the wave length in a vacuum

$$
\omega = \frac{2\pi c}{\lambda}
$$

we can write

$$
n = 1 + \left( \frac{Nq^2}{2\epsilon_0 m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2} \right) \left[ 1 + \left( \frac{2\pi^2 c^2 Nq^2}{\epsilon_0 m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2} \right) \right].
$$

Introducing the constants

$$
A = \frac{Nq^2}{2\epsilon_0 m} \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2}, B = 4\pi^2 c^2 \sum_{j=1}^{\infty} \frac{f_j}{\omega_j^2}
$$

the refractive index can be expressed as

$$
n = 1 + A \left( 1 + \frac{B}{\lambda^2} \right).
$$

This is known as Cauchy’s Formula: the constant $A$ is called the coefficient of Refraction and $B$ is called the coefficient of Dispersion. It applies in most gases, in the optical region.
Absorption Coefficient of water: Water is very slightly blue in color as overtone and combination vibrational absorption bands extend through the red part of the visible spectrum. This absorption spectrum of water as shown in Fig. 9.23 (red light absorbs 100 times more than blue light), together with the five-times greater scattering of blue light over red light, contributes to the blue color of lake, river and ocean waters (Fig. 9.22). Colloidal silica may contribute to the outstanding blue color of certain, often hydrothermal, pools and lakes. Ice is also blue for similar reasons.

9.9 Guided Waves

Wave guide: So far we have studied plane EM waves with infinite extent. Now it is time to study confined EM waves-guided waves. Since in a perfect conductor both the electric and magnetic fields are zero, the wave guide that we use to confine EM waves can be made of perfect conductors.

Consider a wave guide shown in the figure above. A wave guide is hollow in
the inside and the surface is made of a perfect conductor. Then at the boundary (inner surface) the following conditions must be satisfied

$$E_{\parallel} = 0, B_{\perp} = 0$$  (9.290)

Therefore inside the wave guide we need to solve Maxwell’s equations

$$\nabla \cdot \vec{E} = 0, \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$  (9.291)

subject to the above boundary conditions. Since the tangential electric and normal magnetic fields must vanish at the boundary, we must assume that both the electric and magnetic fields depend on the transverse coordinates (i.e. x,y). Hence the general solution satisfying the Maxwell’s equation must be expressed as

$$\vec{E} = \vec{E}_0 (x, y) \exp [i (k z - \omega t)],$$  (9.292)
\[ \vec{B} = \vec{B}_0 (x, y) \exp \{ i (kz - \omega t) \} \]  

Confined EM waves in general have all the three components. So we may write

\[
\vec{E}_0 (x, y) = E_{0x} (x, y) \hat{x} + E_{0y} (x, y) \hat{y} + E_{0z} (x, y) \hat{z},
\]
\[
\vec{B}_0 (x, y) = B_{0x} (x, y) \hat{x} + B_{0y} (x, y) \hat{y} + B_{0z} (x, y) \hat{z}. \tag{9.294}
\]

Then using Eqs. (9.292-9.294) in Maxwell’s equation

\[
\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \tag{9.295}
\]

we have

\[
\begin{bmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
E_{0x} e^{i(kz-\omega t)} & E_{0y} e^{i(kz-\omega t)} & E_{0z} e^{i(kz-\omega t)}
\end{bmatrix}
= - \frac{\partial}{\partial t}
\begin{bmatrix}
B_{0x} e^{i(kz-\omega t)} \\
B_{0y} e^{i(kz-\omega t)} \\
B_{0z} e^{i(kz-\omega t)}
\end{bmatrix} \tag{9.296}
\]

\[
\begin{bmatrix}
e^{i(kz-\omega t)} \frac{\partial E_{0x}}{\partial y} - E_{0y} \frac{\partial e^{i(kz-\omega t)}}{\partial x} \\
e^{i(kz-\omega t)} \frac{\partial E_{0y}}{\partial x} - E_{0x} \frac{\partial e^{i(kz-\omega t)}}{\partial y} \\
e^{i(kz-\omega t)} \frac{\partial E_{0z}}{\partial x} - E_{0x} \frac{\partial e^{i(kz-\omega t)}}{\partial y}
\end{bmatrix}
= i\omega
\begin{bmatrix}
B_{0x} e^{i(kz-\omega t)} \\
B_{0y} e^{i(kz-\omega t)} \\
B_{0z} e^{i(kz-\omega t)}
\end{bmatrix} \tag{9.297}
\]

\[
\begin{bmatrix}
\frac{\partial E_{0x}}{\partial y} - ikE_{0y} \\
\frac{\partial E_{0y}}{\partial x} - ikE_{0x} \\
\frac{\partial E_{0z}}{\partial x} - \frac{\partial E_{0x}}{\partial y}
\end{bmatrix}
= i\omega
\begin{bmatrix}
B_{0x} e^{i(kz-\omega t)} \\
B_{0y} e^{i(kz-\omega t)} \\
B_{0z} e^{i(kz-\omega t)}
\end{bmatrix} \tag{9.298}
\]

\[
\begin{bmatrix}
\frac{\partial E_{0x}}{\partial y} - ikE_{0y} \\
\frac{\partial E_{0y}}{\partial x} - ikE_{0x} \\
\frac{\partial E_{0z}}{\partial x} - \frac{\partial E_{0x}}{\partial y}
\end{bmatrix}
= i\omega
\begin{bmatrix}
B_{0x} \\
B_{0y} \\
B_{0z}
\end{bmatrix} \tag{9.299}
\]
There follows that

\[
\frac{\partial E_{0z}}{\partial y} - ikE_{0y} = i\omega B_{0z}, \quad ikE_{0x} - \frac{\partial E_{0x}}{\partial x} = i\omega B_{0y}, \quad \frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y} = i\omega B_{0z} \quad (9.300)
\]

Similarly using

\[
\nabla \times \bar{B} = \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t}
\]

we get

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
B_{0x}e^{i(kz-\omega t)} & B_{0y}e^{i(kz-\omega t)} & B_{0z}e^{i(kz-\omega t)}
\end{bmatrix}
= \frac{1}{c^2} \frac{\partial}{\partial t}
\begin{bmatrix}
E_{0x} \hat{x}e^{i(kz-\omega t)} & E_{0y} \hat{y}e^{i(kz-\omega t)} & E_{0z} \hat{z}e^{i(kz-\omega t)}
\end{bmatrix}
\]

\[
\left[
\begin{array}{c}
\frac{\partial B_{0z}}{\partial y} - ikB_{0y} \\
\frac{\partial B_{0x}}{\partial y} - ikB_{0y} \\
\frac{\partial B_{0x}}{\partial y} - \frac{\partial B_{0z}}{\partial y}
\end{array}
\right]
\left[
\begin{array}{c}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{array}
\right]
= -i \frac{\varepsilon}{c^2}
\begin{bmatrix}
E_{0x} \hat{x}e^{i(kz-\omega t)} & E_{0y} \hat{y}e^{i(kz-\omega t)} & E_{0z} \hat{z}e^{i(kz-\omega t)}
\end{bmatrix}
\]

\[
\left[
\begin{array}{c}
\frac{\partial B_{0z}}{\partial y} - ikB_{0y} \\
\frac{\partial B_{0x}}{\partial y} - \frac{\partial B_{0z}}{\partial y}
\end{array}
\right]
\left[
\begin{array}{c}
\hat{x} \\
\hat{y}
\end{array}
\right]
= -i \frac{\varepsilon}{c^2}
\begin{bmatrix}
E_{0x} \\
E_{0y}
\end{bmatrix}
\]

There follows that

\[
\frac{\partial B_{0x}}{\partial y} - ikB_{0y} = -i\omega c^2 E_{0x}, \quad ikE_{0x} - \frac{\partial B_{0z}}{\partial x} = -i\omega c^2 E_{0y}, \quad \frac{\partial B_{0y}}{\partial x} - \frac{\partial B_{0z}}{\partial y} = -i\omega c^2 E_{0z}
\]

Combining these equations with the results,

\[
\frac{\partial E_{0z}}{\partial y} - ikE_{0y} = i\omega B_{0x}, \quad ikE_{0x} - \frac{\partial E_{0z}}{\partial x} = i\omega B_{0y}, \quad \frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0z}}{\partial y} = i\omega B_{0z}
\]

we have

\[
\frac{\partial B_{0z}}{\partial y} = -i\omega \frac{c^2}{E_{0x}} + ikB_{0y}, \quad -\frac{\partial E_{0z}}{\partial x} = -ikE_{0x} + i\omega B_{0y}
\]

which gives

\[
E_{0x} = \frac{i}{(\omega/c^2) - k^2} \left( k \frac{\partial E_{0z}}{\partial x} + \frac{\partial B_{0z}}{\partial y} \right),
\]

\[
B_{0y} = \frac{i}{(\omega/c^2) - k^2} \left( k \frac{\partial B_{0z}}{\partial y} + \omega \frac{\partial E_{0z}}{c^2} \right) + ikB_{0x} + \frac{i\omega}{c^2} E_{0y} = \frac{\partial B_{0z}}{\partial x}.
\]
Using the remaining two Maxwell’s equations, we have

\[
\nabla \cdot \vec{E} = \left[ \frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} \right] \exp \left[ i (kz - \omega t) \right] + E_{0z} \frac{\partial \exp \left[ i (kz - \omega t) \right]}{\partial z} = 0
\]

\[
= \frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} + ikE_{0z} = 0
\]

(9.311)

and

\[
\nabla \cdot \vec{B} = \left[ \frac{\partial B_{0x}}{\partial x} + \frac{\partial B_{0y}}{\partial y} \right] \exp \left[ i (kz - \omega t) \right] + B_{0z} \frac{\partial \exp \left[ i (kz - \omega t) \right]}{\partial z} = 0
\]

\[
= \frac{\partial B_{0x}}{\partial x} + \frac{\partial B_{0y}}{\partial y} + ikB_{0z} = 0
\]

(9.312)

so that with the help of the results obtained above, one finds

\[
\frac{\partial}{\partial x} \left[ \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial E_{0z}}{\partial x} + \omega \frac{\partial B_{0z}}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial E_{0z}}{\partial y} - \omega \frac{\partial B_{0z}}{\partial x} \right) \right] + ikE_{0z} = 0
\]

\[
\frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial^2 E_{0z}}{\partial x^2} + \omega \frac{\partial^2 B_{0z}}{\partial x \partial y} \right) + \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial^2 E_{0z}}{\partial y^2} - \omega \frac{\partial^2 B_{0z}}{\partial x \partial y} \right) + ikE_{0z} = 0
\]

\[
\Rightarrow \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right) E_{0z} = 0
\]

(9.313)

and

\[
\frac{\partial}{\partial x} \left[ \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial B_{0z}}{\partial x} - \omega \frac{\partial E_{0z}}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial B_{0z}}{\partial y} + \omega \frac{\partial E_{0z}}{\partial x} \right) \right] + ikB_{0z} = 0
\]

\[
\frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial^2 B_{0z}}{\partial x^2} - \omega \frac{\partial^2 E_{0z}}{\partial y \partial y} \right) + \frac{i}{(\omega/c)^2 - k^2} \left( k \frac{\partial^2 B_{0z}}{\partial y^2} + \omega \frac{\partial^2 E_{0z}}{\partial x \partial y} \right) + ikB_{0z} = 0
\]

\[
\Rightarrow \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right) B_{0z} = 0
\]

(9.314)

Note that

\[
E_{0z} = 0 \Rightarrow \text{Transverse electric (TE) wave}
\]

\[
B_{0z} = 0 \Rightarrow \text{Transverse magnetic (TM) wave}
\]

\[
E_{0z} = 0 \text{ and } B_{0z} = 0 \Rightarrow \text{TEM wave}
\]

(9.315)
9.10 TE waves in a rectangular wave guide

We now apply the above results to a specific wave guide for TE waves. To this end, we may write the magnetic field, \( B_{0z} \), as

\[
B_{0z} = X(x)Y(y)
\]  

so that the equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left( \frac{\omega}{c} \right)^2 - k^2 \right) B_{0z} = 0
\]

(9.317) can be put in the form

\[
Y(y) \frac{d^2X(x)}{dx^2} + X(x) \frac{d^2Y(y)}{dy^2} + \left[ \left( \frac{\omega}{c} \right)^2 - k^2 \right] X(x)Y(y) = 0
\]

(9.318) and dividing the equation by \( X(x)Y(y) \) we find

\[
\frac{1}{X(x)} \frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2Y(y)}{dy^2} + \left[ \left( \frac{\omega}{c} \right)^2 - k^2 \right] = 0.
\]

(9.319)
There follows that
\[-k_x^2 - k_y^2 + \left[ (\omega/c)^2 - k^2 \right] = 0, \tag{9.320}\]
where
\[-k_x^2 = \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2}, \quad -k_y^2 = \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2}. \tag{9.321}\]
We put the equations in the form
\[\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0, \quad \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0 \tag{9.322}\]
and the solutions of which are
\[X(x) = A \cos(k_x x) + B \sin(k_x x), \quad Y(y) = C \cos(k_y y) + D \sin(k_y y). \tag{9.323}\]
Now referring to the boundary conditions, the normal component of the magnetic field must be zero at \(x = 0\) and \(x = a; y = 0\) and \(y = b\) \((B_\perp = 0)\), which means
\[B_{0x}(0, y) = B_{0x}(a, y) = 0, B_{0y}(x, 0) = B_{0y}(x, b) = 0. \tag{9.324}\]
Recalling that
\[B_{0x} = \frac{i}{(\omega/c)^2 - k_x^2} \left( k \frac{\partial B_{0z}}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_{0z}}{\partial y} \right), \tag{9.325}\]
for TE \(E_{0z} = 0\), we have
\[B_{0x} = \frac{i}{(\omega/c)^2 - k_x^2} \left( k \frac{\partial B_{0z}}{\partial x} \right) \tag{9.326}\]
so that
\[\frac{\partial B_{0z}}{\partial x} \bigg|_{x=0,a} = 0 \Rightarrow \frac{\partial X(x)Y(y)}{\partial x} \bigg|_{x=0,a} = 0 \Rightarrow \frac{\partial X(x)}{\partial x} \bigg|_{x=0,a} = 0 \Rightarrow -Ak_x \sin(k_x x) + Bk_x \cos(k_x x) \bigg|_{x=0,a} = 0. \tag{9.327}\]
There follows that
\[B = 0, \quad -Ak_x \sin(k_x a) = 0 \Rightarrow k_x = \frac{n\pi}{a} \tag{9.328}\]
Similarly for the y-component of the magnetic field
\[B_{0y} = \frac{i}{(\omega/c)^2 - k_y^2} \left( k \frac{\partial B_{0z}}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_{0z}}{\partial x} \right), \tag{9.329}\]
for TE waves we have
\[B_{0x} = \frac{i}{(\omega/c)^2 - k_x^2} \left( k \frac{\partial B_{0z}}{\partial x} \right) \tag{9.330}\]
and using the boundary conditions

\[ B_{0y}(x,0) = B_{0y}(x,b) = 0, \quad (9.330) \]

we find

\[ \frac{\partial B_{0z}}{\partial y} \bigg|_{y=0,b} = 0 \Rightarrow \frac{\partial X(x)Y(y)}{\partial y} \bigg|_{y=0,b} = 0 \Rightarrow \frac{\partial Y(y)}{\partial y} = 0 \bigg|_{y=0,b} \]

\[ \Rightarrow A'k_y \sin (k_y y) + B'k_y \cos (k_y y) = 0 \bigg|_{y=0,b} \quad (9.331) \]

There follows that

\[ B = 0, -Ak_y \sin (k_y b) = 0 \Rightarrow k_y = \frac{m\pi}{b}. \quad (9.332) \]

Then substituting the results

\[ k_x = \frac{n\pi}{a}, k_y = \frac{m\pi}{b} \quad (9.333) \]

in the equation

\[ -k_x^2 - k_y^2 + \left[ (\omega/c)^2 - k^2 \right] = 0 \quad (9.334) \]

the wave vector can be expressed as

\[ - \left( \frac{n\pi}{a} \right)^2 - \left( \frac{m\pi}{b} \right)^2 + \left[ (\omega/c)^2 - k^2 \right] = 0 \]

\[ \Rightarrow k = \sqrt{(\omega/c)^2 - \left[ \left( \frac{n\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 \right]} \quad (9.335) \]

or

\[ k = \frac{1}{c} \sqrt{\omega^2 - \omega_{nm}^2}, \quad (9.336) \]

where

\[ \omega_{nm} = c\pi \sqrt{\left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2} \quad (9.337) \]

is known as the cut-off frequency. If the frequency of the wave is less than this frequency it will be attenuated since the wave number becomes imaginary. The wave speed

\[ v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_{nm}^2}{\omega^2}}} > c \quad (9.338) \]

but the group velocity

\[ v_g = \frac{1}{dk/d\omega} = c\sqrt{1 - \frac{\omega_{nm}^2}{\omega^2}} < c \quad (9.339) \]

Qualitative description: Let’s consider an ordinary plane EM wave propagating along the direction \( \theta \) with respect to the positive z-direction as shown in
the figure below. This wave undergoes a successive perfect reflections from the surfaces as the surfaces are made of a conductor. This means the waves form a standing wave along the $x$ and $y$ direction. This means the waves must have a node at the boundaries which is satisfied when

$$n \frac{\lambda_x}{2} = a, m \frac{\lambda_y}{2} = b \Rightarrow \lambda_x = \frac{2a}{n}, \lambda_y = \frac{2b}{m} \quad (9.340)$$

and the corresponding wave numbers given by

$$k_x = \frac{2\pi}{\lambda_x} = \frac{2\pi}{2a} = \frac{n\pi}{a}, k_y = \frac{2\pi}{\lambda_y} = \frac{2\pi}{2b} = \frac{m\pi}{b}. \quad (9.341)$$

which leads to

$$k' = \sqrt{k_x^2 + k_y^2 + k^2} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + k^2}$$

$$k' = \frac{1}{c} \sqrt{(ck)^2 + \left(\frac{c\pi}{a}\right)^2 + \left(\frac{c\pi}{b}\right)^2} \Rightarrow ck' = \sqrt{(ck)^2 + (\omega_{mn})^2}$$

$$\Rightarrow ck = \sqrt{\omega^2 - \omega_{mn}^2}, \quad (9.342)$$

where

$$\omega_{mn} = \sqrt{\left(\frac{c\pi}{a}\right)^2 + \left(\frac{c\pi}{b}\right)^2}, \omega = ck'. \quad (9.343)$$

Only waves incident at selected values of $\theta$ can made it through the wave guide

$$\cos (\theta) = \frac{k}{k'} = \frac{ck}{ck'} = \frac{ck}{\omega}, \Rightarrow \cos (\theta) = \sqrt{\omega^2 - \omega_{mn}^2} = \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}. \quad (9.344)$$

We know that the plane wave travels at speed $c$, but because it is going at an angle $\theta$ to the z-axis, its net velocity is

$$v_g = c \cos (\theta) = c \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}. \quad (9.345)$$
On the other hand the wave velocity which is given by

\[ v = \frac{\omega}{k} \]  (9.346)

becomes

\[ v = \frac{\omega}{k'} \cos (\theta) = \frac{c}{\cos (\theta)} = \frac{c}{\sqrt{1 - \frac{\omega^2}{c^2}}} > c. \]  (9.347)

**Resonant Cavities:** Although an electromagnetic cavity resonator can be of any shape a resonant cavity can be formed from the cylindrical wave guide by placing conductors at the two ends of a waveguide. In a resonant cavity selected values of frequencies that depends on the parameters that determine the size of the cavity and the medium inside the cavity can propagate without lose while other frequencies are being attenuated. The measure of the sharpness of the selected frequency is called the \( Q \) of the cavity. The \( Q \) of the cavity is defined as \( 2\pi \) times the ratio of the time-averaged energy stored in the cavity to the energy loss per cycle

\[ Q = \frac{\omega_0 \text{ Stored energy}}{\text{Power Loss}} \]  (9.348)

The earth and its ionosphere form a resonant cavity. The earth which is made of mostly water has sea water conductivity, \( \sigma \sim 0.1/\Omega m \) while the ionosphere has \( \sigma \sim 10^{-7} - 10^{-4}/\Omega m \). Although this shows that the spherical walls (earth surface and the ionosphere) are far from being a perfect conductor, we idealize the physical reality and consider a model of two perfectly conducting, concentric spheres with radii \( a \) and \( b = a + h \). Using the radius of the earth, \( a = 6400 km \), and the height of the ionosphere above the earth, \( h \sim 100 km \), the lowest frequencies for TE modes are

\[ \omega_{TE} \sim \frac{\pi c}{h} \sim 10^4 \]  (9.349)

where as for the lowest TM modes

\[ \omega_{TM} \sim \frac{c}{h} \sim 10^3. \]  (9.350)

**The Electromagnetic Spectrum:**
CHAPTER 9. ELECTROMAGNETIC WAVES

Electromagnetic Spectrum

Frequency (Hz)

Wavelength

Gamma-rays

X-rays

Ultraviolet

Visible

Near IR

Infra-red

Thermal IR

Far IR

Microwaves

Radar

Radio, TV

AM

Long-waves

1000 MHz

500 MHz

100 MHz

50 MHz

10^12

10^11

10^10

10^9

10^8

10^7

10^6

10^5

10^4

10^3

10^2

10^1

1

400 nm

500 nm

600 nm

700 nm

1 μm

100 μm

1 mm

1 cm

1 m

10 m

100 m

1000 m
Chapter 10

Selected topics I

Up to this point our focus has been how electromagnetic fields be described and behave in various types of media or at the boundaries of two media. We have never discussed so far how electromagnetic fields (or radiation) are actually produced. Of course we know whenever there is a charge there is an electric field. If this charge begins to move the charge would also produce a magnetic field. But the question is how do EM waves or in general EM radiation like be produced. What should happen to the charge ($\rho$) or current ($\mathbf{J}$) localized in some part of space in order to create an EM radiation that can reach to infinity. That is the purpose of this chapter.

10.1 Potentials and Fields

The answer to the question how EM waves are created or what must happen to the charge or current in order to produce EM radiation should come from Maxwell’s equations.

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (10.1) \]

It is important to learn the techniques that allow us to better handle Maxwell’s equations when the charge and current density are not zero especially when these distributions are time dependent. For time dependent current, for example when there is time dependent current density, since $\nabla \times \mathbf{E} \neq 0$, we can not write $\mathbf{E} = -\nabla V$ and that makes solving Maxwell’s equation harder. Fortunately, we have what is knows as Gauge transformation that is based on the properties of the scalar potential $V(\mathbf{r},t)$ and vector potential $\mathbf{A}(\mathbf{r},t)$.

**Scalar and vector potentials**: Even though we can not write $\mathbf{E} = -\nabla V$ in electrodynamics, since $\nabla \cdot \mathbf{B} = 0$ we can still write

\[ \mathbf{B} = \nabla \times \mathbf{A}. \quad (10.2) \]
Substituting this into
\[ \nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \]  
we find
\[ \nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \left( \nabla \times \vec{A} \right) \Rightarrow \nabla \times \left( \vec{E}(\vec{r}, t) + \frac{\partial \vec{A}}{\partial t} \right) = 0 \]  
which can be put in the form
\[ \nabla \times \vec{F}(\vec{r}, t) = 0, \]  
where
\[ \vec{F}(\vec{r}, t) = \vec{E}(\vec{r}, t) + \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \]  
We recall theoretical physics I for an irrotational vector field
\[ \nabla \times \vec{F}(\vec{r}, t) = 0, \]  
is a conservative vector field. For any conservative vector field one can always find a scalar potential \( V(\vec{r}, t) \) such that
\[ \vec{F}(\vec{r}, t) = -\nabla V(\vec{r}, t) \]  
and this leads to
\[ \vec{E}(\vec{r}, t) + \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} = -\nabla V(\vec{r}, t) \Rightarrow \vec{E}(\vec{r}, t) = -\nabla V(\vec{r}, t) + \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}. \]  
Therefore, the electric and magnetic fields in electrodynamics can be expressed in terms of a scalar and a vector potentials as
\[ \vec{E}(\vec{r}, t) = -\nabla V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}, \vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t). \]  
Using these two expressions Gauss’s law and Ampere/Maxwell equations can be rewritten as
\[ \nabla \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\varepsilon_0} \Rightarrow \nabla \cdot \left( -\nabla V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \right) = \frac{\rho(\vec{r}, t)}{\varepsilon_0} \]  
\[ \Rightarrow \nabla^2 V(\vec{r}, t) + \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A}(\vec{r}, t) \right) = -\frac{\rho(\vec{r}, t)}{\varepsilon_0} \]  
and
\[ \nabla \times \vec{B}(\vec{r}, t) = \mu_0 \vec{J}(\vec{r}, t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} \]  
\[ \Rightarrow \nabla \times \left( \nabla \times \vec{A}(\vec{r}, t) \right) = \mu_0 \vec{J}(\vec{r}, t) + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( -\nabla V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \right). \]
Using the relation
\[ \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \]  
(10.13)
we find
\[ \nabla (\nabla \cdot \vec{A}(\vec{r}, t)) - \nabla^2 \vec{A}(\vec{r}, t) = \mu_0 \vec{j}(\vec{r}, t) + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( -\nabla V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \right) \]
\[ \Rightarrow \left( \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{A}(\vec{r}, t) + \nabla \left( \nabla \cdot \vec{A}(\vec{r}, t) + \mu_0 \varepsilon_0 \frac{\partial V(\vec{r}, t)}{\partial t} \right) = \mu_0 \vec{j}(\vec{r}, t). \]
(10.14)

The equation can be expressed in a more symmetric form using
\[ \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \Box^2, \]  
(10.15)
where \( \Box^2 \) is a new operator called the d’Alembertian operator, and a new scalar function \( L(\vec{r}, t) \) defined by
\[ L(\vec{r}, t) = \nabla \cdot \vec{A}(\vec{r}, t) + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = \nabla \cdot \vec{A}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial V}{\partial t} \]
\[ L(\vec{r}, t) = \frac{\partial}{\partial (ct)} \left( \frac{V}{c} \right) + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]  
(10.16)
which leads to
\[ \nabla L(\vec{r}, t) - \Box^2 \vec{A}(\vec{r}, t) = \mu_0 \vec{j}(\vec{r}, t). \]  
(10.17)

Gauss’s law
\[ \nabla \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\varepsilon_0} \]  
(10.18)
using
\[ \vec{E}(\vec{r}, t) = -\nabla V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} \]  
(10.19)
can be written as
\[ \nabla^2 V(\vec{r}, t) + \frac{\partial \left( \nabla \cdot \vec{A}(\vec{r}, t) \right)}{\partial t} = -\frac{\rho(\vec{r}, t)}{\varepsilon_0}. \]  
(10.20)

Using
\[ \Box^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \]  
(10.21)
we may write
\[ \Box^2 V = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \nabla^2 V \Rightarrow \nabla^2 V = \Box^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} \]  
(10.22)
so that Gauss’s law can be put in the form

$$\Box^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} \right) = \Box^2 V - \frac{\partial}{\partial t} \left[ \frac{1}{c^2} \frac{\partial V}{\partial t} - \nabla \cdot \vec{A} \right] = -\frac{\rho}{\epsilon_0}$$

$$\Rightarrow \frac{\partial L}{\partial t} - \Box^2 V = \frac{\rho}{\epsilon_0}.$$  \hspace{1cm} (10.23)

### 10.2 Tensor formulation

Suppose we construct a 4-dimensional Cartesian coordinates, \(x^a\) (time and 3-D space) defined by

\[
x^0 = ct, x^1 = x, x^2 = y, x^3 = z
\]

Let’s consider a 4-dimensional vector potential \(\vec{A}\) with component \(A^a\) (for \(a = 0, 1, 2, 3\)) defined by

\[
\vec{A} = \left( \frac{V}{c}, A_x, A_y, A_z \right) \to A^a
\]

and a 4-dimensional current density, \(\vec{J}\), with components \(A^a\) (for \(a = 0, 1, 2, 3\)) defined by

\[
\vec{J} = (cp(\vec{r}, t), J_x, J_y, J_z) \to J^a.
\]

one can then write

\[
\frac{\partial}{\partial (ct)} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = \frac{\partial}{\partial x^a}
\]

Introducing the 4-unit vectors \((\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)\) along the direction of 4-axes \((x^0 = ct, x^1 = x, x^2 = y, x^3 = z)\) one can express, Amperes/Maxwell’s equation

\[
\nabla L(\vec{r}, t) - \Box^2 \vec{A}(\vec{r}, t) = \mu_0 \vec{J}(\vec{r}, t).
\]

as

\[
\left( \frac{\partial}{\partial x^1} \hat{x}_1 + \frac{\partial}{\partial x^2} \hat{x}_2 + \frac{\partial}{\partial x^3} \hat{x}_3 \right) \cdot L(\vec{r}, t)
\]

\[
-\Box^2 \left[ A^1 \hat{x}_1 + A^2 \hat{x}_2 + A^3 \hat{x}_3 \right] = \mu_0 \left( J^1 \hat{x}_1 + J^2 \hat{x}_2 + J^3 \hat{x}_3 \right). \hspace{1cm} (10.25)
\]

and Gauss’s law

\[
\frac{\partial L(\vec{r}, t)}{\partial (ct)} - \Box^2 V = \frac{\rho}{\epsilon_0}
\]

$$\Rightarrow \frac{\partial L(\vec{r}, t)}{\partial (ct)} - \Box^2 \left( \frac{V}{c} \right) = \frac{\partial}{\partial (ct)} \left( \frac{V}{c^2} \right) = \frac{\rho}{c \epsilon_0} = \frac{\rho}{c} \mu_0 c^2$$

$$\Rightarrow \frac{\partial L(\vec{r}, t)}{\partial (ct)} - \Box^2 \left( \frac{V}{c} \right) = \mu_0 (cp) \Rightarrow \frac{\partial}{\partial (ct)} \hat{x}_0 L(\vec{r}, t) - \Box^2 \left( \frac{V}{c} \hat{x}_0 \right) = \mu_0 (cp \hat{x}_0)$$

$$\Rightarrow \frac{\partial}{\partial x^0} \hat{x}_0 L(\vec{r}, t) - \Box^2 A^0 \hat{x}_0 = \mu_0 J^0 \hat{x}_0$$  \hspace{1cm} (10.27)
where we replaced
\[ x^0 = ct, J^0 = \rho, J^0 = \frac{V}{c}. \]

Upon adding these two equations (Gauss’s and Amper’s laws), we find
\[
-\Box^2 \left[ A^0 \dot{x}_0 + A^1 \dot{x}_1 + A^2 \dot{x}_2 + A^3 \dot{x}_3 \right] = \mu_0 \left[ J^0 \dot{x}_0 + J^1 \dot{x}_1 + J^2 \dot{x}_2 + J^3 \dot{x}_3 \right].
\]

(10.28)

Introducing the 4-dimensional gradient operator as
\[
\Box = \frac{\partial}{\partial (x^0)} \dot{x}_0 + \frac{\partial}{\partial x^1} \dot{x}_1 + \frac{\partial}{\partial x^2} \dot{x}_2 + \frac{\partial}{\partial x^3} \dot{x}_3
\]
one can write
\[
\Box L (\vec{r}, t) - \Box^2 \vec{A} (\vec{r}, t) = \mu_0 \vec{J}.
\]

(10.29)

Introducing the tensor \([g_{ab}]\) (second-rank tensor) defined by
\[
[g_{ab}] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]
we have
\[
\frac{\partial^2 \vec{A}}{\partial (ct)^2} - \frac{\partial^2 \vec{A}}{\partial x^2} - \frac{\partial^2 \vec{A}}{\partial y^2} - \frac{\partial^2 \vec{A}}{\partial z^2}
\]
\[
= \left( \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
\frac{\partial}{\partial x^0} \\
\frac{\partial}{\partial x^1} \\
\frac{\partial}{\partial x^2} \\
\frac{\partial}{\partial x^3}
\end{bmatrix}
\Rightarrow \Box^2 = \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b}
\]
so that
\[
\Box^2 \vec{A} (\vec{r}, t) = \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} \vec{A}^c \dot{x}_c = \sum_{c=0}^{3} \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab} \frac{\partial A^c}{\partial x^a} \frac{\partial}{\partial x^b} \dot{x}_c
\]

Introducing the 4-dimensional gradient operator as
\[
\Box = \frac{\partial}{\partial (x^0)} \dot{x}_0 + \frac{\partial}{\partial x^1} \dot{x}_1 + \frac{\partial}{\partial x^2} \dot{x}_2 + \frac{\partial}{\partial x^3} \dot{x}_3 = \sum_{c=0}^{3} \frac{\partial}{\partial x^c} \dot{x}_c
\]
Noting that
\[ L(\vec{r}, t) = \frac{\partial}{\partial x^0} A^0 + \frac{\partial}{\partial x^1} A^0 + \frac{\partial}{\partial x^2} A^0 + \frac{\partial}{\partial x^3} A^0 = \vec{\nabla} \cdot \vec{A} \]
we can write
\[ \Box L(\vec{r}, t) = \sum_{a=0}^{3} \sum_{c=0}^{3} \frac{\partial^2 A^a}{\partial x^c \partial x^a} = \sum_{a=0}^{3} \sum_{b=0}^{3} \frac{\partial A^a}{\partial x^b} \frac{\partial A^c}{\partial x^b} \]

The equation
\[ \Box L(\vec{r}, t) = \Box^2 \vec{A}(\vec{r}, t) = \mu_0 \vec{J}. \] (10.30)
can be put in the form
\[ \sum_{a=0}^{3} \sum_{c=0}^{3} \frac{\partial^2 A^a}{\partial x^a \partial x^c} - \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab} \frac{\partial A^c}{\partial x^b} = \sum_{c=0}^{3} \mu_0 J^c \hat{x}_c \] (10.31)
\[ \Rightarrow \sum_{c=0}^{3} \left[ \sum_{a=0}^{3} \frac{\partial^2 A^a}{\partial x^c \partial x^a} - \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab} \frac{\partial A^c}{\partial x^a} \frac{\partial A^b}{\partial x^b} \right] \hat{x}_c = \sum_{c=0}^{3} (\mu_0 J^c) \hat{x}_c \] (10.32)

There follows that
\[ \sum_{a=0}^{3} \frac{\partial^2 A^a}{\partial x^a \partial x^c} - \sum_{a=0}^{3} \sum_{b=0}^{3} g_{ab} \frac{\partial A^c}{\partial x^a} \frac{\partial A^b}{\partial x^b} = \mu_0 J^c. \] (10.33)

which can also be expressed as
\[ \frac{\partial^2 A^a}{\partial x^a \partial x^c} - g_{ab} \frac{\partial A^c}{\partial x^a} \frac{\partial A^b}{\partial x^b} = \mu_0 J^c. \] (10.34)

### 10.3 Gauge transformation

Consider two sets of potentials \((V(\vec{r}, t), \vec{A}(\vec{r}, t))\) and \((V'(\vec{r}, t), \vec{A}'(\vec{r}, t))\) which gives identically the same electric and magnetic fields. That means
\[
\vec{E}(\vec{r}, t) = -\nabla V(\vec{r}, t) - \frac{\partial A(\vec{r}, t)}{\partial t}, \quad \vec{E}'(\vec{r}, t) = -\nabla V'(\vec{r}, t) - \frac{\partial A'(\vec{r}, t)}{\partial t} \quad (10.35)
\]
\[ \Rightarrow \vec{E}(\vec{r}, t) = \vec{E}'(\vec{r}, t) \quad (10.36) \]
and
\[ \vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t) = \nabla \times \vec{A}'(\vec{r}, t). \] (10.37)

Suppose these two set of potentials differ by some vector potential \(\vec{\alpha}(\vec{r})\) and scalar \(\beta(\vec{r})\) functions according to
\[
\vec{A}'(\vec{r}, t) = \vec{A}(\vec{r}, t) + \vec{\alpha}(\vec{r}, t), \quad V'(\vec{r}, t) = V(\vec{r}, t) + \beta(\vec{r}, t). \] (10.38)
Then using these expressions one can express the electric and magnetic fields as we must have

\[
\hat{E}'(\vec{r}, t) = -\nabla V'(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}
\]

\[
= -\nabla \left( V(\vec{r}, t) + \beta(\vec{r}, t) \right) - \frac{\partial}{\partial t} \left( \vec{A}(\vec{r}, t) + \vec{\alpha}(\vec{r}, t) \right)
\]

\[
= -\nabla V(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) - \nabla \beta(\vec{r}, t) - \frac{\partial \vec{\alpha}(\vec{r}, t)}{\partial t}
\]

\[
\Rightarrow \hat{E}'(\vec{r}, t) = \hat{E}(\vec{r}, t) - \left( \nabla \beta(\vec{r}, t) + \frac{\partial \vec{\alpha}(\vec{r}, t)}{\partial t} \right) \quad (10.39)
\]

Since we require \( \hat{E}'(\vec{r}, t) = \hat{E}(\vec{r}, t) \),

\[
\nabla \beta(\vec{r}, t) + \frac{\partial \vec{\alpha}(\vec{r}, t)}{\partial t} = 0. \quad (10.40)
\]

Similarly for the magnetic field

\[
\hat{B}'(\vec{r}, t) = \nabla \times \left( \vec{A}(\vec{r}, t) + \vec{\alpha}(\vec{r}, t) \right) = \nabla \times \vec{A}(\vec{r}, t) + \nabla \times \vec{\alpha}(\vec{r}, t) \quad (10.41)
\]

\[
\Rightarrow \hat{B}'(\vec{r}, t) = \hat{B}(\vec{r}, t) + \nabla \times \vec{\alpha}(\vec{r}, t). \quad (10.42)
\]

one finds

\[
\nabla \times \vec{\alpha}(\vec{r}, t) = 0 \quad (10.43)
\]

as we must also have \( \hat{B}'(\vec{r}, t) = \hat{B}(\vec{r}, t) \). Since the vector function \( \vec{\alpha}(\vec{r}, t) \) is irrotational (\( \nabla \times \vec{\alpha} = 0 \)), one can write

\[
\vec{\alpha}(\vec{r}, t) = \nabla \lambda(\vec{r}, t). \quad (10.44)
\]

Substituting Eq. (10.44) into (10.40), one finds

\[
\nabla \left( \beta(\vec{r}, t) + \frac{\partial \lambda(\vec{r}, t)}{\partial t} \right) = 0 \Rightarrow \beta(\vec{r}, t) + \frac{\partial \lambda(\vec{r}, t)}{\partial t} = \kappa(t). \quad (10.45)
\]

indicating function \( \kappa(t) \) depends only on time not position. There follows that

\[
\beta(\vec{r}, t) + \frac{\partial \lambda(\vec{r}, t)}{\partial t} = \kappa(t) \Rightarrow \beta(\vec{r}, t) = \kappa(t) - \frac{\partial \lambda(\vec{r}, t)}{\partial t}. \quad (10.46)
\]

But we can chose another function, \( \lambda' \), defined by

\[
\lambda'(\vec{r}, t) = \lambda(\vec{r}, t) - \int_0^t \kappa(t') \, dt' \Rightarrow \frac{\partial \lambda'(\vec{r}, t)}{\partial t} = \frac{\partial \lambda(\vec{r}, t)}{\partial t} - \kappa(t) \quad (10.47)
\]

\[
\Rightarrow \frac{\partial \lambda(\vec{r}, t)}{\partial t} = \frac{\partial \lambda'(\vec{r}, t)}{\partial t} + \kappa(t) \quad (10.48)
\]

so that

\[
\beta(\vec{r}, t) = \kappa(t) - \frac{\partial \lambda(\vec{r}, t)}{\partial t} = -\frac{\partial \lambda'(\vec{r}, t)}{\partial t}. \quad (10.49)
\]
and
\[ \tilde{\alpha}(\bar{r}, t) = \nabla \lambda(\bar{r}, t) = \nabla \left[ \lambda'(\bar{r}, t) + \int_0^t \kappa(t') \, dt' \right] = \nabla \lambda'(\bar{r}, t). \] (10.50)

Therefore the potentials can be expressed as
\[ \tilde{A'}(\bar{r}, t) = \tilde{A}(\bar{r}, t) + \tilde{\alpha}(\bar{r}, t) = \tilde{A}(\bar{r}, t) + \nabla \lambda'(\bar{r}, t), \]
\[ V'(\bar{r}, t) = V(\bar{r}, t) + \beta(\bar{r}, t) = V(\bar{r}, t) - \frac{\partial \lambda'(\bar{r}, t)}{\partial t}. \] (10.52)

These results show that for any old scalar function \( \tilde{A}(\bar{r}, t) \), we can add \( r_0 \tilde{A}(\bar{r}, t) \) to come up with a different vector potential \( \tilde{A}_0(\bar{r}, t) \), provided we simultaneously subtract \( \frac{\partial \lambda'(\bar{r}, t)}{\partial t} \) from the scalar potential \( \tilde{V}(\bar{r}, t) \). None of this will affect the physical quantities \( \tilde{E}(\bar{r}, t) \) and \( \tilde{B}(\bar{r}, t) \). Gauge transformations are found on these properties of the vector and scalar potential.

**Coulomb Gauge:** In Coulomb Gauge we can choose a vector potential \( \tilde{A}(\bar{r}, t) \) such that
\[ \bar{r} \cdot \tilde{A}(\bar{r}, t) = 0 \] (10.53)
with out changing the electric and magnetic fields. Then in Coulomb gauge, Gauss’s law and Ampere’s law,
\[ \nabla^2 V(\bar{r}, t) + \frac{\partial}{\partial t} \left( \nabla \cdot \tilde{A}(\bar{r}, t) \right) = -\frac{\rho(\bar{r}, t)}{\varepsilon_0} \]
\[ \left[ \nabla^2 \tilde{A}(\bar{r}, t) - \mu_0\varepsilon_0 \frac{\partial^2 \tilde{A}(\bar{r}, t)}{\partial t^2} \right] - \nabla \left[ \nabla \cdot \tilde{A}(\bar{r}, t) + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} \right] = -\mu_0\tilde{J} \] (10.54)
become
\[ \nabla^2 V = -\frac{\rho}{\varepsilon_0}, \]
\[ \nabla^2 \tilde{A} - \mu_0\varepsilon_0 \frac{\partial^2 \tilde{A}}{\partial t^2} = -\mu_0\tilde{J} + \mu_0\varepsilon_0 \nabla \left( \frac{\partial V}{\partial t} \right). \] (10.55)

The first result is Poisson’s equation which we know the solution is given by
\[ V(\bar{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\bar{r}, t')}{|\bar{r} - \bar{r}'|} \, d\tau'. \] (10.56)

This shows that, in Coulomb Gauge, the electric potential depends on the charge distribution at the present time. This indicate that an electron oscillating on on earth which causes change in the potential on earth, according to Coulomb Gauge, a person on the moon almost immediately detects this change. This requires infinite magnitude of speed. This seems in contradictory with special theory of relativity since nothing travels faster than the speed of light. But
10.4. RETARDED POTENTIAL

Don’t get alarmed! Because what we can actually measure on the moon is not the electric potential. It is the electric field which is given by

\[
\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}
\]

where the vector potential is determined from

\[
\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \varepsilon_0 \nabla \left( \frac{\partial V}{\partial t} \right).
\]

in Coulomb Gauge. This shows although it appears that \( V \) depends on the present distribution of the charge, \( \rho (\vec{r}, t) \), since the \( \vec{E} \) field depend on \( \vec{A} \), somehow \( \vec{E} \) will change only after sufficient time has elapsed.

**Lorentz Gauge:** In Lorentz gauge we choose the vector potential such that

\[
\nabla \cdot \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}.
\]

This simplifies the Ampere/Maxwell equation

\[
 \left( \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left( \nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}
\]

into

\[
\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}
\]

and Gauss’s law

\[
\nabla^2 \vec{V} + \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} \right) = -\frac{\rho}{\varepsilon_0}
\]

into

\[
\nabla^2 \vec{V} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{V}}{\partial t^2} = -\frac{\rho}{\varepsilon_0}.
\]

In terms of the d’Alembertian

\[
\Box^2 = \nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}
\]

we can write

\[
\Box^2 \vec{A} = -\mu_0 \vec{J}, \quad \Box^2 \vec{V} = -\frac{\rho}{\varepsilon_0}.
\]

These are called the inhomogeneous wave equations. The Lorentz gauge is commonly used, first because it leads to the wave equation and second because it is a concept independent of the coordinate system chosen and so it fits naturally into the consideration of special relativity.
10.4 Retarded Potential

Consider a volume of continuous charge distribution as shown in Fig. 10.1. For static case the inhomogeneous wave equations

\[ \Box^2 \vec{A} = -\mu_0 \vec{J}, \quad \Box^2 V = -\frac{\rho}{\epsilon_0}. \]  

(10.66)

where

\[ \Box^2 = \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}, \]  

(10.67)

reduces to Poisson’s equation

\[ \nabla^2 \vec{A} = -\mu_0 \vec{J}, \quad \nabla^2 V = -\frac{\rho}{\epsilon_0}. \]  

(10.68)

The solutions are given by

\[ V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d\tau', \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d\tau'. \]  

(10.69)

In the nonstatic case these potentials must be given by

\[ V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d\tau', \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d\tau', \]  

(10.70)

where

\[ t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}. \]  

(10.71)
10.4. RETARDED POTENTIAL

is the retarded time. A simple argument for using the retarded time \( t_r \) instead of the present time \( t \) is the light emitted by distant stars. The light we see during a clear summer night sky comes from a distant stars. The light is an EM wave. EM waves travels in vacuum with a speed, \( c \). Then this wave must travel a distance \( |\vec{r} - \vec{r}'| \) before it reaches to our eye. Therefore, it takes a time \( |\vec{r} - \vec{r}'| / c \) to reach to us. If the present time is \( t \), then the EM wave we receive at the present time \( t \) must be determined by the charge and the current at the retarded time \( t_r = t - |\vec{r} - \vec{r}'| / c \). It can be shown that the potentials

\[
V(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d\tau', \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} d\tau'.
\] (10.72)

satisfy the inhomogeneous wave equations

\[
\Box^2 \vec{A} = -\mu_0 \vec{J}, \quad \Box^2 V = -\frac{\rho}{\epsilon_0},
\] (10.73)

where

\[
\Box^2 = \nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2},
\] (10.74)

Here we will show for the scalar potential only. We want to prove that

\[
\Box^2 V(\vec{r}, t) = \nabla^2 V(\vec{r}, t) - \frac{1}{c} \frac{\partial^2 V(\vec{r}, t)}{\partial t^2} = -\frac{\rho}{\epsilon_0} V(\vec{r}, t) \Rightarrow \nabla^2 V(\vec{r}, t) = \nabla \cdot (\nabla V(\vec{r}, t)) = \frac{1}{c} \frac{\partial^2 V(\vec{r}, t)}{\partial t^2} - \frac{\rho}{\epsilon_0} V(\vec{r}, t).
\] (10.75)

or noting that

\[
\nabla V(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \int \left[ \frac{\nabla \rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} + \rho(\vec{r}', t_r) \nabla \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \right] d\tau',
\] (10.76)

and

\[
\nabla \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}, \quad \nabla \rho(\vec{r}', t_r) = \frac{\partial \rho(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c})}{\partial t} \nabla \left( t - \frac{|\vec{r} - \vec{r}'|}{c} \right)
\] (10.77)

\[
\Rightarrow \nabla \rho(\vec{r}', t_r) = -\frac{1}{c} \frac{\partial \rho(\vec{r}' - \vec{r})}{\partial t} = -\frac{1}{c} \rho \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \right) \quad \text{(10.78)}
\]

we may write

\[
\nabla V(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \int \left[ \left( -\frac{1}{c} \rho \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \right) \right) - \rho(\vec{r}', t_r) \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) \right] d\tau' = -\frac{1}{4\pi \epsilon_0} \int \left[ \left( \frac{1}{c} \rho \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \right) + \rho \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) \right) \right] d\tau'.
\] (10.79)
Thus, we can express the expression for the electric and magnetic fields for the static case as

\begin{align}
\mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') \, d\mathbf{r}' \quad \text{and} \quad \mathbf{B}(r) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}'.
\end{align}

(10.87)
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\[ \vec{E}(r, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(\vec{r}', t_r) \ (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\tau', \quad \vec{B}(r, t) = \frac{1}{4\pi \epsilon_0} \int \frac{\vec{J}(\vec{r}', t_r) \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\tau' \]

(10.88)

we would be wrong! The right form of the electric and magnetic fields satisfying Maxwell’s equations resulting from casual argument published by Oleg Jefimenko in 1966. These equations are obtained by using the retarded electric and vector potentials

\[ \vec{E}(r, t) = -\nabla V((r, t_r)) - \frac{\partial \vec{A}((r, t_r))}{\partial t} \]  \hspace{1cm} (10.89)

Using the result

\[ \nabla V(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \int \left[ -\frac{1}{c} \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^2} - \rho(\vec{r}', t_r) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] \, d\tau' \]

\[ = -\frac{1}{4\pi \epsilon_0} \int \left[ \frac{1}{c} \frac{\dot{\rho}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^2} \right] + \rho(\vec{r}', t_r) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] \, d\tau' \]  \hspace{1cm} (10.90)

and the retarded vector potential

\[ \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \, d\tau'. \]  \hspace{1cm} (10.91)

\[ \frac{\partial \vec{A}((r, t_r))}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\partial \vec{J}(\vec{r}', t_r)}{mt} \frac{\partial t_r}{\partial t} \, d\tau' \]

\[ = \frac{\epsilon_0 \mu_0}{4\pi \epsilon_0} \int \frac{\partial \vec{J}(\vec{r}', t_r)}{mt} \frac{\partial t_r}{\partial t} \, d\tau = \frac{1}{4\pi \epsilon_0} \int \frac{\partial \vec{J}(\vec{r}', t_r)}{mt} \frac{\partial t_r}{\partial t} \, d\tau \]  \hspace{1cm} (10.92)

and

\[ \vec{E}(r, t) = -\nabla V((r, t_r)) - \frac{\partial \vec{A}((r, t_r))}{\partial t} \]

\[ = -\frac{1}{4\pi \epsilon_0} \int \left\{ \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') + \frac{\partial \phi(\vec{r}', t_r)}{\partial t} \frac{\partial t_r}{\partial t} \frac{c}{c^2 |\vec{r} - \vec{r}'|^2} (\vec{r} - \vec{r}') - \frac{\partial J(\vec{r}', t_r)}{\partial t} \frac{\partial t_r}{\partial t} \frac{c}{c^2 |\vec{r} - \vec{r}'|^2} \right\} \, d\tau' \]  \hspace{1cm} (10.93)

\[ \vec{B}(r, t) = \nabla \times \vec{A}(r, t_r) = \frac{\mu_0}{4\pi} \int \left[ \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|^3} + \frac{\partial \vec{J}(\vec{r}', t_r)}{\partial t} \frac{\partial t_r}{\partial t} \frac{c}{c^2 |\vec{r} - \vec{r}'|^2} \right] (\vec{r} - \vec{r}') \, d\tau'. \]  \hspace{1cm} (10.94)

and are called Jefimenko’s equations.
10.5 Electromagnetic radiation

So far we discussed the properties of EM waves and their propagation in both bounded and unbounded matter (dielectric such as glass and conductors such as metals, wave guide etc) but we have not discussed how these waves are generated. Of course we know that these wave made of electric and magnetic fields produced by some sort of charge and current distribution some where in space. But the question is does any kind of charge and current distribution in space produce electric and magnetic fields of the electromagnetic waves? We will address answer to this question in this chapter.

10.5.1 Dipole radiation

*Neither a static charge nor a constant current produce EM waves at infinity.* Consider a spherical surface of radius $r$ ($r \to \infty$) enclosing a source of radiation as shown in the figure below.

We recall the pointing power

$$P = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint \mathbf{E} \times \mathbf{B} \cdot d\mathbf{a}$$

(10.95)

For static case where we have static charge and constant current the fields are

$$E \sim \frac{1}{r^2}, B \sim \frac{1}{r^3}. \quad (10.96)$$

For spherical surface of radius, $r$, the enclosing surface at a distance $r$ from the source is

$$A \sim r^2. \quad (10.97)$$

Therefore the power of the radiation

$$P \sim \frac{1}{r^2} \cdot \frac{1}{r^3} \cdot r^2 \sim \frac{1}{r^3} = 0 \quad (10.98)$$
10.5. ELECTROMAGNETIC RADIATION

For a distance faraway from the source the power is zero. This means static charges and currents can not produce a radiation. Then how can EM radiations are generated?

An oscillating or accelerating charges and oscillating current can generate radiations. Check the Jefimenko’s equation for

\[ E(r, t) = -\frac{1}{4\pi \epsilon_0} \int \left\{ \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') + \frac{\partial \vec{J}(\vec{r}', t_r)}{c |\vec{r} - \vec{r}'|^2} (\vec{r} - \vec{r}') - \frac{\partial \vec{J}(\vec{r}', t_r)}{c^2 |\vec{r} - \vec{r}'|} \right\} d\vec{r}' \]

\[ \Rightarrow E \sim \frac{1}{r} \] (10.99)

and

\[ B(r, t) = \frac{\mu_0}{4\pi} \int \left( \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|^2} + \frac{\partial \vec{J}(\vec{r}', t_r)}{c |\vec{r} - \vec{r}'|} \right) (\vec{r} - \vec{r}') d\vec{r}' \Rightarrow B \sim \frac{1}{r}. \] (10.100)

\[ P \sim \frac{1}{r} \frac{1}{r^2} \neq 0 \] (10.101)

10.5.2 Oscillating electric dipole radiation

Let’s consider two tiny metal spheres separated by a distance \(d\) (center to center) as shown in Fig.10.2. These spheres are connected by a fine conducting wire so that the charges can move back and forth between the two sphere. At a given time \(t\) the charge on the upper sphere is \(q_+(t)\) and the charge on the lower sphere is \(-q_-(t)\). Suppose the charges are oscillating back and forth between the two sphere at an angular frequency, \(\omega\), and the charge at a given time \(t\), can be expressed as

\[ q_+(t) = q_0 \cos(\omega t), \quad q_-(t) = -q_0 \cos(\omega t) \] (10.102)

These two time dependent charge form a physical dipole with time dependent dipole moment \(\vec{p}(t)\) given by

\[ \vec{p}(t) = (q(t) \hat{d}) \hat{z} = q_0 d \cos(\omega t) \hat{z} = p_0 \cos(\omega t) \hat{z} \] (10.103)

where

\[ p_0 = q_0 d \] (10.104)

is the maximum dipole moment. The retarded potential due to the two charges at a position \(\vec{r}'\), is given by

\[ V(\vec{r}', t) = \frac{1}{4\pi \epsilon_0} \left( \frac{q_+(t - t^+_r)}{r_+} + \frac{q_-(t - t^-_r)}{r_-} \right) \] (10.105)

where

\[ r_\pm = \sqrt{\vec{r}^2 + d^2/4 \mp rd \cos \theta}, \]

\[ t^\pm_\mp = \frac{r_\pm}{c} = \frac{1}{c} \sqrt{\vec{r}^2 + d^2/4 \mp rd \cos \theta}, \]

\[ q_+ (t - t^+_r) = q_0 \cos[\omega(t - t^+_r)], \]

\[ q_- (t - t^-_r) = -q_0 \cos[\omega(t - t^-_r)] \] (10.106)
In order to find the potential at a distance very far away (the radiation zone) from these charges so that we would subsequently be able to find the electric and magnetic fields of the EM waves it is necessary to make physically valid approximations. The following three approximations are physically valid approximations for the radiation zone

(a) **Approximation 1**: $d << r$: In this approximation a physical dipole becomes a perfect dipole. Using Taylor series expansion, for $x << 1$ one can make the approximations relation

$$ (1 + x)^{\pm 1/2} \simeq 1 \pm \frac{1}{2} x $$

we can write

$$ r_{\pm} = r \left( 1 \pm \frac{d}{r} \cos \theta + \frac{d^2}{4r^2} \right)^{1/2} \simeq r \left( 1 \pm \frac{d}{r} \cos \theta \right)^{1/2} \simeq r \left( 1 \pm \frac{d}{2r} \cos \theta \right) $$

$$ \frac{1}{r_{\pm}} = \frac{1}{r} \left( 1 \pm \frac{d}{r} \cos \theta + \frac{d^2}{4r^2} \right)^{-1/2} \simeq \frac{1}{r} \left( 1 \pm \frac{d}{r} \cos \theta \right)^{-1/2} $$

$$ \frac{1}{r_{\pm}} \simeq \frac{1}{r} \left( 1 \pm \frac{d}{2r} \cos \theta \right). \quad (10.107) $$

Using these approximations, one can also write

$$ t_{r}^{\pm} = \frac{r_{\pm}}{c} \simeq \frac{r}{c} \left( 1 \pm \frac{d}{2r} \cos \theta \right). \quad (10.108) $$
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Then

\[
\cos \omega (t - t_r^+) = \cos \left\{ \omega t - \frac{r \omega}{c} \left( 1 \mp \frac{d}{2r} \cos \theta \right) \right\} = \\
\cos \omega \left( t - \frac{r}{c} \right) \pm \frac{\omega d}{2c} \cos \theta \\
= \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \cos \left( \frac{\omega d}{2c} \cos \theta \right) \mp \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \sin \left( \frac{\omega d}{2c} \cos \theta \right)
\]

(10.109)

which leads to

\[
q_+ (t - t_r^+) = q_0 \cos \omega \left( t - t_r^+ \right) = \\
= q_0 \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \cos \left( \frac{\omega d}{2c} \cos \theta \right) - q_0 \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \sin \left( \frac{\omega d}{2c} \cos \theta \right),
\]

(10.110)

\[
q_- (t - t_r^-) = -q_0 \cos \omega \left( t - t_r^- \right) = \\
= -q_0 \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \cos \left( \frac{\omega d}{2c} \cos \theta \right) + q_0 \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \sin \left( \frac{\omega d}{2c} \cos \theta \right)
\]

(10.111)

(b) Approximation 2: \( d << c/\omega \): we recall that the wavelength, \( \lambda \), is related to the angular frequency by

\[
\omega = \frac{2\pi c}{\lambda} \Rightarrow \frac{\omega}{\omega} = \frac{\lambda}{2\pi}
\]

(10.113)

For \( d << c/\omega \), indicate that the distance, \( d \), between \( q \) and \(-q\) is very much smaller than the wavelength

\[
d << \frac{c}{\omega} \Rightarrow d << \frac{\lambda}{2\pi} \Rightarrow d << \lambda.
\]

(10.114)

If this condition is met, using small angle approximation for \( x << 1 \)

\[
\sin (x) \simeq x, \cos (x) \simeq 1,
\]

we have

\[
\frac{\omega d}{2c} \cos (\theta) = \frac{d}{2 (c/\omega)} \cos (\theta) = \frac{\pi d}{\lambda} \cos (\theta) << 1
\]

\[
\Rightarrow \sin \left( \frac{\omega d}{2c} \cos (\theta) \right) \simeq \frac{\omega d}{2c} \cos (\theta), \cos \left( \frac{\omega d}{2c} \cos (\theta) \right) \simeq 1
\]

(10.115)

so that

\[
q_+ (t - t_r^+) = q_0 \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{\omega q_0 d}{2c} \cos (\theta) \sin \left( \omega \left( t - \frac{r}{c} \right) \right),
\]

(10.116)

\[
q_- (t - t_r^-) = -q_0 \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{\omega q_0 d}{2c} \sin \left( \omega \left( t - \frac{r}{c} \right) \right)
\]

(10.117)
We now using the above two approximations, we can write the approximate retarded potential as

\[
V(\vec{r}, t) = \frac{q_0}{4\pi \epsilon_0 r} \left\{ \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{d\omega}{2c} \cos(\theta) \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \left( 1 + \frac{d}{2r} \cos(\theta) \right) \\
- \left[ \cos \left( \omega \left( t - \frac{r}{c} \right) \right) + \frac{d\omega}{2c} \cos(\theta) \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \left( 1 - \frac{d}{2r} \cos(\theta) \right) \right] \right\}
\]

(10.118)

Noting that

\[
(x - y)(1 + z) - (x + y)(1 - z) = x + xz - y - yz - (x - xz + y - yz) = 2(xz - y)
\]

(10.119)

one can easily find

\[
V(\vec{r}, t) = \frac{p_0}{4\pi \epsilon_0} \frac{\cos(\theta)}{r} \left[ \frac{\cos \left( \omega \left( t - \frac{r}{c} \right) \right)}{r} \frac{\omega}{c} \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \right]
\]

(10.120)

where

\[
p_0 = q_0 d.
\]

(10.121)

is the maximum dipole moment. For static case, \(\omega \rightarrow 0\), we find what we should expect

\[
V(\vec{r}, t) = \frac{p_0}{4\pi \epsilon_0} \frac{\cos(\theta)}{r^2}.
\]

(10.122)

(c) **Approximation 3:** \(r >> c/\omega\) (radiation zone): We are interested in the radiation zone where \(r\) is very large. In this zone we can write the retarded potential as

\[
V(\vec{r}, t) = -\frac{p_0 \omega}{4\pi \epsilon_0 c} \left( \frac{\cos(\theta)}{r} \right) \sin \left[ \omega \left( t - \frac{r}{c} \right) \right]
\]

(10.123)

*The vector potential:* noting that the current is given by

\[
\vec{I}(t) = \frac{dq}{dt} \hat{z} = -q \omega \sin(\omega t)
\]

(10.124)

for the retarded current we can write

\[
\vec{I}(t) = \frac{dq}{dt} \hat{z} = -q \omega \sin[\omega(t - t_r)] \hat{z}
\]

(10.125)

and the retarded vector potential given by

\[
\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r) \ d\vec{r}'}{|\vec{r} - \vec{r}'|}
\]

(10.126)
for, a line current, can be written as

\[ \tilde{A}(\vec{r}, t) = -\frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{q\omega \sin[\omega (t - t_r)]}{|\vec{r} - \vec{r}'|} dz. \]  

(10.127)

and this leads to

\[ \tilde{A}(\vec{r}, t) \simeq -\frac{\mu_0 p_0 \omega}{4\pi r} \sin \omega (t - r/c) \hat{z}. \]  

(10.128)

**The Electric and the magnetic field in the radiation zone:** In the radiation zone the electric potential is given by

\[ V(\vec{r}, t) = -\frac{\rho_0 \omega}{4\pi \epsilon_0 c} \frac{\cos \theta}{r} \sin \omega (t - r/c) \]  

(10.129)

and the vector potential

\[ \tilde{A}(\vec{r}, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin \omega (t - r/c) \hat{z}. \]  

(10.130)

The electric and magnetic field in the radiation zone can then be determined using

\[ \tilde{E}(\vec{r}, t) = -\nabla V - \frac{\partial \tilde{A}}{\partial t}, \tilde{B}(\vec{r}, t) = \nabla \times \tilde{A}. \]  

(10.131)

First let’s find the gradient of the electric potential in the radiation zone. Taking into account the potential depends on the spherical coordinates \( r \) and \( \theta \), one can write

\[ \nabla V = \hat{r} \frac{\partial V}{\partial r} + \hat{\theta} \frac{\partial V}{\partial \theta} = -\frac{\rho_0 \omega}{4\pi \epsilon_0 c} \left\{ \hat{r} \cos \theta \frac{\partial}{\partial r} \left( \frac{1}{r} \sin \omega (t - r/c) \right) \\
+ \frac{\partial}{\partial \theta} \sin \omega (t - r/c) \frac{\partial \cos \theta}{\partial \theta} \right\} \\
= -\frac{\rho_0 \omega}{4\pi \epsilon_0 c} \left[ \left( \frac{\sin \theta}{r^2} - \cos \theta \frac{\partial}{\partial \theta} \hat{r} \right) \right] \sin \omega (t - r/c) - \frac{\omega \cos \theta}{cr} \cos \omega (t - r/c) \]  

In the radiation zone, \( r >> c/\omega \), we can make the approximation

\[ \frac{1}{r^2} \rightarrow 0 \]  

(10.132)

so that

\[ \nabla V \simeq -\frac{\mu_0 p_0 \omega^2}{4\pi} \cos \omega (t - r/c) \frac{\cos \theta}{r} \hat{r}, \]  

(10.133)

where we replaced

\[ \epsilon_0 c^2 = \epsilon_0 \frac{1}{\epsilon_0 \mu_0} = \frac{1}{\mu_0} \]  

(10.134)
For the time derivative of the vector potential we have
\[
\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{\mu_0 P_0 \omega}{4\pi r} \sin[\omega (t - r/c)] \hat{z} \right] = -\frac{\mu_0 P_0 \omega}{4\pi} \frac{1}{r} \cos[\omega (t - r/c)] \hat{z} \tag{10.135}
\]
\[
\Rightarrow \frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 P_0 \omega^2}{4\pi} \cos[\omega (t - r/c)] \left( \frac{\sin(\theta) \hat{\theta} - \cos(\theta) \hat{\varphi}}{r} \right) \tag{10.136}
\]
where we used
\[
\hat{z} = \cos(\theta) \hat{r} - \sin(\theta) \hat{\varphi}.
\]
Then the electric field, in the radiation zone, becomes
\[
\mathbf{E}(\mathbf{r}, t) = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 P_0 \omega^2}{4\pi} \frac{\sin(\theta)}{r} \cos[\omega (t - r/c)] \hat{\theta}.
\]
For the magnetic field in the radiation zone
\[
\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_\varphi) - \frac{\partial}{\partial \varphi} (A_\theta) \right] \hat{r} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\varphi) - \frac{\partial}{\partial \theta} (r A_\theta) \right] \hat{\varphi} \tag{10.137}
\]
Since
\[
\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 P_0 \omega}{4\pi} \sin[\omega (t - r/c)] \left( \frac{\sin(\theta) \hat{\theta} - \cos(\theta) \hat{\varphi}}{r} \right)
\]
we have
\[
A_\theta (r, \theta) = \frac{\mu_0 P_0 \omega}{4\pi r} \sin(\theta) \sin[\omega (t - r/c)],
\]
\[
A_\varphi (r, \theta) = 0, A_r (r, \theta) = -\frac{\mu_0 P_0 \omega}{4\pi r} \cos(\theta) \sin[\omega (t - r/c)], \tag{10.138}
\]
so that
\[
\mathbf{B}(\mathbf{r}, t) = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial}{\partial \theta} (A_\varphi) \right] \hat{\varphi} - \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\varphi) - \frac{\partial}{\partial \theta} (r A_\theta) \right] \hat{\varphi}
\]
\[
= \frac{\mu_0 P_0 \omega}{4\pi r} \left( \frac{\omega}{c} \sin(\theta) \cos[\omega (t - r/c)] + \frac{1}{r^2} \sin(\theta) \sin[\omega (t - r/c)] \right) \hat{\varphi} \tag{10.139}
\]
Again in the radiation zone \( r >> c/\omega; \)
\[
\frac{1}{r^2} \rightarrow 0 \tag{10.140}
\]
and we find for the magnetic field
\[
\mathbf{B}(\mathbf{r}, t) \simeq \frac{\mu_0 P_0 \omega^2}{4\pi r c} \frac{\sin(\theta)}{r} \cos[\omega (t - r/c)] \hat{\varphi}. \tag{10.141}
\]
The average energy in the radiation zone: Then energy radiated by an oscillating electric dipole is determined by the pointing vector which is given by

$$\vec{S} = \frac{1}{\mu_0} \left( \vec{E} \times \vec{B} \right). \quad (10.142)$$

Using the results for $\vec{E}$ and $\vec{B}$ in the radiation zone,

$$\vec{E}(\vec{r}, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin(\theta)}{r} \cos[\omega(t - r/c)] \hat{\theta},$$

$$\vec{B}(\vec{r}, t) \simeq -\frac{1}{c} \frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin(\theta)}{r} \cos[\omega(t - r/c)] \hat{\phi},$$

we have

$$\vec{S} = \frac{\mu_0}{c} \left( \frac{\mu_0 p_0^2 \omega^4}{4\pi r} \right) \frac{\sin^2(\theta)}{\cos^2[\omega(t - r/c)]} \left( \hat{\theta} \times \hat{\phi} \right)$$

$$\Rightarrow \vec{S} = \frac{\mu_0}{c} \left( \frac{\mu_0 p_0^2 \omega^4}{4\pi r} \right) \sin^2(\theta) \cos^2[\omega(t - r/c)] \hat{r}. \quad (10.143)$$

The time average pointing vector in the radiation zone can be expressed as

$$\langle \vec{S} \rangle = \frac{\mu_0}{c} \left( \frac{\mu_0 p_0^2 \omega^4}{4\pi r} \right) \frac{\sin^2(\theta)}{\cos^2[\omega(t - r/c)]} \hat{r} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2} \frac{\sin^2(\theta)}{r^2} \hat{r}. \quad (10.144)$$

and using

$$\langle \cos^2[\omega(t - r/c)] \rangle = \frac{2\pi}{\omega} \int_0^{2\pi} \cos^2[\omega(t - r/c)] dt = \frac{1}{2}$$

we find

$$\langle \vec{S} \rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2} \frac{\sin^2(\theta)}{r^2} \hat{r}. \quad (10.145)$$

Then the average power of the radiation

$$\langle P \rangle = \int_{\text{Surface}} \langle \vec{S} \rangle \cdot d\vec{a} = \int_{\text{Surface}} \left( \frac{\mu_0 p_0^2 \omega^4}{32\pi^2} \frac{\sin^2(\theta)}{r^2} \hat{r} \right) \cdot \left( r^2 \sin(\theta) d\theta d\phi \hat{r} \right)$$

$$= \int_0^\pi \int_0^{2\pi} \frac{\mu_0 p_0^2 \omega^4}{32\pi^2} \frac{\sin^3(\theta)}{r^2} d\theta d\phi = \int_0^\pi \int_0^{2\pi} \frac{\mu_0 p_0^2 \omega^4}{32\pi^2} \frac{\sin^3(\theta)}{r^2} d\theta d\phi$$

$$\Rightarrow \langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}. \quad (10.147)$$

This result showed that the average power of an electric dipole radiation will never be zero how much far away from it (, assuming there is no absorption along its path). So if there is a very young and hot distant star that has not been discovered yet in our solar system or in any distant galaxy, it would be discovered sooner or later! Because the radiation would never be zero in the radiation zone! It might take thousands, hundreds, millions, or billions of years but it would be here. The radiation of an accelerated charge or a dipole would never die how far the radiation need to travel :)
10.5.3 Magnetic Dipole Radiation

Consider a sinusoidal current in a circular loop given by

\[ I(t) = I_0 \cos(\omega t) \]  \hspace{1cm} (10.148)

The radius of the circular loop is \( b \) on the x-y plane (see the figure below).

\[ \vec{m}(t) = (\pi b^2) I_0 \cos(\omega t) \hat{z} = m_0 \cos(\omega t) \hat{z} \]  \hspace{1cm} (10.149)

where

\[ m_0 = \pi b^2 I_0 \hat{z}. \]  \hspace{1cm} (10.150)

The vector potential in the radiation zone is given by

\[ \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d\vec{r}'. \]  \hspace{1cm} (10.151)

so that for this magnetic dipole we can write

\[ \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos(\omega t_r)}{|\vec{r} - \vec{r}'|} d\vec{l}' \]  \hspace{1cm} (10.152)

Referring to the figure above we may write that

\[ \vec{r}' = b \cos(\phi') \hat{x} + b \sin(\phi') \hat{y} \]

\[ d\vec{l}' = d\vec{r}' = -b \sin(\phi') d\phi' \hat{x} + b \cos(\phi') d\phi' \hat{y} \]  \hspace{1cm} (10.153)
and
\[ |\vec{r} - \vec{r}'| = \sqrt{r'^2 + b^2 - 2rb\cos(\psi)} = \sqrt{r^2 + b^2 - 2r' \cdot \vec{r}'}, \] (10.154)
\[
t_r = t - |\vec{r} - \vec{r}'|/c = t - \sqrt{r'^2 + b^2 - 2r' \cdot \vec{r}'}/c \] (10.155)
so that
\[
\vec{A}(\vec{r}, t) = \mu_0 I_0 \int_0^{2\pi} \cos \left[ \omega \left( t - \frac{1}{c} \sqrt{r'^2 + b^2 - 2r' \cdot \vec{r}'} \right) \right] \left/ \sqrt{r'^2 + b^2 - 2r' \cdot \vec{r}'} \right. \{ -b \sin(\varphi') d\varphi' \hat{x} + b \cos(\varphi') d\varphi' \hat{y} \} . \] (10.156)

Once again referring to the figure above we can write for a point directly above the x-axis
\[ \vec{r} = r \sin(\theta) \hat{x} + r \cos(\theta) \hat{z} \] (10.157)
so that we can write
\[ \vec{r} \cdot \vec{r}' = (r \sin(\theta) \hat{x} + r \cos(\theta) \hat{z}) \cdot b (\cos(\varphi') \hat{x} + \sin(\varphi') \hat{y}) = rb \sin(\theta) \cos(\varphi') \] (10.158)
Therefore, the vector potential can be written as
\[
\vec{A}(\vec{r}, t) = \mu_0 b I_0 \int_0^{2\pi} \cos \left[ \omega \left( t - \frac{1}{c} \sqrt{r'^2 + b^2 - 2rb \sin(\theta) \cos(\varphi')} \right) \right] \left/ \sqrt{r'^2 + b^2 - 2rb \sin(\theta) \cos(\varphi')} \right. \{ -b \sin(\varphi') d\varphi' \hat{x} + b \cos(\varphi') d\varphi' \hat{y} \} . \] (10.159)

Approximation 1: \( b << r \) (A physical magnetic dipole becomes a perfect magnetic dipole): Under this approximation we may write
\[
\sqrt{1 + \frac{b^2}{r^2} - 2\frac{b}{r} \sin(\theta) \cos(\varphi')} \simeq \sqrt{1 - 2\frac{b}{r} \sin(\theta) \cos(\varphi')} \simeq 1 - \frac{b}{r} \sin(\theta) \cos(\varphi')
\]
and
\[
\frac{1}{\sqrt{r'^2 + b^2 - 2rb \sin(\theta) \cos(\varphi')}} \simeq \frac{1}{r} \left[ 1 + \frac{b}{r} \sin(\theta) \cos(\varphi') \right]
\]
which leads to
\[
\cos \left\{ \omega \left[ t - \frac{r}{c} \sqrt{1 + \frac{b^2}{r^2} - 2\frac{b}{r} \sin(\theta) \cos(\varphi')} \right] \right\} \simeq \cos \left\{ \omega \left[ t - \frac{r}{c} \left( 1 - \frac{b}{r} \sin(\theta) \cos(\varphi') \right) \right] \right\} = \cos \left[ \omega \left( t - \frac{r}{c} \right) + \frac{\omega b}{c} \sin(\theta) \cos(\varphi') \right]
\]
Then
\[
\cos \left[ \omega \left( t - \frac{r}{c} \right) + \frac{\omega b}{c} \sin(\theta) \cos(\varphi') \right] = \cos \left[ \omega \left( t - \frac{r}{c} \right) \cos \left[ \frac{\omega b}{c} \sin(\theta) \cos(\varphi') \right] \right]\]
\[ - \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \sin \left[ \frac{\omega b}{c} \sin(\theta) \cos(\varphi') \right] \] (10.160)
Approximation 2: \( b \ll c/\omega \) : Recalling that the wavelength is related to the frequency by
\[
\omega = \frac{2\pi c}{\lambda}
\]
this approximation mean that
\[
b \ll c/\omega \Rightarrow \omega b \ll c.
\]
Under this approximation, we have
\[
\sin \left( \frac{\omega b}{c} \sin (\theta) \cos (\varphi') \right) \simeq \frac{\omega b}{c} \sin (\theta) \cos (\varphi')
\]
\[
\cos \left( \frac{\omega b}{c} \sin (\theta) \cos (\varphi') \right) \simeq 1
\]
so that
\[
\Rightarrow \cos \left[ \omega \left( t - \frac{r}{c} \right) + \frac{\omega b}{c} \sin (\theta) \cos (\varphi') \right] = \cos \left[ \omega \left( t - \frac{r}{c} \right) \right]
\]
\[
- \frac{\omega b}{c} \sin (\theta) \cos (\varphi') \sin \left[ \omega \left( t - \frac{r}{c} \right) \right]
\]
(10.163)
Using these approximations we can write the approximate retarded vector potential as
\[
\vec{A}(\vec{r}, t) = \frac{\mu_0 b I_0}{4\pi r} \int_0^{2\pi} \left\{ \left[ \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{\omega b}{c} \sin (\theta) \cos (\varphi') \sin \left( \omega \left( t - \frac{r}{r} \right) \right) \right] \right\}
\]
\[
\times \left[ 1 + \frac{b}{r} \sin (\theta) \cos (\varphi') \right] \left[ -\sin (\varphi') d\varphi' \hat{x} + \cos (\varphi') d\varphi' \hat{y} \right].
\]
(10.164)
\[
\vec{A}(\vec{r}, t) = \frac{\mu_0 b I_0}{4\pi r} \int_0^{2\pi} \left\{ \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{\omega b}{c} \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \sin (\theta) \cos (\varphi') \right\}
\]
\[
+ \frac{b}{r} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \sin (\theta) \cos (\varphi')
\]
\[
- \frac{\omega b^2}{cr} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \sin^2 (\theta) \cos^2 (\varphi') \left[ -\sin (\varphi') d\varphi' \hat{x} + \cos (\varphi') d\varphi' \hat{y} \right].
\]
(10.165)
Base on the above two approximations we have
\[
\frac{\omega b^2}{cr} = \left( \frac{\omega b}{c} \right) \frac{b}{r} << 1.
\]
(10.166)
and we can approximate the above expression as
\[
\vec{A}(\vec{r}, t) \approx \frac{\mu_0 b I_0}{4\pi r} \int_0^{2\pi} \left\{ \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{\omega b}{c} \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \sin (\theta) \cos (\varphi') \right\}
\]
\[
+ \frac{b}{r} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \sin (\theta) \cos (\varphi') \left[ -\sin (\varphi') d\varphi' \hat{x} + \cos (\varphi') d\varphi' \hat{y} \right].
\]
(10.167)
Noting that the x-component
\[ A_x(\vec{r}, t) = -\mu_0 b I_0 \frac{\cos \left( \omega \left( t - \frac{r}{c} \right) \right)}{4\pi r} \int_0^{2\pi} \sin (\varphi') d\varphi' \]
\[ + \sin (\theta) \left[ \frac{b}{r} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) - \frac{\omega b}{c} \sin \left( \omega \left( t - \frac{r}{c} \right) \right) \right] \int_0^{2\pi} \cos (\varphi') \sin (\varphi') d\varphi' \]
(10.168)
and the y-component
\[ A_y(\vec{r}, t) = \frac{\mu_0 b I_0}{4\pi r} \left\{ \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \int_0^{2\pi} \cos (\varphi') d\varphi' - \frac{\omega b}{c} \sin (\theta) \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \right\} \]
\[ \times \int_0^{2\pi} \cos^2 (\varphi') d\varphi' + \frac{b}{r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \sin (\theta) \int_0^{2\pi} \cos^2 (\varphi') d\varphi' \]  
(10.169)
and using the results
\[ \int_0^{2\pi} \sin [\varphi'] \cos [\varphi'] d\varphi' = 0 \]
\[ \int_0^{2\pi} \cos [\varphi'] \cos [\varphi'] d\varphi' = 0 \]
\[ \int_0^{2\pi} \sin [\varphi'] d\varphi' = \int_0^{2\pi} \cos [\varphi'] d\varphi' = \pi \]
we find
\[ A_x(\vec{r}, t) = 0 \]  
(10.170)
and
\[ A_y(\vec{r}, t) = \frac{\mu_0 b^2 I_0}{4\pi r} \left\{ \frac{-\omega}{c} \sin (\theta) \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] + \frac{1}{r} \sin (\theta) \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \right\}. \]
(10.171)
Therefore the vector potential becomes
\[ \vec{A}(\vec{r}, t) = \frac{\mu_0 \pi b^2 I_0}{4\pi r} \sin (\theta) \left( \frac{1}{r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] - \frac{\omega}{c} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \right) \hat{y}. \]
(10.172)
or in terms of the magnetic dipole moment
\[ \vec{A}(\vec{r}, t) = \frac{\mu_0 m_0}{4\pi r} \sin (\theta) \left( \frac{1}{r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] - \frac{\omega}{c} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \right) \hat{y}. \]
(10.173)
\[ \vec{A}(\vec{r}, t) = \frac{\mu_0 m_0}{4\pi r} \sin (\theta) \left( \frac{1}{r} \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] - \frac{\omega}{c} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \right) \hat{\varphi}. \]  
(10.174)
where we take into account that \( \vec{A} \), in general points along the \( \varphi \) direction if we chose the point described by the vector \( \vec{r} \) is off the x-z plane. In the static case where \( \omega \to 0 \), we find

\[
\vec{A}(\vec{r}, t) = \frac{\mu_0 \epsilon_0}{4\pi r^2} \varphi.
\]

\[\text{(10.175)}\]

**Approximation 3: \( r \gg c/\omega \) (radiation zone):** We are interested in the radiation zone where \( r \) is very large. In this zone we can write the retarded potential as

\[
\vec{A}(\vec{r}, t) = -\frac{\mu_0 \epsilon_0 \omega^2}{4\pi c} \left( \frac{\sin (\theta)}{r} \right) \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \varphi.
\]

\[\text{(10.176)}\]

**The electric field:** For a radiation emitted by a magnetic dipole the electric field depends only on the vector potential. Since the total charge in the conducting loop (dipole) is zero, the retarded potential in the radiation zone is zero. Thus we may write the electric field as

\[
\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 \epsilon_0 \omega^2}{4\pi c} \left( \frac{\sin (\theta)}{r} \right) \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \varphi
\]

\[\text{(10.177)}\]

**The magnetic field:** The magnetic field in the radiation zone can be expressed as

\[
\vec{B} = \nabla \times \vec{A} = -\frac{\mu_0 \epsilon_0 \omega^2}{4\pi c^2} \left( \frac{\sin (\theta)}{r} \right) \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \hat{\varphi}
\]

\[\text{(10.178)}\]

**The Average Power in the radiation zone:** Then energy radiated by an oscillating magnetic dipole is determined by the pointing vector which is given by

\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}.
\]

\[\text{(10.179)}\]

Using the results for \( \vec{E} \) and \( \vec{B} \), in the radiation zone, obtained above, we have

\[
\vec{S} = \frac{\mu_0 \epsilon_0 \omega^2}{4\pi c} \left( \frac{\sin (\theta)}{r} \right) \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \varphi \\
\times \left[ -\frac{\mu_0 \epsilon_0 \omega^2}{4\pi c^2} \left( \frac{\sin (\theta)}{r} \right) \cos \left[ \omega \left( t - \frac{r}{c} \right) \right] \right] \hat{\varphi}.
\]

\[\text{(10.180)}\]

Noting that the time average for

\[
\langle \cos^2 [\omega (t - r/c)] \rangle = \frac{1}{2}
\]

\[\text{(10.181)}\]

The time average pointing vector in the radiation zone can be expressed as

\[
\langle \vec{S} \rangle = \frac{\mu_0 \epsilon_0^2 \omega^4 \sin^2 (\theta)}{32\pi^2 c^3} \frac{\sin (\theta)}{r^2} \vec{r}.
\]

\[\text{(10.182)}\]
ELECTROMAGNETIC RADIATION

The time average power

\[
\langle P \rangle = \oint_{\text{Surface}} \mathbf{S} \cdot d\mathbf{a} = \oint_{\text{Surface}} \left( \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \frac{\sin^2 (\theta)}{r^2} \right) \cdot (r^2 \sin (\theta) d\theta d\varphi) \\
= \langle P \rangle = \int_0^\pi \int_0^{2\pi} \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3} \sin^3 (\theta) d\theta d\varphi
\]

would become

\[
\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}. \tag{10.184}
\]

**Electrical vs magnetic dipole radiation:** Recalling that the power emitted by an electrical dipole in the radiation zone is

\[
\langle P_E \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c^3} \tag{10.185}
\]

and that of a magnetic dipole is

\[
\langle P_B \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} \tag{10.186}
\]

we have

\[
\frac{\langle P_B \rangle}{\langle P_E \rangle} = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} \left/ \frac{\mu_0 p_0^2 \omega^4}{12\pi c^3} \right. \tag{10.187}
\]

\[
\frac{\langle P_B \rangle}{\langle P_E \rangle} = \frac{1}{c^2} \frac{m_0^2}{p_0^2}. \tag{10.188}
\]

Using

\[
p_0 = qd \tag{10.189}
\]

and the corresponding current

\[
I_0 = \omega q \tag{10.190}
\]

we may write

\[
p_0 = I_0 d/\omega. \tag{10.191}
\]

Assuming the same magnitude of current, \(I_0\), in the magnetic dipole, we have

\[
m_0 = I_0 (\pi b^2) \tag{10.192}
\]

and the ratio of the power becomes

\[
\frac{\langle P_B \rangle}{\langle P_E \rangle} = \frac{1}{c^2} \left( \frac{I_0 (\pi b^2)}{I_0 d/\omega} \right)^2 = \frac{\omega^2 (\pi b^2)^2}{c^2 d^2}. \tag{10.193}
\]

For

\[
\pi b \simeq d \tag{10.194}
\]
we find
\[
\frac{\langle P_B \rangle}{\langle P_E \rangle} = \frac{\omega^2}{c^2} \left( \frac{\pi b^2}{c^2} \right)^2 = \left( \frac{\omega b}{c} \right)^2.
\]  
(10.195)

but we know that from the second approximation for magnetic dipole radiation
that (Approximation 2: \(b << c/\omega \Rightarrow \frac{\omega b}{c} << 1\)) which tells us
\[
\frac{\langle P_B \rangle}{\langle P_E \rangle} << 1 \Rightarrow \langle P_B \rangle << \langle P_E \rangle.
\]  
(10.196)
Chapter 11

Relativistic Electrodynamics

Unlike Newtonian mechanics, classical electrodynamics is already consistent with special relativity. Maxwell’s equations and the Lorentz force law can be applied legitimately in any inertial system. The difference is that what one observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical. This section proves you a deeper understanding of the structure of electrodynamics—law that had seemed arbitrary and unrelated before take on a kind of coherence and inevitability when approaching from the point of view of relativity.

11.1 Review of Special Theory of relativity

11.1.1 The Lorentz coordinate transformations

For two observers $O$ and $O'$ shown in the figure below Lorentz coordinate transformations are given by

\[ y' = y, z' = z, x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, t' = \frac{t + \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}. \]
11.1.2 Relativistic length contraction

Proper length and length contraction: Rest length of a body is determined by measuring the difference between the spatial coordinates of the endpoints of the body. Since the body is not moving, these measurements can be made at any time. Consider a ruler oriented along the $x-x'$ direction as shown in the figure below. The ruler is at rest with respect to observer $O$. Now the observer $O'$ moves with a velocity $v$ in the $x-x'$ direction. Let the ends of the ruler be designated by A and B. Then using Lorentz transformation, the length of the ruler as observed by observer $O'$, $L$, can be expressed as

$$x_B - x_A = \frac{x_B - x_A + v(t_B' - t_A')}{\sqrt{1 - v^2/c^2}}$$  \hspace{1cm} (11.1)

$$L_0 = \frac{L + v(t_B' - t_A')}{\sqrt{1 - v^2/c^2}}.$$  \hspace{1cm} (11.2)

where

$$x_B' - x_A' = L_0$$  \hspace{1cm} (11.3)

is the proper length of the ruler as measured by $O'$. If $x_A$ and $x_B$ are measured at the same time, so that $t_B - t_A = 0$, then

$$L = L_0 \sqrt{1 - v^2/c^2}.$$  \hspace{1cm} (11.4)

Since $\sqrt{1 - v^2/c^2} < 1$, we have $L < L_0$, so that the length of the moving ruler is measured by $O$ to be contracted. This result is called Lorentz-Fitzgerald contraction.

11.1.3 Relativistic time dilation

Proper time: If an observer, say $O$ determines that two events $A$ and $B$ occur at the same location, the time interval between these two events can be determined by $O$ with a single clock. This time interval, $t_B - t_A = \Delta t_0$, as measured by $O$ with his single clock, is called the proper time interval between the events.

Time dilation: suppose the same two events $A$ and $B$ are viewed by a second observer $O'$, moving with a velocity $v$ with respect to $O$. The second observer will necessarily determine that the two events occur at different locations and will therefore have to use two different, properly synchronized clocks to determine the time separation $t_B' - t_A' = \Delta t'$ between A and B. Using Lorentz time transformation,

$$\Delta t' = \frac{\Delta t_0 + \frac{v}{c^2} (x_B - x_A)}{\sqrt{1 - v^2/c^2}}.$$  \hspace{1cm} (11.5)

Since $O$ determines that the two events occur at the same location, $x_B - x_A = 0$

$$\Delta t' = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}.$$  \hspace{1cm} (11.6)
Since $\sqrt{1 - v^2/c^2} < 1$, we have $\Delta t' > \Delta t_0$, so that the time interval between the two events as measured by $O'$ is dilated (enlarged). Here the single clock was taken to be at rest with respect to $O$. The same result would be obtained if the single clock were taken to be at rest with respect to $O'$. Thus, in general, suppose a single clock advances through a time interval $\Delta t_0$. If this clock is moving with a velocity $v$ with respect to an observer, he will determine that his two clocks advances through a time interval $\Delta t$ given by

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \gamma \Delta t_0.$$ (11.7)

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$ (11.8)

**Example 10.1** The twin abducted by an alien... A mother remembers the birth that she gave at her home to an identical twin boys 80 years ago. She also recalls the unbelievable event that took place immediately after she gave birth. One of the twin had been abducted by an alien... Immediately after the alien abducted the baby had left her place (Earth) with a spaceship traveling with a speed $v = 0.999c$. The mother of the twins is a kind of physics nerd and knew pretty much about special theory of relativity. So she sneaked an atomic clock into the alien spaceship just before it took off. What would be the age of the abducted twin as measured by clock she sneaked into the spaceship?

**Solution:**

$$\Delta t = 80y \sqrt{1 - 0.999^2} = 3.5\text{ years}.$$ (11.9)

*Note: Such kind of paradoxical result is verified daily in high energy physics laboratories. At the Fermi National Accelerator laboratory charged pions with energy 200GeV are produced and transported 300 meters with less than 3% loss because of decay. With a life time of $\Delta t_0 = 2.56 \times 10^{-8}s$, the Galilean decay distance is $c\Delta t_0 = 7.7\text{ meters}$. Therefore without time dilation the fraction of pions that would survive would be

$$\text{Fraction} = \frac{N(\Delta t_0)}{N_0} = e^{\frac{-300}{140}} \approx 10^{-17} \text{ of the Pions}$$ (11.10)

But at 200GeV, we have

$$200GeV = \gamma E_0$$ (11.11)

and for charged pions, the rest mass energy is about $E_0 = 140MeV$, which gives

$$\gamma = \frac{200}{140} \times 10^3 MeV \approx 1400,$$ (11.12)

So the mean free path which is given by

$$\Delta l = c\Delta t' = c\gamma \Delta t_0 \approx 11km$$ (11.13)
11.1.4 Relativistic velocity Transformations

Consider an observer $O'$ moving along the common $x - x'$ axis at a constant velocity $v$ with respect to a second observer, $O$. Each observers measures the velocity of a single particle with $O$ recording $(u_x, u_y, u_z)$ and $O'$ recording $(u'_x, u'_y, u'_z)$ for the components of the particles velocity. These velocities are related by

$$ u'_x = \frac{u_x - v}{1 - (v/c^2) u_x}, \quad u'_y = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - (v/c^2) u_x}, \quad u'_z = \frac{u_z \sqrt{1 - v^2/c^2}}{1 - (v/c^2) u_x}. \quad (11.14) $$

$v$ is positive if $O'$ moves in the positive $x$-direction and negative $O'$ moves in the negative $x$-direction. Inverting these equations give

$$ u_x = \frac{u'_x + v}{1 + (v/c^2) u'_x}, \quad u_y = \frac{u'_y \sqrt{1 - v^2/c^2}}{1 + (v/c^2) u'_x}, \quad u_z = \frac{u'_z \sqrt{1 - v^2/c^2}}{1 + (v/c^2) u'_x}. \quad (11.15) $$

Relativistic Force Transformations:

If a particle acted by parallel force with components $F^\parallel$ and $F_{\perp}$ (parallel and perpendicular to its velocity $\vec{v}$) measured by an observer on an inertial frame $O$. The force measured by an observer in the $O'$ reference frame that travels with the same velocity $\vec{v}$ of the particle (where the particle is instantaneously at rest), would be

$$ F'_{\perp} = \sqrt{1 - \frac{v^2}{c^2}} F_{\perp}, \quad F'^{\parallel} = F^{\parallel} \quad (11.16) $$

11.2 Magnetism as a relativistic Phenomenon

Consider a string of positive charge described by a charge density $\lambda$ moving along the positive $x$-direction with velocity $v$ as shown in the figure below. Superimposed on the positive charge there is a negative charge with the same charge density $\lambda$ moving in the negative $x$-direction with the same magnitude of velocity. The the current

$$ I = 2\lambda v \quad (11.17) $$

Now consider a point charge $q$ moving along the positive $x$-direction with a velocity $u < v$. In the fixed reference frame where we measured the velocities $(O)$, since the total charge is zero the electrical force on the point charge is zero. But what about on a reference frame, $O'$, moving with a velocity $u$ along with the point charge? to answer these question we apply special theory of relativity. Using the relativistic velocity transformations from the previous section

$$ u'_x = \frac{u_x - v}{1 - (v/c^2) u_x}, \quad (11.18) $$

For the positive line charge the speed as measured by an observer on the $O$ frame we have $u_x = v \hat{x}$ since it is moving along the positive $x$-direction and for
11.2. MAGNETISM AS A RELATIVISTIC PHENOMENON

the negative line charge $u_x = -v \hat{x}$ since it is moving in the negative x-direction. The frame (which is a frame attached to the point charge) is moving along the positive x axis with speed which means we have $v = u \hat{x}$. Therefore for the speed of the positive and negative line charges as observed in the $O'$ frame, we may write

$$\vec{v}_+ = \frac{v - u}{1 - (vu/c^2)} \hat{x}, \quad \vec{v}_- = -\frac{v + u}{1 + (vu/c^2)} \hat{x},$$

so that the magnitudes would be come

$$v_+ = \frac{v - u}{1 - vu/c^2}, \quad v_- = \frac{v + u}{1 + vu/c^2} \Rightarrow v_\pm = \frac{v \mp u}{1 \mp vu/c^2} \quad (11.19)$$

This shows that the negative charge moves faster than the positive charge. Suppose the two line charges have a proper length $L_0$, then from Lorentz length contraction, the length of the positive and negative charges as measured by an observer in the $O'$ frame would be

$$L_+ = L_0 \sqrt{1 - v_+^2/c^2},$$

and

$$L_- = L_0 \sqrt{1 - v_-^2/c^2},$$

respectively. Then the corresponding line charge densities

$$\lambda_\pm = \pm \frac{q}{L_\pm} = \pm \frac{q}{L_0 \sqrt{1 - v_\pm^2/c^2}} = \pm \frac{\lambda_0}{\sqrt{1 - v_\pm^2/c^2}} = \pm \gamma_\pm \lambda_0 \quad (11.20)$$

where $\lambda_0$ is the proper line charge density (in its own rest system)

$$\lambda_0 = \frac{q}{L_0}, \quad \gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}} \quad (11.21)$$

This charge density is different from the charge density in the $O$ reference frame, which is

$$\lambda = \frac{q}{L} = \frac{q}{L_0 \sqrt{1 - v^2/c^2}} \Rightarrow \lambda = \frac{\lambda_0}{\sqrt{1 - v^2/c^2}} = \gamma \lambda_0. \quad (11.22)$$

Noting that

$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}} = \frac{1}{\sqrt{1 - (vu/c^2)^2}} = \frac{1}{\sqrt{1 - \left(\frac{v \mp u}{c \mp vu/c^2}\right)^2}} \quad (11.23)$$

This charge density is different from the charge density in the $O$ reference frame, which is

$$\lambda = \frac{q}{L} = \frac{q}{L_0 \sqrt{1 - v^2/c^2}} \Rightarrow \lambda = \frac{\lambda_0}{\sqrt{1 - v^2/c^2}} = \gamma \lambda_0. \quad (11.24)$$

This charge density is different from the charge density in the $O$ reference frame, which is

$$\lambda = \frac{q}{L} = \frac{q}{L_0 \sqrt{1 - v^2/c^2}} \Rightarrow \lambda = \frac{\lambda_0}{\sqrt{1 - v^2/c^2}} = \gamma \lambda_0. \quad (11.25)$$

Noting that

$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}} = \frac{1}{\sqrt{1 - \left(\frac{v \mp u}{c \mp vu/c^2}\right)^2}} \quad (11.26)$$
\[ \gamma_{\pm} = \frac{c^2 \mp uw}{\sqrt{(c^2 \mp uw)^2 - c^2 (v \mp u)^2}} = \frac{c^2 \mp uw}{\sqrt{c^4 + (vu)^2 - c^2 (v^2 + u^2)}} \]

\[ = \frac{c^2 \mp uw}{\sqrt{c^4 - c^2 (v^2 + u^2)}} = \frac{c^2 \mp uw}{\sqrt{(1 - \frac{u^2}{c^2})} \sqrt{c^4 - c^2 u^2}} \]

\[ = \frac{1}{c^2 \sqrt{(1 - \frac{u^2}{c^2})} \sqrt{1 - \frac{u^2}{c^2}}} = \frac{1 \mp \frac{u^2}{c^2}}{\sqrt{1 - u^2/c^2}} \]

(11.27)

where

\[ \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \]

(11.28)

Then the line charges

\[ \lambda_{\pm} = \pm \gamma_{\pm} \lambda_0 \]

(11.29)

can be put in the form

\[ \lambda_{\pm} = \pm \gamma \lambda_0 \frac{1 \mp \frac{u^2}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}. \]

(11.30)

Then the net line charge distribution in the \( O' \) from would be

\[ \lambda_{tot} = \lambda_+ + \lambda_- \]

\[ = \gamma \lambda_0 \left( \frac{1 - \frac{u^2}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{1 + \frac{u^2}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = - \frac{2 \gamma \lambda_0 \frac{u}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} = - \frac{2 \lambda \frac{u}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} \]

(11.31)

where we replaced \( \gamma \lambda_0 = \lambda \). The above results lead to the conclusion that as a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

The electrical Force: the electrical field due to this net line charge at a distance \( s \) from the line charge can be expressed as

\[ E' = \frac{\lambda_{tot}}{2 \pi \epsilon_0 s} \]

(11.32)

and the electrical force on the point charge \( q \) in the \( O' \) frame, due to this field, can be expressed as

\[ F'_\perp = F'_{ei} = qE' = - \frac{q}{2 \pi \epsilon_0 s} \frac{2 \lambda \frac{u}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} = - \frac{q}{\pi \epsilon_0 s} \frac{\lambda \frac{u}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}. \]

(11.33)

If we replace

\[ \frac{1}{c^2} = \mu_0 \epsilon_0 \]

(11.34)
11.3 HOW THE FIELDS TRANSFORM

we have

\[ F'_\perp = F'_{el} = -\frac{\mu_0 q \omega w}{\pi s} \frac{1}{\sqrt{1 - u^2/c^2}}. \]  

(11.35)

Recalling that

\[ I = 2 \lambda v \]  

(11.36)

we may write

\[ F'_\perp = F'_{el} = -\frac{\mu_0 I}{2 \pi s} \frac{qu}{\sqrt{1 - u^2/c^2}}. \]  

(11.37)

If there is a force on \( O' \) then there must be on \( O \) frame. The transformation rules for forces states that

\[ F'_\perp = \frac{1}{\gamma} F_\perp, F_\parallel = F_\parallel \]  

(11.38)

the component of \( F \) parallel to the motion is unchanged. Therefore the force experienced by the charge on the \( O \) frame can be expressed as

\[ F_\perp = -\frac{\mu_0 I q u}{2 \pi s} = qu \left( \frac{\mu_0 I}{2 \pi s} \right), F_\perp = -quB \]  

(11.39)

we got magnetic force!

11.3 How the Fields Transform

A parallel plate capacitor: let’s consider two large plane conductors of length \( l_0 \) and width \( w_0 \) and separated by a distance \( d_0 \) as measured by an observer on an inertial frame \( O \) (see the figure below) which is stationary. Each of these plates carry a total charge \( q_0 \) (bottom plate) and \(-q_0 \) top plate. The charge density on each plate on this frame is \( \pm \sigma_0 \). Applying Gauss’s law, we have shown that the electric field is given by

\[ \vec{E}_0 = \frac{\sigma_0}{\epsilon_0} \hat{y} \]  

(11.40)

where

\[ \sigma_0 = \frac{q_0}{l_0 w_0}. \]  

(11.41)

Now let’s consider a reference frame \( O' \) that travels with a velocity \( v_0 \) along the positive x-direction (see figure above). Relative to this frame, the charges will then be moving with \( v_0 \) in the negative x-direction. The length and width of the plates as measured by an observer on the \( O' \) frame, using Lorentz transformation, can write

\[ l = l_0 \sqrt{1 - v_0^2/c^2}, w = w_0, d = d_0. \]  

(11.42)

Therefore, the surface charge density in the \( O' \) frame would be

\[ \sigma = \frac{q_0}{lw} = \frac{q_0}{l_0 w_0 \sqrt{1 - v_0^2/c^2}} \Rightarrow \sigma = \frac{\sigma_0}{\sqrt{1 - v_0^2/c^2}} \]  

(11.43)
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and the electric field in \( O' \) frame, which is given by

\[ E = \frac{\sigma}{\epsilon_0 x_0} \]  

(11.44)

becomes

\[ E_{\perp} = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \frac{\sigma_0}{\epsilon_0 x_0} \Rightarrow E_{\perp} = \gamma_0 E_0. \]  

(11.45)

where

\[ \gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \]  

(11.46)

This electric field is normal to the direction of the velocity. Let’s see what would happen to the component of the electric field parallel to the velocity of frame \( O \), if there is any. To this end, we put the plates on the \( y - z \) plane (see the figure below).

When the \( O' \) frame moves with velocity \( v' \) along positive x-direction, what contracts is \( d_0 \) and that will not affect the electric field. Therefore the parallel components will not change.

**Example 10.2** Electric field of a point charge in a uniform motion. A point charge \( q \) is at rest at the origin in system \( O \). What is the electric field of this charge in system \( O' \), which moves to the right at speed \( v_0 \)?

Let’s consider the figure shown below with point charge at the origin in the \( O \) frame. For an observer in the \( O \) frame located a position described by the vector

\[ \vec{r} = x_0 \hat{x} + y_0 \hat{y} + z_0 \hat{z} \]  

(11.47)
the electric field is given by

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \]

\[ \vec{E}_0 = \frac{1}{4\pi\varepsilon_0} \frac{q}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} (x_0 \hat{x} + y_0 \hat{y} + z_0 \hat{z}) \]

which leads to

\[ E_{x0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} x_0, E_{y0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} y_0, \]

\[ E_{z0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} z_0. \]

For an observer traveling with a velocity

\[ v = v_0 \hat{x} \]

the charge is moving with a velocity

\[ v = -v_0 \hat{x} \]

The electric field for this observer, using the results obtained earlier we may write as

\[ E_x = E_{x0} = \frac{1}{4\pi\varepsilon_0} \frac{qx_0}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \]

\[ E_y = \frac{1}{\sqrt{1 - v_0^2/c^2}} E_{y0} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q y_0}{(x_0^2 + y_0^2 + z_0^2)^{3/2}} \]

\[ E_z = \frac{1}{\sqrt{1 - v_0^2/c^2}} E_{z0} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q z_0}{(x_0^2 + y_0^2 + z_0^2)^{3/2}}. \]
But this result is in terms of the uncontracted length \((x_0, y_0, z_0)\), we must be able to express the field in terms of the position of the point \(p\) and the charge as measured by an observer in \(O'\) frame. From the figure shown, using Lorentz transformations we may write

\[
x' + v_0 t = x_0 \sqrt{1 - \frac{v_0^2}{c^2}}, y' = y_0, z' = z_0
\]

(11.54)

If we represent the contracted \(x\) position of the charge by \(R_x, R_y,\) and \(R_z,\) we may write

\[
R_x = x' + v_0 t = x_0 \sqrt{1 - \frac{v_0^2}{c^2}}, R_y = y_0, R_y = z_0
\]

\[
\Rightarrow \frac{R_x}{\sqrt{1 - \frac{v_0^2}{c^2}}} = \frac{x' + v_0 t}{\sqrt{1 - \frac{v_0^2}{c^2}}} = x_0,
\]

(11.55)

which leads to

\[
E_x = E_{x0} = \frac{1}{4\pi\varepsilon_0} \frac{q\gamma R_x}{\left(\gamma^2 R_x^2 + R_y^2 + R_z^2\right)^{3/2}},
\]

\[
E_y = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} E_{y0} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q R_y}{\left(\gamma^2 R_x^2 + R_y^2 + R_z^2\right)^{3/2}},
\]

\[
E_z = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} E_{z0} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q R_z}{\left(\gamma^2 R_x^2 + R_y^2 + R_z^2\right)^{3/2}}.
\]

(11.56)

The electric field would then be

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q}{\left(\gamma^2 R_x^2 + R_y^2 + R_z^2\right)^{3/2}} (R_x \hat{x} + R_y \hat{y} + R_z \hat{z})
\]

(11.57)
11.3. \textit{How the Fields Transform} \hfill 399

\begin{equation}
\vec{E}_0 = \frac{1}{4\pi \epsilon_0} \frac{\gamma q}{(\gamma^2 R_x^2 + R_y^2 + R_z^2)^{3/2}} \vec{R} \tag{11.58}
\end{equation}

if the angle between the velocity vector and vector $R$ is $\theta$, we may write

\begin{align*}
R_x &= R \cos(\theta), R_y = R \sin(\theta) \cos(\varphi) \hat{y}, R_z = R \sin(\theta) \sin(\varphi) \hat{z}, \\
\Rightarrow R_y^2 + R_z^2 &= R^2 \sin^2(\theta) \tag{11.59}
\end{align*}

so that for an observer in the $O'$ frame the electric field can be expressed as

\begin{equation}
\vec{E}_0 = \frac{1}{4\pi \epsilon_0} \frac{\gamma q}{(\gamma^2 R^2 \cos^2(\theta) + R^2 \sin^2(\theta))^{3/2}} \vec{R} \tag{11.60}
\end{equation}

\textit{Electric and magnetic field:}

(a) \textit{The transformation of the $E_y$ and $B_z$ components:} Let’s consider how the fields are transformed when both the electric and magnetic fields do exist. To see that we consider a third reference frame $\hat{O}$ in addition to the two reference frames $O$ and $O'$ as shown in the figure below.

For an observer on $O'$ frame in addition to the electric field

\begin{equation}
E_y = \frac{\sigma}{\epsilon_0} \tag{11.61}
\end{equation}

there is a magnetic field due to the surface current generated by the motion of the surface charges in the negative $x$ direction relative to observer $O'$. This current density on the two plates is given by which we determine using Ampere’s law given by

\begin{equation}
\vec{K}_\pm = \mp \sigma v_0 \hat{x} \tag{11.62}
\end{equation}

and the corresponding magnetic fields

\begin{equation}
\vec{B}_\pm = -\frac{1}{2} \mu_0 \sigma v_0 \hat{z}. \tag{11.63}
\end{equation}

Then the net magnetic field in between the plates

\begin{equation}
B_z = -\mu_0 \sigma v_0 \tag{11.64}
\end{equation}
The third frame $\tilde{O}$ is traveling with a velocity $\tilde{v}$ along the positive x-direction relative to the $O'$ frame. The velocity of this frame relative to $O$ frame is given by Lorentz transformation

$$\tilde{v} = \frac{v + v_0}{1 + (vv_0/c^2)}$$ (11.65)

Since the observer on this frame travels with a velocity $\tilde{v}$ in the positive x-direction the dimensions of the plates would be

$$\tilde{l} = l_0\sqrt{1 - \tilde{v}^2/c^2}, \tilde{w} = w_0, \tilde{d} = d_0$$ (11.66)

Which leads to a surface charge density

$$\tilde{\sigma} = \frac{q_0}{l\tilde{w}} \Rightarrow \tilde{\sigma} = \frac{\sigma_0}{\sqrt{1 - \tilde{v}^2/c^2}} = \tilde{\gamma}\sigma_0$$ (11.67)

where

$$\tilde{\gamma} = \frac{1}{\sqrt{1 - \tilde{v}^2/c^2}}$$ (11.68)

and for the electric and magnetic fields

$$\tilde{E}_y = \frac{\tilde{\sigma}}{\epsilon_0} = \frac{\tilde{\gamma}\sigma_0}{\epsilon_0}$$ (11.69)

$$\tilde{B}_z = -\mu_0\tilde{\sigma}\tilde{v} = -\tilde{\gamma}\mu_0\sigma_0\tilde{v}.$$ (11.70)

Or in terms of the surface charge density in frame $O'$

$$\sigma = \gamma_0\sigma_0$$ (11.71)

we can write

$$\tilde{E}_y = \left(\frac{\tilde{\gamma}}{\gamma_0}\right) \frac{\sigma}{\epsilon_0}, \tilde{B}_z = -\left(\frac{\tilde{\gamma}}{\gamma_0}\right) \mu_0\sigma\tilde{v}$$ (11.72)

Noting that

$$\frac{\tilde{\gamma}}{\gamma_0} = \frac{1}{\sqrt{1 - \tilde{v}^2/c^2}} = \sqrt{1 - \frac{v^2_0/c^2}{1 - \tilde{v}^2/c^2}}$$ (11.73)

Using

$$\tilde{v} = \frac{v + v_0}{1 + (vv_0/c^2)}$$ (11.74)

we may write

$$\tilde{\gamma} \gamma_0 = \sqrt{\frac{1 - v_0^2/c^2}{1 - (v + v_0)^2/(1 + vv_0/c^2)}} = \gamma \left(1 + \frac{vv_0}{c^2}\right)$$ (11.75)

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$ (11.76)
11.3. HOW THE FIELDS TRANSFORM

Using this result the electric field becomes

$$\vec{E}_y = \gamma \left( 1 + \frac{v\nu_0}{c^2} \right) \frac{\sigma}{\epsilon_0} = \gamma \left( \frac{\sigma}{\epsilon_0} - \frac{v}{\mu_0\epsilon_0 c^2} (\mu_0\sigma v_0) \right)$$  \hspace{1cm} (11.77)

In the $O'$ frame we know that the $E$ and $B$ fields are given by

$$E_y = \frac{\sigma}{\epsilon_0}, B_z = -\mu_0\sigma v_0.$$  \hspace{1cm} (11.78)

Using these results we may write

$$\vec{E}_y = \gamma \left( E_y - \frac{v}{c^2\epsilon_0 \mu_0} B_z \right) .$$  \hspace{1cm} (11.79)

Similarly for the magnetic field we find

$$\vec{B}_z = -\gamma \left( 1 + \frac{v\nu_0}{c^2} \right) \mu_0 \sigma \frac{v + \nu_0}{1 + (v\nu_0/c^2)} = \gamma \left( -\mu_0\sigma v_0 - \mu_0\epsilon_0 v \frac{\sigma}{\epsilon_0} \right)$$  \hspace{1cm} (11.80)

Using the $B$ and $E$ fields in the $O'$ frame

$$\vec{B}_z = \gamma (B_z - \epsilon_0 \mu_0 v E_y) .$$  \hspace{1cm} (11.81)

Or

$$\vec{E}_y = \gamma (E_y - v B_z) , \vec{B}_z = \gamma \left( B_z - \frac{v}{c^2} E_y \right) .$$  \hspace{1cm} (11.82)

(b) The transformation of the $E_z$ and $B_y$ components: To find out how the $E_z$ and $B_y$ transform we simply align the same capacitor parallel to the $x$-$y$ plane instead of the $x$-$z$ plane. As shown in the figure below:

For an observer on $O'$ frame the electric field

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z} .$$  \hspace{1cm} (11.83)

Which means

$$E_z = \frac{\sigma}{\epsilon_0} .$$  \hspace{1cm} (11.84)
where \( \sigma \) is the charge density observed by an \( O' \) which is related to the proper charge density by

\[
\sigma = \gamma_0 \sigma_0. \tag{11.85}
\]

There is a surface current generated by the motion of the surface charges on the plates relative to observer \( O' \). Relative to \( O' \) the plates are moving in the \(-x\) direction. Thus the current density on the two plates is can be expressed as

\[
\vec{J}_\perp = \mp \sigma \sigma_0 \hat{x} \tag{11.86}
\]

and the corresponding magnetic fields (in between the plates)

\[
\vec{B}_z = B_\perp = \frac{1}{2} \mu_0 \sigma \sigma_0 \hat{y}. \tag{11.87}
\]

Then the net magnetic field in between the plates

\[
\vec{B} = \vec{B}_z + \vec{B}_\perp = \mu_0 \sigma \sigma_0 \hat{y}. \tag{11.88}
\]

which means

\[
B_y = \mu_0 \sigma \sigma_0. \tag{11.89}
\]

The third frame \( O \) is traveling with a velocity \( \vec{v} \) along the positive \( x \)-direction relative to the \( O' \) frame. The velocity of this frame relative to \( O \) frame is given by Lorentz transformation

\[
\vec{v} = \frac{\vec{v} + \vec{v}_0}{1 + (\vec{v} \cdot \vec{v}_0/c^2)}. \tag{11.90}
\]

Since the observer on this frame travels with a velocity \( \vec{v} \) in the positive \( x \)-direction the dimensions of the plates would be

\[
\vec{l} = l_0 \sqrt{1 - \vec{v}^2/c^2}, \vec{w} = w_0, \vec{d} = d_0 \tag{11.91}
\]

Which leads to a surface charge density

\[
\vec{\sigma} = \frac{\sigma_0}{\vec{l} \vec{w}} = \frac{\sigma_0}{l_0 w_0 \sqrt{1 - \vec{v}^2/c^2}} = \frac{\sigma_0}{\sqrt{1 - \vec{v}^2/c^2}} = \gamma \sigma_0 \tag{11.92}
\]

where

\[
\gamma = \frac{1}{\sqrt{1 - \vec{v}^2/c^2}}, \tag{11.93}
\]

and for the electric and magnetic fields

\[
\vec{E}_z = \frac{\vec{\sigma}}{\epsilon_0} = \frac{\gamma \sigma_0}{\epsilon_0}, \vec{B}_y = \mu_0 \vec{\sigma} \vec{v} = \gamma \mu_0 \sigma_0 \vec{v}. \tag{11.94}
\]

Or in terms of the surface charge density in frame \( O' \) (\( \sigma \))

\[
\sigma = \gamma_0 \sigma_0 \Rightarrow \sigma_0 = \frac{\sigma}{\gamma_0} \tag{11.95}
\]
11.3. HOW THE FIELDS TRANSFORM

we may write

\[ \tilde{E}_z = \left( \frac{\tilde{\gamma}}{\gamma_0} \right) \frac{\sigma}{\epsilon_0} \tilde{B}_y = \left( \frac{\tilde{\gamma}}{\gamma_0} \right) \mu_0 \tilde{\sigma} \tilde{v} \]  

(11.96)

We recall

\[ \frac{\tilde{\gamma}}{\gamma_0} = \gamma \left( 1 + \frac{vv_0}{c^2} \right) \]  

(11.97)

so that

\[ \tilde{E}_z = \gamma \left( 1 + \frac{vv_0}{c^2} \right) \frac{\sigma}{\epsilon_0} = \gamma \left( \frac{\sigma}{\epsilon_0} + \frac{v}{\mu_0 \epsilon_0 c^2} (\mu_0 \sigma v_0) \right) \]  

(11.98)

In the \( O' \) frame we know that the \( E \) and \( B \) fields are given by

\[ E_z = \frac{\sigma}{\epsilon_0}, \quad B_y = \mu_0 \sigma v_0. \]  

(11.99)

using these results we may write

\[ \tilde{E}_z = \gamma \left( E_z + \frac{v}{c^2 \epsilon_0 \mu_0} B_y \right), \]  

(11.100)

Similarly for the magnetic field we find

\[ \tilde{B}_y = \gamma \left( 1 + \frac{vv_0}{c^2} \right) \mu_0 \sigma \left( \frac{v + v_0}{1 + (vv_0/c^2)} \right) = \gamma \left( \mu_0 \sigma v_0 + \mu_0 \epsilon_0 v \frac{\sigma}{\epsilon_0} \right) \]  

(11.101)

using the \( B \) and \( E \) fields in the \( O' \) frame

\[ \tilde{B}_y = \gamma (B_y + \epsilon_0 \mu_0 v E_z) \]  

(11.102)

Or

\[ \tilde{E}_z = \gamma (E_z + v B_y), \quad \tilde{B}_y = \gamma \left( B_y + \frac{v}{c^2} E_z \right), \]  

(11.103)

As for the \( x \) components

\[ \tilde{E}_x = E_x, \quad \tilde{B}_x = B_x \]  

(11.104)

To show for the magnetic field we consider a solenoid with length \( L \) and number of turns \( N \) (with \( n = N/L \) turns per unit length) carrying a current \( I = \frac{\Delta q}{\Delta t} \) as measured by an observer in \( O' \) frame. The magnetic field in this frame is given by

\[ B_x = \mu_0 \frac{N}{L} I = \mu_0 n I \]  

(11.105)

Now if we consider an observer in the \( O \) frame traveling with a velocity \( v \) relative to \( O' \) along the \( x \)-direction. For this observer length is contracted and time is dilated. This leads to

\[ \tilde{n} = \frac{N}{L} \left( \frac{N}{\sqrt{1 - v^2/c^2} L} \right) = \gamma n \]  

(11.106)

and the current

\[ \tilde{I} = \frac{\Delta q}{\Delta t'} = \frac{\Delta q}{\sqrt{1 - v^2/c^2}} = \sqrt{1 - v^2/c^2} \frac{\Delta q}{\Delta t} = \frac{1}{\gamma} I \]  

(11.107)
Then for the observer in the $\tilde{O}$ the magnetic field would be
\[
\tilde{B}_x = \mu_0 n \tilde{I} = \mu_0 \gamma n \frac{1}{\gamma} I = \mu_0 n I = B_x
\] (11.108)
which is the same.

11.4 The Field Tensor

The Galilean and Lorentz transformation: Consider two observers sitting at the origins of an $O$ frame and the other in the $O'$ frame with their clock synchronized initially (at time $t' = t = 0$) the two origins coincides.

Then the $O'$ frame begins to move along the positive $x$-direction with a velocity $v$. The observer in $O'$ observed an event at a position described by $(x', y', z')$ in his frame at a later time $t'$ measured by a clock in his pocket.

According to the Galilean transformation these measurements are related by
\[
x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t
\] (11.109)
11.4. THE FIELD TENSOR

Taking into account the fact that length contracted and time dilated, we may write
\[ x' = \frac{1}{\sqrt{1 - v^2/c^2}} (x - vt) \]
\[ y' = y, \quad z' = z, \quad t' = \frac{1}{\sqrt{1 - v^2/c^2}} \left(t - \frac{v}{c^2} x\right) \]  
(11.110)

these equations are called Lorentz transformation. To go from \( O' \) back to \( O \) we can do the algebra and show that
\[ x = \frac{1}{\sqrt{1 - v^2/c^2}} (x' + vt'), \quad y = y', \quad z = z', \quad t = \frac{1}{\sqrt{1 - v^2/c^2}} \left(t' + \frac{v}{c^2} x\right) \]  
(11.111)

The structure of space-time and Lorentz transformation matrix: replacing \((x, y, z)\) by \((x^1, x^2, x^3)\) and introducing a new length \(x^0 = ct\) which represents fourth dimension (time) but in a different way (length). It is measured in meters. For this dimension one meter corresponds to the time that light takes to travel one meter. Introducing the variables
\[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \beta = \frac{v}{c} \]  
(11.112)

the Lorentz transformation can be expressed as
\[ \bar{x}^0 = \gamma (x^0 - \beta x^1), \quad \bar{x}^1 = \gamma (x^1 - \beta x^0), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3, \]  
(11.113)

where we represent the primes by bars. These are "Four-vectors" which can be put using a four-by-four matrix as
\[
\begin{pmatrix}
\bar{x}^0 \\
\bar{x}^1 \\
\bar{x}^2 \\
\bar{x}^3
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}
\]  
(11.114)

Using Greek indices \((\mu, \nu)\) running from 0 to 3, this can be expressed as
\[ \bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda_\mu^\nu x^\nu = \Lambda_\mu^\nu x^\nu \]  
(11.115)

where \(\Lambda\) is the Lorentz transformation matrix. Using the convention that repeated indices represent summation usually this is written as
\[ \bar{x}^\mu = \Lambda_\mu^\nu x^\nu \]  
(11.116)

The Field transformation: In the previous lecture we have seen that the electric and magnetic field vectors as measured by an observer \( O \) (rest frame) \((E_x, E_y, E_z)\) and an observer \( \bar{O} \) in a frame moving with a velocity \( v \) in the positive x-direction \((\bar{E}_x, \bar{E}_y, \bar{E}_z)\) are transformed according to
\[ \bar{E}_x = E_x, \quad \bar{E}_y = \gamma (E_y - vB_z), \quad \bar{E}_z = \gamma (E_z + vB_y), \]  
(11.117)
\[ \vec{B}_x = B_x, \vec{B}_y = \gamma \left( B_y + \frac{v}{c^2} E_z \right), \vec{B}_z = \gamma \left( B_z - \frac{v}{c^2} E_y \right). \] (11.118)

This is a six component object, and it is represented by an \textit{antisymmetric, second-rank tensor}. A (second-rank) tensor is an object with two indices which transform with two factors of the Lorentz transformation matrix, \( \Lambda \) according to

\[ \bar{t}^{\mu \nu} = \sum \sum \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda \sigma} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda \sigma} \] (11.119)

where \( \Lambda^\mu_\mu \) is the entry in row \( \mu \), column \( \nu \). The Lorentz transformation matrix, if the \( \bar{O} \) frame is moving in the x-direction at speed \( v \), is given by

\[ \Lambda = \begin{pmatrix} \gamma & -\gamma \beta & 0 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (11.120)

Consider the 4-dimensional matrix

\[ t^{\mu \nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}. \] (11.121)

If this matrix is antisymmetric second-rank tensor it obeys the relation

\[ t^{\mu \nu} = -t^{\nu \mu} \] (11.122)

which leads to

\[ t^{\mu \nu} = \begin{pmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{pmatrix}. \] (11.123)

which has six distinct elements of the original 16. Then the antisymmetric field tensor in the \( O \) and \( \bar{O} \) frame can be expressed as

\[ F^{\mu \nu} = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & F^{13} \\ -F^{02} & -F^{12} & 0 & F^{23} \\ -F^{03} & -F^{13} & -F^{23} & 0 \end{pmatrix}. \] (11.124)

\[ \bar{F}^{\mu \nu} = \begin{pmatrix} 0 & \bar{F}^{01} & \bar{F}^{02} & \bar{F}^{03} \\ -\bar{F}^{01} & 0 & \bar{F}^{12} & \bar{F}^{13} \\ -\bar{F}^{02} & -\bar{F}^{12} & 0 & \bar{F}^{23} \\ -\bar{F}^{03} & -\bar{F}^{13} & -\bar{F}^{23} & 0 \end{pmatrix}. \] (11.125)

and the field transformation equation

\[ \bar{F}^{\mu \nu} = \sum_{\lambda=0}^{3} \sum_{\sigma=0}^{3} \Lambda^\mu_\lambda \Lambda^\nu_\sigma F^{\lambda \sigma} \] (11.126)
11.4. THE FIELD TENSOR

First let’s consider $\mu = 0, \nu = 1$. This leads to

$$\bar{F}^{01} = \sum_{\lambda=0}^{3} \sum_{\sigma=0}^{3} A^0_\lambda A^1_{\lambda \sigma} F^{\lambda \sigma} = \sum_{\lambda=0}^{3} A^0_\lambda \left( A^0_{\lambda 0} F^{\lambda 0} + A^1_{\lambda 1} F^{\lambda 1} + A^2_{\lambda 2} F^{\lambda 2} + A^3_{\lambda 3} F^{\lambda 3} \right)$$  \hspace{1cm} (11.127)

recalling that for $A^\mu_\nu$, $\mu$ is the row number and $\nu$ is column number, from the matrix

$$\Lambda = \begin{pmatrix} \gamma & -\gamma \beta & 0 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \hspace{1cm} (11.128)$$

we find

$$A^0_0 = -\gamma \beta, A^1_1 = \gamma, A^1_2 = 0, A^1_3 = 0.$$  \hspace{1cm} (11.129)

so that

$$\bar{F}^{01} = A^0_0 \left( -\gamma \beta F^{\lambda 0} + \gamma F^{\lambda 1} \right). \hspace{1cm} (11.130)$$

Expanding this, we have

$$\bar{F}^{01} = A^0_0 \left( -\gamma \beta F^{00} + \gamma F^{01} \right) + A^1_1 \left( -\gamma \beta F^{11} + \gamma F^{11} \right)$$

$$\quad + A^2_2 \left( -\gamma \beta F^{20} + \gamma F^{21} \right) + A^3_3 \left( -\gamma \beta F^{30} + \gamma F^{31} \right)$$  \hspace{1cm} (11.131)

so that using

$$A^0_0 = \gamma, A^1_1 = -\gamma \beta, A^2_2 = 0 + A^3_3 = 0, F^{00} = 0, F^{11} = 0, F^{10} = -F^{01} \hspace{1cm} (11.132)$$

we find

$$\bar{F}^{01} = \gamma^2 \left( 1 - \beta^2 \right) F^{01}$$  \hspace{1cm} (11.133)$$

Substituting

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \beta = \frac{v}{c}. \hspace{1cm} (11.134)$$

we get

$$\bar{F}^{01} = F^{01} \hspace{1cm} (11.135)$$

Following the same procedure we can show that

$$\bar{F}^{01} = F^{01}, \bar{F}^{02} = \gamma \left( F^{02} - \beta F^{12} \right), \bar{F}^{03} = \gamma \left( F^{03} - \beta F^{31} \right) \hspace{1cm} (11.136)$$

$$\bar{F}^{23} = F^{23}, \bar{F}^{31} = \gamma \left( F^{31} - \beta F^{03} \right), \bar{F}^{12} = \gamma \left( F^{12} - \beta F^{02} \right). \hspace{1cm} (11.137)$$

Comparison of

$$\bar{F}^{01} = F^{01}, \bar{F}^{02} = \gamma \left( F^{02} - \beta F^{12} \right), \bar{F}^{03} = \gamma \left( F^{03} - \beta F^{31} \right) \hspace{1cm} (11.138)$$

with

$$\bar{E}_x = E_x, \bar{E}_y = \gamma (E_y - v B_z), \bar{E}_z = \gamma (E_z + v B_y), \hspace{1cm} (11.139)$$
and
\[ F^{23} = F^{23}, F^{31} = \gamma (F^{31} - \beta F^{03}), F^{12} = \gamma (F^{12} - \beta F^{02}). \] (11.140)

with
\[ B_x = B_x, B_y = \gamma \left( B_y + \frac{v}{c^2} E_z \right), B_z = \gamma \left( B_z - \frac{v}{c^2} E_y \right). \] (11.141)

Gives the Field Matrix of the form
\[
F_{\mu\nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & B_y & -B_x & 0
\end{pmatrix}.
\] (11.142)

On the other hand if we compare
\[ F^{01} = F^{01}, F^{02} = \gamma (F^{02} - \beta F^{12}), F^{03} = \gamma (F^{03} - \beta F^{31}) \] (11.143)

with
\[ \bar{B}_x = B_x, \bar{B}_y = \gamma \left( B_y + \frac{v}{c^2} E_z \right), \bar{B}_z = \gamma \left( B_z - \frac{v}{c^2} E_y \right). \] (11.144)

and
\[ \bar{F}^{23} = F^{23}, \bar{F}^{31} = \gamma (F^{31} - \beta F^{03}), \bar{F}^{12} = \gamma (F^{12} - \beta F^{02}). \] (11.145)

with
\[ \bar{E}_x = E_x, \bar{E}_y = \gamma (E_y - v B_z), \bar{E}_z = \gamma (E_z + v B_y). \] (11.146)

we find a dual tensor \( G^{\mu\nu} \) given by
\[
G^{\mu\nu} = \begin{pmatrix}
0 & B_x & B_y & B_z \\
-B_x & 0 & -E_z/c & E_y/c \\
-B_y & E_z/c & 0 & E_x/c \\
-B_z & -E_y/c & E_x/c & 0
\end{pmatrix}.
\] (11.147)

this is called the dual tensor.

### 11.5 Electrodynamics in Tensor Notation

**Current Density 4-vector:** Consider a small charge \( Q \) in space traveling with a velocity \( v \). In the charge rest frame if the volume that this charge occupied is \( V_0 \), then the proper charge density can be expressed as
\[
\rho_0 = \frac{Q}{V_0}.
\] (11.148)
11.5. ELECTRODYNAMICS IN TENSOR NOTATION

However, the relativistic charge density, $\rho$, due to Lorentz contraction along the direction of the velocity, would be

$$\rho = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{Q}{V_0} = \frac{1}{\sqrt{1 - v^2/c^2}} \rho_0$$

and the relativistic current density

$$\vec{J} = \rho \vec{v} = \rho_0 \frac{\vec{v}}{\sqrt{1 - v^2/c^2}}.$$  \hfill (11.150)

The three components of the relativistic current density ($J_x, J_y, J_z$) along with the relativistic charge density multiplied by the speed of light ($cp$) makes the current density 4-vector which is represented as

$$J^\mu = (cp, J_x, J_y, J_z).$$ \hfill (11.151)

where

$$J^0 = cp, J^1 = J_x, J^2 = J_y, J^3 = J_z$$ \hfill (11.152)

**The Continuity Equation:** We recall the continuity equation which relates the charge density with the current density

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = -\frac{\partial (cp)}{\partial (ct)} \hfill (11.153)$$

$$\Rightarrow \frac{\partial (cp)}{\partial (ct)} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0 \hfill (11.154)$$

$$\sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu} = 0 \hfill (11.155)$$

or just simply

$$\frac{\partial J^\mu}{\partial x^\mu} = 0$$ \hfill (11.156)

This is the four-dimensional divergence of $J^\mu$.

**Maxwell’s equations:** We recall the four Maxwell’s equations

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho, \nabla \cdot \vec{B} = 0,$$ \hfill (11.157)

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$ \hfill (11.158)

All these equations can be written using the filed tensor and the current density 4-vector as

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \hfill (11.159)$$

where

$$J^\mu = (cp, J_x, J_y, J_z).$$ \hfill (11.160)
CHAPTER 11. RELATIVISTIC ELECTRODYNAMICS

\[
F^{\mu\nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & B_y & -B_x & 0
\end{pmatrix},
\]

(11.161)

and

\[
G^{\mu\nu} = \begin{pmatrix}
0 & B_x & B_y & B_z \\
-B_x & 0 & -E_z/c & E_y/c \\
-B_y & E_z/c & 0 & E_x/c \\
-B_z & -E_y/c & E_x/c & 0
\end{pmatrix}.
\]

(11.162)

A. Gauss’s law: Let \( \mu = 0 \), this gives

\[
\frac{\partial F^{0\nu}}{\partial x^\nu} = \mu_0 J^0 = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \mu_0 J^0
\]

(11.163)

Noting that

\[ J^0 = c\rho, F^{00} = 0, F^{01} = E_x/c, F^{02} = E_y/c, F^{03} = E_z/c \]

we find

\[
\frac{1}{c} \left[ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] = \mu_0 c^2 \rho
\]

(11.165)

\[
\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho
\]

(11.166)

B. Ampere’s law: For \( \mu = 1 \), have

\[
\frac{\partial F^{1\nu}}{\partial x^\nu} = \mu_0 J^1 = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \mu_0 J^1
\]

(11.167)

Noting that

\[ J^1 = J_x, F^{10} = -E_x/c, F^{11} = 0, F^{12} = B_z, F^{13} = -B_y \]

we find

\[
\left[ -\frac{1}{c^2} \frac{\partial E_x}{\partial t} + \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \right] = \mu_0 J_x
\]

\[
\Rightarrow \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \frac{1}{c^2} \frac{\partial E_x}{\partial t}
\]

(11.169)

For \( \mu = 2 \), we have

\[
\frac{\partial F^{2\nu}}{\partial x^\nu} = \mu_0 J^2 = \frac{\partial F^{20}}{\partial x^0} + \frac{\partial F^{21}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{23}}{\partial x^3} = \mu_0 J^2
\]

(11.170)

Noting that

\[ J^2 = J_y, F^{20} = -E_y/c, F^{21} = -B_z, F^{22} = 0, F^{23} = B_x \]

(11.171)
we find
\[-\frac{1}{c^2} \frac{\partial E_y}{\partial t} + \left( -\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} \right) = \mu_0 J_y \] (11.172)
\[\Rightarrow \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_y + \frac{1}{c^2} \frac{\partial E_y}{\partial t} \] (11.173)
We can similarly show that for \( \mu = 3 \)
\[\frac{\partial F^{3\nu}}{\partial x^\nu} = \mu_0 J^3 \Rightarrow \frac{\partial B_y}{\partial x} - \frac{\partial B_z}{\partial y} = \mu_0 J_z + \frac{1}{c^2} \frac{\partial E_z}{\partial t} \] (11.174)
gives and find Ampere’s law
\[\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \] (11.175)

C. The ”no-name law” \((\nabla \cdot \vec{B} = 0)\): To find this and Last (Faraday’s law) Maxwells equation we use the dual tensor. For \( \mu = 0 \), we have
\[\frac{\partial G^{0\nu}}{\partial x^\nu} = 0 \Rightarrow \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} = 0 \] (11.176)
Using
\[G^{00} = 0, G^{01} = B_x, G^{02} = B_y, G^{03} = B_z \] (11.177)
we find
\[\Rightarrow \frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} = 0 \Rightarrow \nabla \cdot \vec{B} = 0 \] (11.178)

D. Faraday’s law: for \( \mu = 1 \)
\[\frac{\partial G^{1\nu}}{\partial x^\nu} = 0 \Rightarrow \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = 0 \] (11.179)
Using
\[G^{10} = -B_x, G^{11} = 0, G^{12} = -E_z/c, G^{13} = E_y/c \] (11.180)
we find
\[\Rightarrow -\frac{1}{c} \frac{\partial B_x}{\partial t} - \left( \frac{1}{c} \frac{\partial E_z}{\partial y} - \frac{1}{c} \frac{\partial E_y}{\partial z} \right) = 0 \]
\[\Rightarrow -\frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = \left( \nabla \times \vec{E} \right)_x \] (11.181)
similarly for \( \mu = 2 \) and \( \mu = 3 \), we find
\[\Rightarrow -\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = \left( \nabla \times \vec{E} \right)_y \] (11.182)
and
\[\Rightarrow -\frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = \left( \nabla \times \vec{E} \right)_z \] (11.183)
so that we find Faraday’s law

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]  \hspace{1cm} (11.184)

The Lorentz force: in terms of the field tensor \( F^{\mu \nu} \) and the proper velocity \( \eta^\mu \), Minkowski force on a charge \( q \) is given by

\[ K^\mu = q \eta^\mu F^{\mu \nu} \]  \hspace{1cm} (11.185)

where

\[ \eta^0 = \frac{c}{\sqrt{1 - v^2/c^2}}, \eta^1 = \frac{v_x}{\sqrt{1 - v^2/c^2}}, \eta^2 = \frac{v_y}{\sqrt{1 - v^2/c^2}}, \eta^3 = \frac{v_z}{\sqrt{1 - v^2/c^2}}, \]

for \( \mu = 1 \), we have

\[ K^1 = q \eta^0 F^{1 \nu} = q (-\eta^0) F^{10} + q \eta^1 F^{11} + q \eta^2 F^{12} + q \eta^3 F^{13} \]  \hspace{1cm} (11.187)

so that using

\[ F^{10} = -E_x/c, F^{11} = 0, F^{12} = B_z, F^{13} = -B_y \]  \hspace{1cm} (11.188)

we find

\[ K^1 = \frac{q E_x}{\sqrt{1 - v^2/c^2}} + \frac{q v_y B_z}{\sqrt{1 - v^2/c^2}} - \frac{q v_z B_y}{\sqrt{1 - v^2/c^2}} \]  \hspace{1cm} (11.189)

\[ K^1 = \frac{q}{\sqrt{1 - v^2/c^2}} [E_x + (v_y B_z - v_z B_y)] \]  \hspace{1cm} (11.190)

\[ K^1 = \frac{q}{\sqrt{1 - v^2/c^2}} [E_x + (\vec{v} \times \vec{B})_z] \]  \hspace{1cm} (11.191)

similarly for \( \mu = 2 \) and \( \mu = 3 \), one can easily find

\[ K^2 = \frac{q}{\sqrt{1 - v^2/c^2}} [E_y + (\vec{v} \times \vec{B})_y] \]  \hspace{1cm} (11.192)

and

\[ K^3 = \frac{q}{\sqrt{1 - v^2/c^2}} [E_z + (\vec{v} \times \vec{B})_z] \]  \hspace{1cm} (11.193)

Thus we can write

\[ \vec{F} = q \left[ \vec{E} + (\vec{v} \times \vec{B}) \right] \]  \hspace{1cm} (11.194)

Referring to how the force is transformed in relativistic electrodynamics, we can easily note that

\[ \vec{F}^* = q \left[ \vec{E} + (\vec{v} \times \vec{B}) \right] \]
is the Lorentz force.

Relativistic Potentials: the scalar and vector potential together form a 4-vector represented by $A^\mu$

$$A^\mu = (V/c, A_x, A_y, A_z) \quad (11.195)$$

in terms of 4-vector potential the field tensor can be written as

$$F^{\mu \nu} = \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \quad (11.196)$$

Note that the differentiation is with respect to the covariant vector $x^\mu$ and $x^\nu$. This change the sign of the zeroth component, $x^0 = -c^0$. Applying this relation the field equations

$$\frac{\partial F^{\mu \nu}}{\partial x^\nu} = \mu_0 J^\mu \quad (11.197)$$

become

$$\frac{\partial}{\partial x^\nu} \left[ \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \right] = \mu_0 J^\mu \quad (11.198)$$

$$\frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\nu}{\partial x^\nu} \right) - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\nu} \right) = \mu_0 J^\mu \quad (11.199)$$

Using Lorentz gauge

$$\nabla \cdot \vec{A} = \frac{1}{c^2} \frac{\partial V}{\partial t} \Rightarrow \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = -\frac{\partial (V/c)}{\partial (ct)} \quad (11.200)$$

$$\Rightarrow \frac{\partial (V/c)}{\partial (ct)} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 \quad (11.201)$$

$$\frac{\partial A^\mu}{\partial x^\mu} = 0 \text{ or } \frac{\partial A^\nu}{\partial x^\nu} = 0 \quad (11.202)$$

we may write the field equation

$$\frac{\partial}{\partial x^\mu} \left( \frac{\partial A^\nu}{\partial x^\nu} \right) - \frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\nu} \right) = \mu_0 J^\mu \quad (11.203)$$

as

$$\frac{\partial}{\partial x^\nu} \left( \frac{\partial A^\mu}{\partial x^\nu} \right) = -\mu_0 J^\mu \quad (11.204)$$

or

$$\Box^2 A^\mu = -\mu_0 J^\mu \quad (11.205)$$

where

$$\Box^2 = \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \quad (11.206)$$

is the d’Albertian.