Theoretical physics IV- *Introduction to General Relativity* 

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Contents

Introduction ix

1 The spacetime of special relativity 1
   1.1 Inertial frames, Galilean and Lorentz transformations ........... 1
   1.2 Axes rotation vs Lorentz transformation .......................... 6
   1.3 The interval and the lightcone ................................. 10
   1.4 Length contraction and time dilation ............................ 14
   1.5 Invariant hyperbolae ............................................. 16
   1.6 Worldline and proper time ........................................ 18
   1.7 The Doppler effect ................................................. 22
   1.8 Velocity and acceleration ........................................ 25

2 Manifolds 33
   2.1 What is a Manifold? ............................................... 33
   2.2 Curves and surfaces in a Manifold ............................... 35
   2.3 Coordinate transformations and summation convention ............ 38
   2.4 The Riemannian geometry ........................................... 42
   2.5 Intrinsic and extrinsic geometry and the metric ..................... 45
   2.6 Length, area, and volume .......................................... 52
   2.7 Local Cartesian coordinates and tangent space ...................... 60
   2.8 The signature of a manifold ....................................... 65

3 Vector Calculus on manifolds 69
   3.1 The tangent vector ................................................. 69
   3.2 The basis vectors ................................................... 70
   3.3 The metric function and coordinate transformations ............... 71
      3.3.1 Raising and lowering vector indices ........................... 77
   3.4 The inner product and null vectors ................................ 78
   3.5 The affine connections ............................................. 79
   3.6 Local geodesic and Cartesian coordinates .......................... 87
   3.7 The gradient, the divergence, the curl on a manifold ............... 89
   3.8 Intrinsic derivative of a vector along a curve .................... 92
   3.9 Parallel transport ................................................... 93
   3.10 Null curves, non-null curves, and affine parameter ............... 94


3.11 The calculus of variation-(a review from Theoretical Physics I) 96
3.11.1 Geodesic and stationary points 96
3.11.2 The geodesic in Euclidean space 96
3.11.3 The general problem 99
3.12 The geodesic on a manifold 103
3.13 Stationary property of the non-null geodesic 104

4 Tensor Calculus on manifolds 109
4.1 Tensors fields and rank of a tensor 109
4.2 The metric tensor revisited 114
4.3 Mapping tensors into tensors 115
4.4 Elementary tensor operations 116
4.5 Tensors and coordinate transformations 118
4.6 Tensor equations and the quotient theorem 119
4.7 Covariant derivatives of a tensor 120
4.8 Intrinsic derivative 122

5 Special relativity using tensors 125
5.1 The Minkowski spacetime in Cartesian coordinates 125
5.1.1 The metric tensor and the Affine connection 125
5.1.2 The Lorentz transformation 126
5.1.3 Four vector and Lorentz transformation 129
5.2 The four-momentum of a particle 132
5.3 Four momentum of a photon and the Doppler effect 134
5.4 Relativistic mechanics for a massive particle 139
5.5 Relativistic collision and Compton scattering 142
5.6 Accelerating observers and the tetrads 144

6 Electromagnetism 147
6.1 The Lorentz force 147
6.2 The charge and the current density 149
6.3 The electromagnetic field equations 151
6.4 Electromagnetism in the Lorentz gauge 154
6.5 Electromagnetism in arbitrary coordinates 158
6.6 Equation of motion for a charged particle 160

7 The equivalence principle and spacetime curvature 163
7.1 Newtonian gravity and the equivalence principle 163
7.2 Gravity as spacetime curvature and local Cartesian coordinates 164
7.3 Observers in a curved spacetime 167
7.4 Weak gravitational fields and the Newtonian limit 168
7.5 Electromagnetism in curved spacetime 173
7.6 The curvature tensor 174
7.7 The Einstein Tensor 179
7.8 Curvature and parallel transport 180
8 The gravitational field equations 187
  8.1 The energy-momentum tensor 187
  8.2 A perfect fluid 189
  8.3 Conservation of energy and momentum 191
  8.4 Classical limit 193
  8.5 The Einstein equations 195
    8.5.1 The Einstein field equations in vacuum 198
    8.5.2 The Einstein field equations in the weak-field limit 199
Preface

This material is my lecture note for General Relativity course (PHYS 4800) at Middle Tennessee State University.
Introduction
Chapter 1

The spacetime of special relativity

In this chapter we review the basic notions underlying in the Newtonian and special relativistic viewpoints of space and time.

1.1 Inertial frames, Galilean and Lorentz transformations

If we want to describe an event that occurred somewhere on this planet, we must be able to tell exactly where (space) it occurred and when (time). This means we must be able to tell the time at which the event occurred and the exact position of this event in this space (three dimensional space) as measured from some origin, $O$, on some reference frame $S$ or from another origin on another reference frame $S'$. The reference frame $S'$ could be in motion relative to $S$ in an arbitrary manner. These reference frames could be our home, our car that we are driving, a spacecraft traveling in the deep outer space, or the international space station orbiting our planet. Suppose an event in space described by the coordinates $(x, y, z)$ is recorded at time $t$ by an observer at $O$ on reference frame $S$, as shown in Fig. 1.1. Using "four dimensional space", we may express this event as $(t, x, y, z)$. Let’s say this same event for an observer $O'$ on reference frame $S$ is recorded at a different time $t'$ at a different point in space $(t', x', y', z')$.

Now the question is how these two observation are related to one another if one of the reference frame is moving with some velocity $v$. Well, before we relate $(t', x', y', z')$ and $(t, x, y, z)$, it is very important to know about the reference frames. Here we will consider only, what is know as, inertial reference frames. Inertial reference frames are reference frames moving with a constant velocity (constant magnitude and direction) with respect to one another. These are none
2

CHAPTER 1. THE SPACETIME OF SPECIAL RELATIVITY

Figure 1.1: Two inertial reference frames. Inertial frame $S'$ is moving with a constant velocity along the positive x relative to inertial frame $S$.

accelerating frame of references

\[
\frac{d^2 X}{dt^2} = \frac{d^2 Y}{dt^2} = \frac{d^2 Z}{dt^2} = 0
\]

and

\[
\frac{d^2 X'}{dt'^2} = \frac{d^2 Y'}{dt'^2} = \frac{d^2 Z'}{dt'^2} = 0.
\]

Consider the inertial reference frame $S'$ in Fig (1.1). It is moving with a constant velocity $v$ in the positive $x$ direction

\[
\vec{v} = v\hat{x}
\]

relative to $O$. Obviously, reference $S$ is moving with the same magnitude of velocity but in the opposite direction for an observer $O'$

\[
\vec{v} = -v\hat{x}
\]

Taking this velocity into account, we can at least expect the $x$ position of the event for an observer $O'$ ($x'$) would depend on both $x$ and $t$. We make a linear relationship assumption

\[
x' = Dt + Ex
\]

Let’s make a similar assumption for time $t'$ too, why not?

\[
t' = At + Bx
\]

For $y'$ and $z'$, obviously

\[
y' = y, z' = z.
\]
1.1. INERTIAL FRAMES, GALILEAN AND LORENTZ TRANSFORMATIONS

Since we require that for \( x' = 0 \),

\[
x = vt
\]
we have

\[
x' = Dt + Ex \Rightarrow 0 = Dt + Evt \Rightarrow D = -Ev \quad (1.7)
\]

For \( x = 0 \), we also require

\[
x' = -vt'
\]
which leads to

\[
x' = Dt + Ex \Rightarrow -vt' = Dt \Rightarrow D = -v \frac{t'}{t} \quad (1.8)
\]

Substituting \( x = 0 \) into

\[
t' = At + Bx
\]
we find

\[
t' = At \quad (1.10)
\]
so that

\[
D = -v \frac{t'}{t} = -Av. \quad (1.11)
\]

From Eqs. (1.7) and (1.11) we see that

\[
E = A, D = -Av
\]
so that an inertial reference frames, we may write

\[
t' = At + Bx, \quad (1.12)
\]
\[
x' = A(x - vt), \quad (1.13)
\]
\[
y' = y, \quad (1.14)
\]
\[
z' = z, \quad (1.15)
\]

*Galilean Transformation*: in the Galilean transformation time is absolute that mean \( t \) is independent of the reference frame. Thus we have

\[
A = 1, B = 0
\]

which leads to

\[
t' = t, \quad (1.16)
\]
\[
x' = x - vt, \quad (1.17)
\]
\[
y' = y, \quad (1.18)
\]
\[
z' = z. \quad (1.19)
\]

If we take the time derivative of the \( x \) coordinate, we find

\[
\frac{dx'}{dt'} = \frac{dx}{dt} - v \Rightarrow u_x' = u_x - v \quad (1.20)
\]
which is an equation that can be obtained from a common sense of relative velocities. If you differentiate the velocity,
\[
\frac{du_x'}{dt'} = \frac{du_x}{dt}
\]
(1.21)
since the velocity of the reference frames is constant. This means the acceleration is the same in both \(S\) and \(S'\).

**Homework Problem 1:** You heard in the News that there are two events happened somewhere in this planet. Suppose event one occurred at time \(t_1\) at a point in space \((x_1, y_1, z_1)\), which we may describe using spacetime coordinates \((t_1, x_1, y_1, z_1)\), as recorded by an observer on an inertial reference frame \(S\). The second event occurred at a later time \(t_2\) at another point in space \((x_2, y_2, z_2)\) as recorded by the same observer. Show that the time difference
\[
\Delta t = t_2 - t_1
\]
(1.22)
and the quantity
\[
(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2
\]
(1.23)
are, separately, invariant under any Galilean transformation. Note that
\[
\Delta x = x_2 - x_1, \Delta y = y_2 - y_1, \Delta z = z_2 - z_1
\]
(1.24)
You must show that
\[
\Delta t' = \Delta t, (\Delta r')^2 = (\Delta r)^2
\]
Lorentz transformation: In the special theory of relativity Einstein abandoned the postulate of an absolute time and replaced it by the postulate that the speed of light \(c\) is the same in all inertial frame. Next we will derive the Lorentz transformation using Einstein postulate about the speed of light. To this end, we may rewrite Eqs. (1.12)-(1.15)
\[
ct' = c(At + Bx),
\]
(1.25)
\[
x' = A(x - vt),
\]
(1.26)
\[
y' = y,
\]
(1.27)
\[
z' = z.
\]
(1.28)
Suppose an observer one on an inertial frame \(S\) measured the distance traveled a photon to be \(\Delta r\) over a time interval \(\Delta t\). Another observer another inertial frame, \(S'\), measured the distance and time interval to be \(\Delta r'\) and \(\Delta t'\), respectively. Thus according to Einstein’s postulate, the speed of this photon is the same. Assume that the measurement took place in vacuum, one can then write
\[
\frac{\Delta r}{\Delta t} = c = \frac{\Delta r'}{\Delta t'}
\]
There follows that
\[
(c\Delta t)^2 = (\Delta r)^2 \Rightarrow (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0
\]
(1.29)
\[
(c\Delta t')^2 = (\Delta r')^2 \Rightarrow (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = 0
\]
(1.30)
1.1. INERTIAL FRAMES, GALILEAN AND LORENTZ TRANSFORMATIONS

Using Eqs. (1.25)-(1.28), we may write

\[
(c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2
= c^2 (A\Delta t + B\Delta x)^2 - A^2 (\Delta x - v\Delta t)^2 - (\Delta y')^2 - (\Delta z')^2
= c^2 [A^2 (\Delta t)^2 + B^2 (\Delta x)^2 + 2AB (\Delta t) (\Delta x)]
- A^2 [(\Delta x)^2 + v^2 (\Delta t)^2 - 2v (\Delta x) (\Delta t)] - (\Delta y')^2 - (\Delta z')^2
= c^2 A^2 (\Delta t)^2 + c^2 B^2 (\Delta x)^2 + 2c^2 AB (\Delta t) (\Delta x) - A^2 (\Delta x)^2
- A^2 v^2 (\Delta t)^2 + 2A^2 v (\Delta x) (\Delta t) - (\Delta y')^2 - (\Delta z')^2
\]

(1.31)

\[
\Rightarrow (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2
+ 2A [c^2 B + Av] (\Delta x) (\Delta t) = 0
\]

(1.32)

Now referring to Eq. (1.29), we have

\[
(c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2
= (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2
\]

so that using the result in Eq. (1.32)

\[
c^2 A^2 - A^2 v^2 = c^2 A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma
\]

(1.33)

\[
2A [c^2 B + Av] = 0 \Rightarrow c^2 B + Av = 0 \Rightarrow B = -\frac{v}{c^2} A = -\gamma \frac{v}{c}
\]

(1.34)

where

\[
\beta = \frac{v}{c},
\]

(1.35)

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}.
\]

(1.36)

Substituting Eqs. (1.33) and (1.34) into Eqs. 1.25) and (1.26), the Lorentz transformation is given by

\[
ct' = c (At + Bx) = c \left( \gamma t - \frac{\gamma \beta}{c} x \right) \Rightarrow ct' = \gamma (ct - \beta x),
\]

(1.37)

\[
x' = \gamma (x - vt) \Rightarrow x' = \gamma (x - c\beta t),
\]

(1.38)

\[
y' = y,
\]

(1.39)

\[
z' = z.
\]

(1.40)

The Lorentz transformation is also known as the boost in the \(x\)-direction. For the case \(v << c\), we have

\[
\beta \approx 0, \ \gamma \approx 1
\]
the Lorentz transformation reduces to
\begin{align}
  t' &= t, \\
  x' &= x - vt, \\
  y' &= y, \\
  z' &= z.
\end{align}
(1.41) (1.42) (1.43) (1.44)
which agrees with the Galilean transformation.

**Homework Problem 2**: Consider the two events in problem 1 described by the spacetime coordinates \((t_1, x_1, y_1, z_1)\) and \((t_2, x_2, y_2, z_2)\). Show that the interval between these two events squared which is defined by
\[
  (\Delta s)^2 = (ct_1 - ct_2)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2
\]
(1.45)
is invariant under the Lorentz transformation squared.

The interval in Eq. (1.45) shows that space and time are united in a four dimensional continuum called spacetime. The geometry which is characterized by Eq. (1.45) is known as *Minkowski geometry*. The spacetime of special relativity is not Euclidean because of the minus sign. It is often called pseudo Euclidean.

**Note**: The inverse transformation whether it is Galilean or Lorentz, the equations are given by replacing \(v\) by \(-v\).
\begin{align}
  ct &= \gamma (ct' + \beta x'), \\
  x &= \gamma (x' + c\beta y'), \\
  y &= y', \\
  z &= z'.
\end{align}
(1.46) (1.47) (1.48) (1.49)
Eqs. (1.46)- (1.49) are Lorentz transformation from the \(S'\) frame to \(S\).

### 1.2 Axes rotation vs Lorentz transformation

Let’s consider the geometry that we are familiar with the *Euclidean geometry* in Cartesian coordinates. We call this coordinate system \(S\). In this coordinate system a point can be described by the coordinates \((x, y, z)\). Suppose we rotate the x-y-z coordinate system about the z axis, as shown in Fig. 1.2, by an angle \(\theta\) in a counterclockwise direction to form another coordinate system (reference frame) \(S'\). In this rotated coordinate system the same point is described by \((x', y', z')\). Let’s omit for the z-coordinate and find the \(x'\) and \(y'\) in terms of \(x\) and \(y\). From Fig. 1.2 [*Theoretical Physics I*], one can easily show that
\begin{align}
  y' &= -x \sin (\theta) + y \cos (\theta), \\
  x' &= x \cos (\theta) + y \sin (\theta),
\end{align}
(1.50) (1.51)
or in a Matrix form
\[
  \begin{bmatrix}
    y' \\
    x'
  \end{bmatrix} = \begin{bmatrix}
    \cos (\theta) & -\sin (\theta) \\
    \sin (\theta) & \cos (\theta)
  \end{bmatrix} \begin{bmatrix}
    y \\
    x
  \end{bmatrix}.
\]
(1.52)
1.2. AXES ROTATION VS LORENTZ TRANSFORMATION

Figure 1.2: A Cartesian coordinate system rotated about the z-axis in a counterclockwise direction by an angle $\theta$. We named the y-axis "ct".

Let’s replace $y'$ by $ct'$ and $y$ by $ct$ so that

$$
\begin{bmatrix}
ct' \\
x'
\end{bmatrix} = \begin{bmatrix}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{bmatrix} \begin{bmatrix}
ct \\
x
\end{bmatrix}.
$$

(1.53)

We recall the "boost" (Lorentz transformation) from Eqs. (1.37)- (1.38),

$$
\begin{align*}
ct' &= \gamma (ct - \beta x), \\
x' &= \gamma (x - c\beta t), \\
y' &= y, \\
z' &= z.
\end{align*}
$$

(1.54)\quad(1.55)\quad(1.56)\quad(1.57)

that can be put, using matrices, in the form

$$
\begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
\gamma & -\gamma\beta & 0 & 0 \\
-\gamma\beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix}.
$$

(1.58)

Let’s omit the $y$ and $z$ coordinate, we can write the Lorentz transformation as

$$
\begin{bmatrix}
ct' \\
x'
\end{bmatrix} = \begin{bmatrix}
\gamma & -\gamma\beta \\
-\gamma\beta & \gamma
\end{bmatrix} \begin{bmatrix}
ct \\
x
\end{bmatrix}.
$$

(1.59)

At least comparing the diagonal elements in Eqs. (1.53) and (1.59), one may be tempted to say, $\gamma = \cos (\psi)$, $\psi$ is some angle of rotation. However, since

$$
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \begin{cases} 
1, & v = 0 \\
\infty, & v = c
\end{cases}
$$
trigonometric functions can not be an option to consider. But we know that the hyperbolic function
\[ \cosh (\psi) = \frac{e^\psi + e^{-\psi}}{2} = \begin{cases} 1, & \psi = 0 \\ \infty, & \psi \to \infty \end{cases} \] (1.60)

So let’s say that
\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \cosh (\psi) \] (1.61)

so that one finds
\[ 1 - \frac{v^2}{c^2} = 1 - \frac{1}{\cosh^2 (\psi)} \Rightarrow \beta = \frac{v}{c} = \sqrt{\frac{\cosh^2 (\psi) - 1}{\cosh^2 (\psi)}} = \tanh (\psi) \]
\[ \Rightarrow \psi = \tanh^{-1} (\beta). \]

The parameter \( \psi \) is called the rapidity parameter. Note that for
\[ \beta = \frac{v}{c} = \begin{cases} 1, & v = c \\ 0, & v = 0 \end{cases} \] (1.62)

we find
\[ \psi = \tanh^{-1} (\beta) = \begin{cases} \infty, & \beta = 1 \\ 0, & \beta = 0 \end{cases} \] (1.63)

Now using
\[ \gamma = \cosh (\psi), \gamma \beta = \sinh (\psi), \] (1.64)

we can express the "Boost" as
\[
\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh (\psi) & -\sinh (\psi) \\ -\sinh (\psi) & \cosh (\psi) \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}. \] (1.65)

Now comparing Eqs. (1.52) and (1.65), we can see that the boost has essentially the same structure as coordinate-axis rotation except the trigonometric functions are replaced by hyperbolic function. Therefore, the physics of relativity (for an inertial frames of reference) is a coordinate transformation in the Minkowski spacetime! We can easily see this by considering the \( S \) and \( S' \) frame of references in the standard configuration (i.e. \( S' \) is moving with a constant velocity \( v \) in the positive \( x \)-direction). Let’s omit the \( y \) and \( z \) coordinates so that an event can be described by the coordinates \((ct, x)\) in an inertial frame \( S \) and by coordinates \((ct', x')\) in the \( S' \) inertial frame. In Fig. 1.3 \( S' \) frame is constructed by rotating \( ct \) axis in a clockwise direction and the \( x \) axis in a counterclockwise direction by an angle \( \psi \), which is given by Eq. (1.63), as shown in Fig. ??.

Now if we decompose \( ct \) and \( x \) into components along \( ct' \) and \( x' \), we can easily see that
\[
ct' = ct \cosh (\psi) - x \sinh (\psi),
\]
\[
x' = -ct \sinh (\psi) + x \cosh (\psi). 
]
1.2. AXES ROTATION VS LORENTZ TRANSFORMATION

Note that the decomposition of the vectors form a parallelogram not a rectangle like the Cartesian coordinate axes rotation. Now using $y' = y$ and $z' = z$ for the boost, we have

\[
\begin{align*}
ct' &= ct \cosh(\psi) - x \sinh(\psi), \\
x' &= -ct \sinh(\psi) + x \cosh(\psi), \\
y' &= y, \\
z' &= z.
\end{align*}
\]

or using matrices

\[
\begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix} =
\begin{bmatrix}
\cosh(\psi) & -\sinh(\psi) & 0 & 0 \\
-\sinh(\psi) & \cosh(\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix},
\]

which is the Lorentz transformation.

Though generally, the inertial reference frame, $S'$ moves with a constant velocity, $\vec{v}$, in an arbitrary direction and the origin could be off with respect to the origin of the inertial reference frame $S$ at the initial time $t = t' = 0$, and also the axes could be rotated (see Fig. 1.4), we will consider only the "boost" (the $S'$ reference frame in Fig. 1.1). This is because the displacement and the rotation does not introduce any new physics. It is the boost only, which is also known as standard configuration, that brings a new physics due to the relative motion. Now let’s look how we can get the inertial reference frame $S'$ from $S$ by decomposing the transformation. First translate the origin $O$ by $\vec{R}$. This makes the origin of $S$ and $S'$ coincide. Rotate the x-axis by $\alpha$. This lines up the velocity $\vec{v}$ with the x-axis. A final rotation lines up the inertial frame $S'$ with $S$. That makes the $S'$ the standard configuration shown in Fig. 1.1.
Homework Problem 3: Using the Lorentz transformation
\[
\begin{bmatrix}
ct' \\
x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
cosh(\psi) & -\sinh(\psi) & 0 & 0 \\
-\sinh(\psi) & \cosh(\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
c t \\
x \\
y \\
z
\end{bmatrix}.
\]
(1.67)

Show that the interval squared between the two events in problem 1 is invariant.

1.3 The interval and the lightcone

We have proved that the interval
\[
(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2
\]
(1.68)
in the Minkowski spacetime is invariant under the boost. From the expression in Eq. (1.68), we can also see that it could be positive, negative or zero. This sign which is defined as
\[
\begin{cases}
(\Delta s)^2 > 0 & \text{timelike} \\
(\Delta s)^2 = 0 & \text{lightlike} \\
(\Delta s)^2 < 0 & \text{spacelike}
\end{cases}
\]

Timelike: we can find an inertial frame in which the two events occur at the same spacial coordinate.

Spacelike: we can find an inertial reference frame on which the two events occur at the same time coordinate.

Before we see what is the lightcone is, let’s consider a cone in the Euclidean geometry shown in Fig.1.5. A point on the surface of a cone with coordinates
1.3. THE INTERVAL AND THE LIGHTCONE

Figure 1.5: A cone of radius $R$ and height $H$ with its apex centered at the origin.

$(x, y, z)$ in cylindrical coordinates is defined by

$$x = R \frac{z}{H} \cos(\varphi), \quad y = R \frac{z}{H} \sin(\varphi),$$

where $H$ is the height of a cone with radius $R$ with apex centered at the origin $(0,0,0)$. The opening angle $\vartheta$ of a right cone is the vertex angle made by a cross section through the apex and center of the base

$$\vartheta = 2 \tan^{-1} \left( \frac{R}{H} \right).$$

The equation of a cone can then be expressed as

$$x^2 + y^2 = \left( \frac{R}{H} \right)^2 z^2.$$  \hfill (1.71)

For a cone with apex centered at $(x_1, y_1, z_1)$ instead of $(0,0,0)$ as shown in Fig. 1.6, one can write

$$x - x_1 = R \frac{z - z_1}{H} \cos(\varphi), \quad y - y_1 = R \frac{z - z_1}{H} \sin(\varphi),$$

or

$$\Delta x = R \frac{\Delta z}{H} \cos(\varphi), \quad \Delta y = R \frac{\Delta z}{H} \sin(\varphi),$$

where

$$\Delta x = x - x_1, \quad \Delta y = y - y_1, \quad \Delta z = z - z_1.$$

The equation of the cone becomes

$$(\Delta x)^2 + (\Delta y)^2 = (\Delta z)^2 \left( \frac{R}{H} \right)^2.$$  \hfill (1.74)
CHAPTER 1. THE SPACETIME OF SPECIAL RELATIVITY

Figure 1.6: A cone of radius \( R \) and height \( H \) with its apex centered at \((x_1, y_1, z_1)\).

Introducing the constant
\[
c = \frac{R}{H}
\]
we may write
\[
\frac{(\Delta x)^2 + (\Delta y)^2}{c^2} = (\Delta z)^2
\]

Now let’s consider the Minkowski spacetime with axes \((x, y, ct)\) with the \(z\)-axis omitted. Then the interval
\[
(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2
\]
becomes
\[
(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2
\]
For a lightlike \((\Delta s)^2 = 0\), we find
\[
\frac{(\Delta x)^2 + (\Delta y)^2}{c^2} = (\Delta t)^2
\]
which is equation of a cone and it is known as the lightcone. We note that
\[
(c\Delta t) - (\Delta x)^2 - (\Delta y)^2 = \begin{cases} 
(\Delta s)^2 > 0, & \text{Outside (timelike)} \\
(\Delta s)^2 = 0, & \text{On the lightlike (lightlike)} \\
(\Delta s)^2 < 0, & \text{Inside (spacelike)}
\end{cases}
\]

Now let’s omit both \(y\) and \(z\) axis and consider the four different events, \(\text{Event 1, Event 2, Event 3, and Event 4}\) described by two coordinates \((ct_1, x_1), (ct_2, x_2), (ct_3, x_3),\) and \((ct_4, x_4)\)The physical interpretation of spacelike \((\Delta s)^2 < 0\) and timelike \((\Delta s)^2 > 0\) is what we stated earlier. When the interval is Timelike,
1.3. THE INTERVAL AND THE LIGHTCONE

Figure 1.7: Spacetime diagram illustrating the lightcone of Event 1 in relation of three other events occurred inside, on, and outside the cone.

Figure 1.8: Spacetime diagram
we can find an inertial frame in which the two events occur on the same spatial coordinate; and when it is Spacelike, we can find an inertial reference frame on which the two events occur at the same time coordinate. In order to understand this let’s consider the boost \( S' \) with both the \( y' \) and \( z' \) coordinates omitted. In Fig. 1.8 we can see that in the reference frame \( S' \) Event 1 and Event 2 which are timelike and occurred at different time and different place in reference frame, we can see that on the boost frame both have the same spatial coordinates. This meant we found an inertial reference frame (the boost) on which the two events occurred at the same point in space. On the other hand Event 1 and Event 4 which are spacelike events on the \( S \) inertial reference frame, on the \( S' \) frame both events have the same time coordinates. This means we are able to find an inertial reference frame on which Event 1 and Event 4 occurred at the same time.

1.4 Length contraction and time dilation

Let’s assume that the boost \( S' \) be the Orion spacecraft traveling in the deep outer space with a constant velocity, \( \vec{v} \), away from our planet (Inertial reference frame \( S \)) as shown in Fig. 1.9. Suppose this spacecraft carries a couple leaving this planet behind. The man is standing on a box while the woman is on the floor inside the spacecraft as shown in Fig. 1.9. The direction of the velocity is in the positive x-direction as shown in Fig. 1.9. The woman measures the man’s height (proper length) and found it to be \( l_0 \), that we may express it as

\[
l_0 = x'_2 - x'_1, \tag{1.80}
\]
where \( x'_1 \) and \( x'_2 \) are the x-coordinates of his feet and his head as measured by the woman. Using the Lorentz transformation

\[
x' = \gamma (x - c\beta t),
\]

(1.81)

we have

\[
l_0 = x'_2 - x'_1 = \gamma (x_2 - c\beta t_2) - \gamma (x_1 - c\beta t_1) \\
\Rightarrow l_0 = \gamma [(x_2 - x_1) - c\beta (t_2 - t_1)]
\]

(1.82)

Suppose the woman took the measurement for \( x_2 \) and \( x_1 \) at the same time, we have

\[ t_1 = t_2 = t \]

so that one can write the proper length (height) of the man as

\[
l_0 = \gamma (x_2 - x_1) = \gamma l \Rightarrow l = \frac{l_0}{\gamma} = l_0 \sqrt{1 - \frac{v^2}{c^2}},
\]

(1.83)

where

\[ l = x_2 - x_1. \]

is the height of the man measured by an observer on the earth (the \( S \) inertial frame). Obviously, the observer on earth measures the Man’s contracted height (length) by a factor of \( 1/\gamma \).

In their journey in the deep space suppose they got pregnant and the man had to deliver their baby. Their is a clock aboard on the spacecraft and the man measured the time interval between the beginning of the labor and the arrival of their baby. Let this time be \( T_0 = \Delta t' = t'_2 - t'_1 \), where \( t'_1 \) is the time at which the labor began (Event 1) and \( t'_2 \) (Event 2) is the time at which the baby arrived as recorded. Similarly, an observer on earth (frame \( S \)) recorded the time of these two events \( t_2 \) and \( t_1 \), respectively, using his own Clock. Using the inverse Lorentz transformation for time, where \( v \) is replaced by \(-v\) in Eq. (1.46), one can write

\[
t_1 = \gamma \left( t'_1 + vx'_1 \right), t_2 = \gamma \left( t'_2 + vx'_2 \right),
\]

(1.84)

so that for, \( T = \Delta t = t_2 - t_1 \), one finds

\[
T = \gamma \left[ (t'_2 - t'_1) + v (x'_2 - x'_1) \right],
\]

(1.85)

Suppose the observer on \( S' \) recorded the two events at the same place

\[
x'_2 = x'_1 = x,
\]

(1.86)

then using

\[
T_0 = \Delta t' = t'_2 - t'_1.
\]
we find

\[ T = \gamma T_0 = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow T_0 = \sqrt{1 - \frac{v^2}{c^2}} T \] (1.87)

The result in Eq. (1.87) shows that the moving clock (on the \( S' \) frame) ticks more slowly by a factor of \( \sqrt{1 - \frac{v^2}{c^2}} \) than the clock on a rest frame (the \( S \) frame).

**Homework Problem 4**: Suppose the couples on the spacecraft celebrated their child (a girl) sweet sixteen birthday as measured by a clock on board the spacecraft (\( S' \)). The girl is about \( 1.6 \text{m} \) tall as measured by her parents. Assume the spacecraft is traveling with constant velocity, \( v = 0.8c \), where \( c \) is the speed of light in vacuum.

(a) What would be the age of the girl as measured by an observer on earth (\( S \) inertial frame).

(b) How tall is the girl as measured by an observer on earth (\( S \) inertial frame).

### 1.5 Invariant hyperbolae

Let’s consider the interval in the Minkowski spacetime

\[(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2.\] (1.88)

For \((\Delta y)^2 = (\Delta z)^2 = 0\), we can write

\[(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2.\] (1.89)

in the \( S \) frame and also

\[(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2.\] (1.90)

in the \( S' \) frame as the interval is invariant. We can calibrate the length scale for

![Figure 1.10: The invariant hyperbolae.](image-url)
1.5. INVARIANT HYPERBOLAE

Figure 1.11: Length contraction and time dilation.

The graph for these equations is shown in Fig. 1.10 (using Mathematica). Let the point described by the coordinates \((ct = 0, x = 0)\) be \(O\), \((ct = 0, x = 1)\) be \(A\), and \((ct = 1, x = 0)\) be \(B\) respectively. \(OA\) represents a unit length and \(OB\) a unit time on the \(S\) frame (Fig. 1.11). We know that the interval is invariant in the Minkowski spacetime. This means if we rotate the \(ct\) axis clockwise and \(x\) axis counterclockwise by \(\psi = \sinh^{-1}\left[\frac{v}{c}\right]\), we find the boost (the \(S'\) inertial reference frame) with coordinates \(ct'\) and \(x'\) on which the interval remains invariant

\[
(c \Delta t')^2 - (\Delta x')^2 = \pm 1 \quad (1.92)
\]

We can easily see the length contraction and time dilation from this invariance and Fig. 1.11.

Consider a proper length on the \(S'\) inertial reference frame which is \(OC\). Suppose \(OC\) is a meterstick \((OC = l_0 = 1m)\) which is the length measured by
an observer on the $S'$ frame). For an observer on the $S$ frame, the length $l$ is given by

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} = 1m \sqrt{1 - \frac{v^2}{c^2}} < 1m$$

Now draw a line of constant $x'$ through point $C$ that indicates a unit length on the $S'$ ($x' = 1$). As we can see from the diagram in Fig. 1.11, this line intersects with the $x$-axis on the $S$ inertial reference frame at a point where $x < 1$. This confirms that the length of the meterstick is less than a meter for an observer on the $S$ frame.

Now let’s consider the time dilation. We recall that

$$T_0 = \sqrt{1 - \frac{v^2}{c^2}} T$$

where $T$ is the time interval between two events as measured by a clock on the $S$ frame and $T_0$ is the time interval between these two same events as measured by a clock on the $S'$ frame. Suppose we consider two events happen at $x' = 0$ on the $S'$ frame (event $O$ and event $D$). Suppose the time interval between these two events as measured by a clock on the $S'$ is 1 sec, i.e. $T_0 = 1s$

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1s}{\sqrt{1 - \frac{v^2}{c^2}}} > 1s$$

As one can clearly see from Fig. 1.11, Point D on the hyperbolae has a time coordinate (in the $S$ frame) that is greater than one, $T > 1$.

1.6 Worldline and proper time

Consider two events (Event 1 and Event 2). The line joining these two events in the Minkowski spacetime is known as the worldline. It could be a straight or wiggly line. If these two events are separated by an infinitesimally interval, $ds$, and the two events are described by $(t, x, y, z)$ and $(t + dt, x + dx, y + dy, z + dz)$, we can write

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$  \hspace{1cm} (1.93)

The invariant interval between Event 1 and Event 2 along an arbitrary path (straight or wiggly) is given by

$$\Delta s = \int_{1}^{2} ds = \int_{1}^{2} \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}.$$  \hspace{1cm} (1.94)

From our discussion in the previous sections, we recall

$$\begin{align*}
\left(\Delta s\right)^2 & > 0, \quad \text{timelike} \\
\left(\Delta s\right)^2 & = 0, \quad \text{lightlike} \\
\left(\Delta s\right)^2 & < 0, \quad \text{spacelike}
\end{align*}$$  \hspace{1cm} (1.95)
and the invariant interval in the Minkowski spacetime would become

$$\Delta s = \int_1^2 ds = \begin{cases} 
\text{real for timelike} \\
0 \text{ for lightlike} \\
\text{imaginary for spacelike}
\end{cases}$$

The fact that we get an imaginary for $\Delta s$ in the case of spacelike, it means

![Figure 1.12: The worldline of a photon (solid line) and a massive particle (broken line).](image)

the worldline for a particle must lie within the lightcone as shown in Fig. 1.12. This is also required by relativistic mechanics as it prohibits the acceleration of a massive particle to speeds greater than or equal to the speed of light $c$. Let’s

![Figure 1.13: An alien spaceship traveling with a speed $v$ along the positive x-direction relative to the earth.](image)

consider the inertial reference frame at rest, $S$, be our planet earth. Suppose a man saw an alien spaceship passing by him with a velocity, $v$, in the positive $x$-direction, as shown in Fig. 1.13. The man saw the alien’s spaceship at a position
(x, y, z) at time t near the earth. In the Minkowski spacetime we describe this even by (t, x, y, z). The worldline of the alien spaceship can be described by a curved line (inside a cone). Assuming the velocity of the alien’s spaceship, v, changes magnitude and direction along the x-axis, the worldline for the alien could be the wiggly line shown in Fig. 1.14 passing through a light cones in the Minkowski spacetime. This wiggly worldline of the alien is defined by the interval, ds,

$$\begin{align*}
\text{ds}^2 &= c^2 \text{dt}^2 - \text{dx}^2, \\
\text{that leads to a function, } \text{x(t)}, \text{ that depends on time, } t. \text{ Another convenient alternative way of expressing the worldline is to use some invariantly defined time parameter, } \tau, \text{ that monotonically changes over the alien (or the particle) world-} \\
\text{line. We can then use this parameter to define the worldline using, } (t(\tau), x(\tau)). \text{ This parameter, } \tau, \text{ is known as the proper time. Generally, it is defined as}
\end{align*}$$

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.\tag{1.98}$$

For dy = dz = 0, we have

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 \Rightarrow d\tau^2 = dt^2 \left(1 - \frac{v^2}{c^2}\right).$$
so that the proper time becomes

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt = dt/\gamma_v. \tag{1.99}$$

Let’s consider two events, Event 1 and Event 2, associated to the alien in the spaceship that are recorded by the same observer on earth. Let Event 1 be when the observer spot the alien for the first time, \((t, x, y, z)\). Event 2 be, after a few seconds, when the alien began waving and dancing as he is flying away from the observer (without abducting him...), \((t + \Delta t, x + \Delta x, y, z)\), as shown in Fig. 1.15. The proper time between these two events is given by

\[
\Delta \tau = \int_t^{t+\Delta t} \sqrt{1 - \frac{v^2}{c^2}} dt = \Delta \tau = \sqrt{1 - \frac{v^2}{c^2}} \Delta t. \tag{1.100}
\]

Note that here the alien’s spaceship is considered to have a constant velocity relative to the observer on earth.

Let’s imagine that the alien turned the spaceship engine off between this two events so that, \(v = 0\). The proper time becomes

$$\Delta \tau = \Delta t, \tag{1.101}$$

which is the same as the time interval between these two events as measured by the observer clock on earth (i.e. the frame \(S\), that is at rest). It means the proper time, \(\tau\), is just the time coordinate recorded by clocks at rest. Generally, if at any instant in the history of the particle (or the alien in this case), we introduce an instantaneous rest frame \(S'\) such that the particle is momentarily at rest in the \(S'\), then the proper time \(\tau\) is simply the time recorded by the clock that moves along with the particle (or in this case the alien spacecraft). Therefore, the proper time, \(\tau\), is an invariantly defined quantity.
1.7 The Doppler effect

Let’s reconsider the alien in his spaceship and the observer on earth. At the rear end of the spaceship there is a high power laser (not harmful). After the alien spaceship receded millions miles away from the earth towards his home planet, the alien turned this laser directed at the observer on earth (the $S$ frame that is at rest). The spaceship is still moving with a velocity $v$ in the positive $x$-direction relative to the observer on earth (see Fig. 1.16). Let’s consider four events describing a photon emitted from the alien spaceship recorded by the observer on earth. These events described two successive wavecrests of the photon. The two successive wavecrests, when the photon is emitted are referred as Event 1 (E1) and Event 2 (E2) and recorded as $(t_1, x_1)$ and $(t_2, x_2)$, respectively, by the observer on earth. When these wavecrests received by the observer on earth are referred as Event 3 (E3) and Event 4 (E4) and recorded as $(t_3, x)$ and $(t_4, x)$, respectively (See Fig 1.17). Note that the $x$-coordinates for E3 and E4 are the same as the man received the photon at the same position. From Eq. (1.100), the proper time for E1 and E2 as measured by an observer on $S'$ (the alien) moving with a speed $v$ is given by

$$
\Delta \tau_{21} = \sqrt{1 - \frac{v^2}{c^2}} (t_2 - t_1) = \sqrt{1 - \frac{v^2}{c^2}} \Delta t_{21}. 
$$}

(1.102)

On the other hand the proper time for an observer on the $S$ frame (the man on earth) for E3 and E4, since, $v = 0$, Eq. (1.100) gives

$$
\Delta \tau_{43} = t_4 - t_3 = \Delta t_{43}. 
$$}

(1.103)
Along the worldline joining $E_1$ to $E_3$ and $E_2$ to $E_4$, (see Fig. 1.18) the interval is zero since we are considering a photon (Lightlike, $ds = 0$)

$$c^2 ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0. \quad (1.104)$$

For $dy = dz = 0$, we have

$$c^2 dt^2 - dx^2 = 0. \quad (1.105)$$

so that for the worldline joining $E_1$ and $E_3$, we can write

$$\int_{t_1}^{t_2} c dt = - \int_{t_1}^{t_3} dx \Rightarrow \int_{t_1}^{t_2} c dt = - \int_{x_1}^{x_2} dx. \quad (1.106)$$

The minus sign, as it can be seen from Fig. 1.17, is because of the photon is traveling in the negative direction. Similarly, for the worldline connecting $E_2$ and $E_4$, we can write

$$\int_{t_2}^{t_4} c dt = - \int_{t_2}^{t_4} dx \Rightarrow \int_{t_2}^{t_4} c dt = - \int_{x_3}^{x_4} dx. \quad (1.107)$$

Noting that

$$\int_{t_2}^{t_4} c dt = \int_{t_2}^{t_1} c dt + \int_{t_1}^{t_3} c dt + \int_{t_3}^{t_4} c dt = - \int_{t_1}^{t_2} c dt + \int_{t_2}^{t_3} c dt + \int_{t_3}^{t_4} c dt$$

$$= -c \Delta t_{21} + c \Delta t_{43} + \int_{t_1}^{t_3} c dt$$
and using Eq. (1.106), one finds
\[
\int_{t_2}^{t_4} c dt = -c \Delta t_{21} + c \Delta t_{43} - \int_{x_1}^{x} dx.
\]  
(1.108)

Substituting this into Eq. (1.107), we have
\[
-c \Delta t_{21} + c \Delta t_{43} - \int_{x_1}^{x} dx = - \int_{x_2}^{x} dx,
\]  
(1.109)

so that using
\[
\int_{x_2}^{x} dx = \int_{x_2}^{x_1} dx + \int_{x_1}^{x} dx.
\]  
(1.110)

one finds
\[
-c \Delta t_{21} + c \Delta t_{43} - \int_{x_1}^{x} dx = - \int_{x_2}^{x_1} dx - \int_{x_1}^{x} dx
\]
\[
\Rightarrow c(\Delta t_{43} - \Delta t_{21}) = - \int_{x_2}^{x_1} dx = x_2 - x_1 = \Delta x_{21}
\]  
(1.111)

There follows that
\[
\Delta t_{43} = \left(1 + \frac{1}{c} \frac{\Delta x_{21}}{\Delta t_{21}}\right) \Delta t_{21} = \left(1 + \frac{v}{c}\right) \Delta t_{21}.
\]  
(1.112)
1.8. VELOCITY AND ACCELERATION

Using Eqs. (1.102) and (1.103), we have

\[ \frac{\Delta \tau_{43}}{\Delta \tau_{21}} = \frac{\Delta t_{43}}{\sqrt{1 - \frac{v^2}{c^2}} \Delta t_{21}} \]

so that upon substituting Eq. (1.112), one finds

\[ \frac{\Delta \tau_{43}}{\Delta \tau_{21}} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1 + \frac{v}{c}}{\sqrt{(1 + \frac{v}{c})(1 - \frac{v}{c})}}. \]  

(1.113)

Thus the time interval between the two successive wavecrests as observed by an observer on earth, \( S \) rest frame (\( \Delta \tau_{43} \)) and the alien on the spaceship, \( S' \) moving with speed \( v \), (\( \Delta \tau_{21} \)) are related by

\[ \frac{\Delta \tau_{43}}{\Delta \tau_{21}} = \frac{1 + \frac{v}{c}}{\sqrt{(1 + \frac{v}{c})(1 - \frac{v}{c})}} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}. \]  

(1.114)

Note that this times are the period of the emitted photon that can be related to the corresponding frequencies for an observer on \( S \) (\( f \)) and on \( S' \) (\( f' \)) by

\[ \Delta \tau_{43} = 1/f, \Delta \tau_{21} = 1/f' \]

we find

\[ \frac{f}{f'} = \frac{1 - \frac{v}{c}}{\sqrt{1 + \frac{v}{c}}}. \]  

(1.115)

This is **Doppler-effect formula**.

1.8 Velocity and acceleration

**Velocity**: Let’s consider two spacecraft commanded by two aliens as shown in Fig.1.19. The first spaceship commanded by Alien 1 is spotted by an observer on earth (\( S \) frame) and Alien 2 commanding the second spaceship is flying with a constant speed \( v \) along the positive x-direction (\( S' \) frame). For Alien 1 spaceship the observer on earth recorded \((t, x(t), y(t), z(t))\) which Alien 2 in the second spaceship recorded \((t', x(t'), y(t'), z(t'))\). The velocity for the first spaceship (Alien 1) as recorded by an observer on earth, \((u_x, u_y, u_z)\), is determined by taking the time derivative of the position coordinates.

\[ u_x(t) = \frac{dx(t)}{dt}, u_y(t) = \frac{dy(t)}{dt}, u_z(t) = \frac{dz(t)}{dt}. \]  

(1.116)

Using the inverse Lorentz transformation

\[ ct = \gamma (ct' + \beta x'), x' = \gamma (x' + c\beta t'), y = y', z = z', \]  

(1.117)
we can write

\[
\gamma (c dt' + \beta dx') = \gamma \left( c + \beta \frac{dx'}{dt'} \right) dt' = \gamma \left( c + \beta u_x' \right) dt',
\]

\[
\Rightarrow dt = \gamma \left( 1 + \frac{vu_x'}{c^2} \right) dt',
\]  

(1.118)

\[
dx = \gamma (dx' + c\beta dt') = \gamma \left( \frac{dx'}{dt'} + c\beta \right) dt' = \gamma (u_x' + v) dt',
\]

(1.119)

\[
dy = dy', dz = dz',
\]

(1.120)

so that

\[
\frac{dx}{cdt} = \frac{\gamma (dx' + c\beta dt')}{\gamma (cdt' + \beta dx')},
\]

(1.121)

\[
\frac{dy}{cdt} = \frac{dy'}{\gamma (cdt' + \beta dx')},
\]

(1.122)

\[
\frac{dz}{cdt} = \frac{dz}{\gamma (cdt' + \beta dx')}.
\]

(1.123)
This can be rewritten as

\[
\begin{align*}
  u_x &= \frac{dx}{dt} = \frac{\gamma (u'_x + v)}{\gamma \left(1 + \frac{v u'_x}{c^2}\right)} \frac{dt'}{dt}, \\
  u_y &= \frac{dy}{dt} = \frac{u'_y \sqrt{1 - \frac{v^2}{c^2}}}{\gamma \left(1 + \frac{v u'_x}{c^2}\right)} \frac{dt'}{dt}, \\
  u_z &= \frac{dz}{dt} = \frac{u'_z \sqrt{1 - \frac{v^2}{c^2}}}{\gamma \left(1 + \frac{v u'_x}{c^2}\right)} \frac{dt'}{dt},
\end{align*}
\]

(1.124)  

(1.125)  

(1.126)

The inverse transformation for the velocity, replacing \( v \) by \( -v \), is given by

\[
\begin{align*}
  u'_x &= \frac{u_x - v}{1 - \frac{v^2}{c^2}}, \\
  u'_y &= \frac{u_y \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}}, \\
  u'_z &= \frac{u_z \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}}.
\end{align*}
\]

(1.127)

**Homework Problem 5:** Consider three inertial reference frames \( S, S', \) and \( S'' \). Suppose \( S' \) is related to \( S \) by a boost of speed \( v \) in the \( x \)-direction and that \( S'' \) is related to \( S' \) by a boost of speed \( u' \) in the \( x' \)-direction. Using the rapidity parameter defined as

\[
\psi_v = \tanh^{-1} \left( \frac{v}{c} \right), \quad \psi_{u'} = \tanh^{-1} \left( \frac{u'}{c} \right),
\]

(1.128)

show that

**(a)**

\[
\begin{align*}
  ct'' &= ct \cosh (\psi_v + \psi_{u'}) - x \sinh (\psi_v + \psi_{u'}), \\
  x' &= -ct \sinh (\psi_v + \psi_{u'}) + x \cosh (\psi_v + \psi_{u'}), \\
  y' &= y, \\
  z' &= z.
\end{align*}
\]

**(b)**

\[
\begin{align*}
  u &= ct \tanh (\psi_v + \psi_{u'}) = \frac{\tanh (\psi_v) + \tanh (\psi_{u'})}{1 + \tanh (\psi_v) \tanh (\psi_{u'})} = \frac{u' + v}{1 + u'v/c^2}.
\end{align*}
\]

**Acceleration:** The acceleration of Alien 1 (spaceship 1) as observed by Alien 2 in the spaceship \( (S') \) frame can also be determined from the corresponding velocities

\[
\begin{align*}
  a'_x &= \frac{du'_x}{dt'}, \\
  a'_y &= \frac{du'_y}{dt'}, \\
  a'_z &= \frac{du'_z}{dt'}.
\end{align*}
\]

(1.129)
Using the transformation for velocity

\[
\begin{align*}
  u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}}, \\
  u'_y &= \frac{u_y \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}}, \\
  u'_z &= \frac{u_z \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}},
\end{align*}
\]

(1.130)

we have

\[
\begin{align*}
  du'_x &= d \left[ \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \right] = \frac{du_x}{1 - \frac{vu_x}{c^2}} + \frac{du_x \frac{v}{c^2}}{(1 - \frac{vu_x}{c^2})^2} = \frac{du_x (1 - \frac{v^2}{c^2})}{(1 - \frac{vu_x}{c^2})^2}, \\
  \Rightarrow du'_x &= \frac{du_x}{\gamma_v^2 (1 - \frac{vu_x}{c^2})^2},
\end{align*}
\]

(1.131)

\[
\begin{align*}
  du'_y &= d \left[ \frac{u_y \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}} \right] = \frac{1}{\gamma_v} \left[ \frac{du_y}{1 - \frac{vu_x}{c^2}} + \frac{u_y \frac{v}{c^2} du_x}{(1 - \frac{vu_x}{c^2})^2} \right],
\end{align*}
\]

(1.132)

\[
\begin{align*}
  du'_z &= d \left[ \frac{u_z \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}} \right] = \frac{1}{\gamma_v} \left[ \frac{du_z}{1 - \frac{vu_x}{c^2}} + \frac{u_z \frac{v}{c^2} du_x}{(1 - \frac{vu_x}{c^2})^2} \right],
\end{align*}
\]

(1.133)

where

\[ \gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \]

(1.134)

Similarly, using the transformation for time

\[
\begin{align*}
  ct' &= \gamma_v (ct - \beta x) \Rightarrow dt' = \gamma_v (dt - \frac{v}{c^2} dx) = \gamma_v \left( 1 - \frac{v}{c^2} \right) \frac{dx}{dt} = \gamma_v \left( 1 - \frac{v}{c^2} \right) dt
\end{align*}
\]

(1.135)

\[
\Rightarrow dt' = \gamma_v \left( 1 - \frac{vu_x}{c^2} \right) dt
\]

(1.136)

then the acceleration becomes

\[
\begin{align*}
  a'_x &= \frac{du'_x}{dt'} = \frac{du_x}{\gamma_v^2 (1 - \frac{vu_x}{c^2})^2} \left[ \gamma_v \left( 1 - \frac{vu_x}{c^2} \right) dt \right] = \frac{a_x}{\gamma_v^3 (1 - \frac{vu_x}{c^2})^3},
\end{align*}
\]

(1.137)

\[
\begin{align*}
  a'_y &= \frac{du'_y}{dt'} = \frac{1}{\gamma_v} \left[ \frac{du_y}{1 - \frac{vu_x}{c^2}} + \frac{u_y \frac{v}{c^2} du_x}{(1 - \frac{vu_x}{c^2})^2} \right] \left[ \gamma_v \left( 1 - \frac{vu_x}{c^2} \right) dt \right] \\
  \Rightarrow a'_y &= \frac{1}{\gamma_v^2 (1 - \frac{vu_x}{c^2})^2} a_y + \frac{u_y v}{\gamma_v^2 c^2 (1 - \frac{vu_x}{c^2})^3} a_x
\end{align*}
\]

(1.138)

\[
\begin{align*}
  a'_z &= \frac{du'_z}{dt'} = \frac{1}{\gamma_v} \left[ \frac{du_z}{1 - \frac{vu_x}{c^2}} + \frac{u_z \frac{v}{c^2} du_x}{(1 - \frac{vu_x}{c^2})^2} \right] \left[ \gamma_v \left( 1 - \frac{vu_x}{c^2} \right) dt \right] \\
  \Rightarrow a'_z &= \frac{1}{\gamma_v^2 (1 - \frac{vu_x}{c^2})^2} a_z + \frac{u_z v}{\gamma_v^2 c^2 (1 - \frac{vu_x}{c^2})^3} a_x
\end{align*}
\]

(1.139)
For the first spaceship let’s consider the case

\[ u_x = u(t), u_y = u_z = 0 \Rightarrow a_x = a = \frac{du}{dt}, a_y = a_z = 0, \quad (1.140) \]

so that its velocity and acceleration as observed by the alien in the second spaceship be

\[ u'_x = \frac{u - v}{1 - \frac{cu}{c^2}u_x}, u'_y = 0, u'_z = 0 \]
\[ (1.141) \]

and

\[ a'_x = a' = \frac{a}{\gamma^3 \left(1 - \frac{cu}{c^2}\right)^3}, a'_y = 0, a'_z = 0, \]
\[ (1.142) \]

respectively. Suppose Alien 1 in the first spaceship makes continuous record of his accelerometer reading, \( f(\tau) \), (the rate of change of his velocity per unit time—the proper time, \( \tau \)). Alien 2 in the second spaceship (\( S' \) frame) that was traveling with a constant velocity, \( v \), relative to the observer on earth (\( S \) frame) begin to accelerate such that it would catch up with Alien 1, \( v = u(t) \). Then the speed of the first spaceship (Alien 1) relative to the second spaceship (Alien 2) becomes

\[ u'_x = \frac{u - v}{1 - \frac{cu}{c^2}u_x} = 0. \]
\[ (1.143) \]
Under this condition the second spaceship ($S'_0$ frame) is referred as us an *instantaneous reference frame* (IRF). From what we know about the proper time, we note that

$$\delta \tau = \delta \tau'.$$  \hfill (1.144)

Then the acceleration of the first spaceship as read by Alien 1 from his accelerometer, $f(\tau)$, and measured by Alien 2 ($S'$ frame), $a'_x$, are related by

$$a' = \frac{du'}{dt'} = \frac{du'}{d\tau} = f(\tau).$$  \hfill (1.145)

For $v = u(t)$, we have

$$a' = \frac{a}{\gamma_v^3 \left(1 - \frac{u^2}{c^2}\right)^3} = \frac{a \left(1 - \frac{u^2}{c^2}\right)^{3/2}}{\left(1 - \frac{u^2}{c^2}\right)^3} \Rightarrow a = a' \left(1 - \frac{u^2}{c^2}\right)^{3/2}$$

$$\Rightarrow \frac{du}{dt} = \frac{du'}{dt'} \left(1 - \frac{u^2}{c^2}\right)^{3/2},$$ \hfill (1.146)

so that using Eq. (1.145), one finds

$$\frac{du}{dt} = \left(1 - \frac{u^2}{c^2}\right)^{3/2} f(\tau)$$ \hfill (1.147)

Noting that the proper time on the first spaceship (Alien 1)

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}} dt,$$ \hfill (1.148)

we may rewrite Eq. (1.147) as

$$\frac{du}{d\tau} = \left(1 - \frac{u^2}{c^2}\right)^{3/2} f(\tau).$$ \hfill (1.149)

Introducing the transformation defined by

$$u(\tau) = c \tanh[\psi(\tau)]$$ \hfill (1.150)

we have

$$\frac{cd\tau}{d\tau} \tanh[\psi(\tau)] = (1 - \tanh^2[\psi(\tau)]) f(\tau)$$

$$\Rightarrow \left[ \frac{c}{\cosh^2[\psi(\tau)]} \frac{d\psi}{d\tau} \right] = \frac{f(\tau)}{\cosh^2[\psi(\tau)]} \Rightarrow c \frac{d\psi}{d\tau} = f(\tau)$$

$$\Rightarrow \psi(\tau) = \frac{1}{c} \int_{\tau_0}^{\tau} f(\xi) d\xi.$$ \hfill (1.151)

From Eqs. (1.148) and (1.150), we note that

$$\frac{dt}{d\tau} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = \cosh[\psi(\tau)], \quad \frac{dx}{d\tau} = c \tanh[\psi(\tau)].$$ \hfill (1.152)
Suppose the first spaceship is accelerating at a constant rate (i.e. $f(\xi) = k$), so that

$$\psi(\tau) = \frac{1}{c} \int_0^\tau f(\xi) \, d\xi = \frac{k\tau}{c},$$  \hspace{1cm} (1.153)

we find for the time and position the first spaceship as function of the proper time

$$\frac{dt}{d\tau} = \cosh \left( \frac{k\tau}{c} \right) \Rightarrow t(\tau) = t_0 + \frac{c}{k} \sinh \left( \frac{k\tau}{c} \right),$$  \hspace{1cm} (1.154)

$$\frac{dx}{d\tau} = \frac{dx}{d\tau} \frac{dt}{d\tau} = u(\tau) \frac{dt}{d\tau} = c \sinh \left( \frac{k\tau}{c} \right) \Rightarrow x(\tau) = x_0 + \frac{c}{k} \left[ \cosh \left( \frac{k\tau}{c} \right) \right],$$  \hspace{1cm} (1.155)

For $t_0 = x_0 = 0$, these equations can be rewritten as

$$\left( \frac{kt(\tau)}{c} \right) = \sinh \left( \frac{k\tau}{c} \right), \quad \frac{kx(\tau)}{c} = \cosh \left( \frac{k\tau}{c} \right) - 1.$$  \hspace{1cm} (1.156)

We recall the interval is given by

$$(c\Delta t)^2 - (\Delta x)^2 = (\Delta s)^2 \Rightarrow \left( \frac{c\Delta t}{\Delta s} \right)^2 = \left( \frac{\Delta x}{\Delta s} \right)^2 = 1.$$  \hspace{1cm} (1.157)

which we may write in terms of the proper time, $\tau$, as

$$\left( \frac{c\Delta t}{\Delta s} \right)^2 = 1 + \left( \frac{k\tau}{c} \right)^2 \Rightarrow \left( \frac{\Delta x}{\Delta s} \right)^2 = \left( \frac{k\tau}{c} \right)^2.$$  

The parametric graph for the worldlines, for Alien 1 in the first spaceship that is uniformly accelerating, $f(\tau) = k$, (the red curve) and for the observer on earth at, $x = 0$, that is at rest on the $S$ frame (green). The blue line is the event horizon for Alien 1. Events that are beyond the event horizon will never be seen by Alien 1.
Figure 1.21: The worldline for the accelerating spaceship (red) and the observer on earth (green). The dotted blue line is the horizon for the accelerating spaceship.

Figure 1.22: The light cones for the accelerating spaceship (red) and the observer on earth (green). The dotted blue line is the horizon for the accelerating spaceship.
2.1 What is a Manifold?

Consider a ridged meterstick pinned at the north and south poles inside a hollow sphere as shown in Fig. 2.1. Initially, $t = 0$, the sphere is at rest and suddenly begins to rotate. It is free to rotate about the x-axis, y-axis, or z-axis. Let’s say we want to describe the angular position of the center of mass of the meter stick over a period of time, $\tau = 10s$, with a time interval of $2s$. How many independent parameters, that depend on time, do we need to describe the angular position of the center of mass of the meterstick relative to its initial position at $t = 0$? Well the answer is simple. We need three independent parameters, the Euler axial.
angles, \((\alpha^1(t), \alpha^2(t), \alpha^3(t))\) which describes the rotation about the \(x, y,\) and \(z\) axes at a given instant of time. Then over the 10 second interval we have a set that consist of 5 points

\[
\{ [\alpha^1(2), \alpha^2(2), \alpha^3(2)], [\alpha^1(4), \alpha^2(4), \alpha^3(4)], [\alpha^1(6), \alpha^2(6), \alpha^3(6)], \\
\alpha^1(8), \alpha^2(8), \alpha^3(8)], [\alpha^1(10), \alpha^2(10), \alpha^3(10)] \}
\]

We can make the time interval infinitesimal to continuously describe the angular position of the center of mass of the meterstick. The resulting set of points form a **Manifold of dimension three**.

Let’s consider another example of a Manifold. In classical mechanics you may have studied what is called the phase space. In this space you can describe the state of a particle over a period of time using the three coordinates of space locating the position of the particle and the three coordinates of speed (or momentum) describing how fast the particle is moving at a given instant of time. In Cartesian coordinate system we use the independent parameters \((x, y, z)\) for position and \((\dot{x}, \dot{y}, \dot{z})\) for the speed of the particle. I can represent these independent parameters by \((x^1(t), x^2(t), x^3(t), x^4(t), x^5(t), x^6(t))\). So when you describe the state of the particle say from \(t = 0\) to \(t = t_0\), you can use infinitesimal time interval so that you will have a set of points that can be parameterized continuously in terms of \((x^1(t), x^2(t), \ldots x^6(t))\). These set of points form a **Manifold of dimension six**.

Now let’s apply this to the Minkowski spacetime where we have three space coordinates \((x(t), y(t), z(t))\) and time, \(t\). This forms a four dimensional manifold with four coordinates each parametrized by the proper time \(\tau\), \((x^1(\tau), x^2(\tau), x^3(\tau), x^4(\tau))\). Therefore, a manifold is any set that can be continuously parameterized. Therefore, an \(N\) dimensional manifold, \(M\), of points is one for which \(N\) independent real coordinates \((x^1, x^2, x^3, \ldots, x^N)\) are required to specify any point completely.

A **manifold is Continuous**: if you pick any point, \(p\), on the Manifold and you can find another points whose coordinates differ infinitesimally from the point \(p\).

A **manifold is differentiable**: if you pick any point, \(p\), on the Manifold and you can find a scalar function \(\phi\) that is differentiable at that point \(p\).

**Coordinates of a Manifold** \(M\): a point in an \(N\)-Dimensional Manifold is represented by the coordinates \((x^1, x^2, x^3, \ldots, x^N)\) which we represent by \(x^a\) where it is understood that \(a = 1, 2, 3, \ldots N\).

**Degeneracy in a Manifold**: sometimes it may not be possible to cover the whole manifold with only one none-degenerate coordinate system. Example is a plane in polar coordinate system \((\rho, \varphi)\). A plane is a two dimension Manifold. (called \(R^2\)). A plane in polar coordinates has a degeneracy at the origin since \(\varphi\) is indeterminate at the origin. (Fig. 2.2)

**Coordinate patches**: these are coordinate systems that covers a portion of the Manifold where we have degeneracy. For example the surface of a sphere is a two dimensional Manifold (called \(S^2\)). It can be described by two independent coordinates \((x^1 = \theta, x^2 = \varphi)\) except at two points on the Manifold. These are
2.2 Curves and Surfaces in a Manifold

Both curves and surfaces on a Manifold are defined \textit{parametrically}. That means we use some common parameters. For example, a curve in the phase space that we saw earlier, can be defined in terms of the time parameter, \( t \). Another example, a curve in the 4D Minkowski spacetime manifold is defined by the interval

\[
 ds^2 (\tau) = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{2.1}
\]
Generally, we use a parameter, $u$, to define a curve.

A curve: a curve in a Manifold of dimension $N$ is define by a parametric equation

$$x^a = x^a(u), \text{ where } a = 1, 2, 3...N. \quad (2.2)$$

For example the curve shown in pink in Fig. 2.4 is defined by

$$x^1(u) = \cos^5(u), \quad x^2(u) = 0.4u^2, \quad x^3(u) = 0.4u^3, \quad (2.3)$$

and it needs only one parameter, $u$.

A surface: a surface in a Manifold of dimension $N$ (which also referred as a submanifold) has $M$ degrees of freedom which is always less than the dimension of the Manifold ($M < N$) and therefore it depends on $M$ parameters that we represent by $(u^1, u^2, u^3,...u^M)$ and is defined by the parametric equation

$$x^a = x^a(u^1, u^2, u^3,...u^M), \text{ where } a = 1, 2, 3...N. \quad (2.4)$$

Hypersurface: a surface of dimension $M$ in a Manifold of dimension $N$ with $M = N - 1$. In this case the $N - 1$ parameters can be eliminated from the $N$ equations and you can find one equation

$$f(x^1, x^2, x^3,...x^N) = 0. \quad (2.5)$$

The surface shown in blue in Fig. 2.4 in the 3D manifold needs two parameters to define it

$$x^1 = 2 \cos(u_1), \quad x^2 = \sin(u_2), \quad x^3 = u_2. \quad (2.6)$$

Note that this surface is in 3D manifold and it parameterized by two coordinates $(u, v)$, $(M = N - 1 = 3 - 1 = 2$, it is a hypersurface).
2.2. CURVES AND SURFACES IN A MANIFOLD

Example 2.1 Let’s consider the 3-D Euclidean Manifold. A sphere is a Hypersurface since $M = 2$, (Fig.2.5). A point on a sphere is defined by

$$x^2 + y^2 + z^2 = a^2,$$  \hspace{1cm} (2.7)

where $a$ is the radius of the sphere. We note that in this case the surface of the sphere is a Hypersurface that can be defined by the equation

$$g(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2 - a^2 = 0,$$  \hspace{1cm} (2.8)

where we used $(x^1, x^2, x^3)$ for $(x, y, z)$. Introducing the parameters $u_1, u_2,$ and $u_3$ defined by

$$x^1 = u_1 \sin (u_2) \cos (u_3), x^2 = u_1 \sin (u_2) \sin (u_3), x^3 = u_1 \cos (u_2)$$

we can write the equation that define the surface of the sphere ($M = 2$) that is embedded in a 3D manifold ($N = 3$) using only one parameter (by eliminating the $N - 1 = 2$, parameters) as

$$g(x^1, x^2, x^3) = (u_1 \sin (u_2) \cos (u_3))^2 + (u_1 \sin (u_2) \sin (u_3))^2 + (u_1 \cos (u_2))^2 - a^2 = 0 \Rightarrow g(x^1, x^2, x^3) = u_1^2 - a^2 = 0$$  \hspace{1cm} (2.9)

which is the property of a hypersurface in a manifold.

Therefore a point is restricted to lie in a hypersurface ($M = N - 1$ dimensional submanifold embedded in $N$-dimensional Manifold), then the points coordinate must satisfy Eq. (2.5). We come up with a similar generalization to this for a point that belong to any surface with dimension $M$ in a Manifold of
CHAPTER 2. MANIFOLDS

Let’s consider the 3-D Euclidean Manifold. A point in this Manifold can be represented using Cartesian coordinates \((x, y, z)\) which we shall represent by \((x^1, x^2, x^3)\). This same point can also be represented using spherical coordinates \((r, \theta, \varphi)\) that shall represent by \((x^1, x^2, x^3)\). Now the question is how we relate the Cartesian coordinates with the spherical coordinates or vice versa. In terms of these notations, one can write

\[
\begin{align*}
g_1 (x^1, x^2, x^3, \ldots, x^N) &= 0, \\
g_2 (x^1, x^2, x^3, \ldots, x^N) &= 0, \\
g_3 (x^1, x^2, x^3, \ldots, x^N) &= 0, \\
&
\end{align*}
\]

or

\[
\begin{align*}
g_{N-M} (x^1, x^2, x^3, \ldots, x^N) &= 0
\end{align*}
\]

2.3 Coordinate transformations and summation convention

Let’s consider the 3-D Euclidean Manifold. A point in this Manifold can be represented using Cartesian coordinates \((x, y, z)\) which we shall represent by \((x^1, x^2, x^3)\). This same point can also be represented using spherical coordinates \((r, \theta, \varphi)\) that shall represent by \((x^1, x^2, x^3)\). Now the question is how we relate the Cartesian coordinates with the spherical coordinates or vice versa. In terms of these notations, one can write

\[
\begin{align*}
r &\rightarrow r (x, y, z), \text{ or } x^1 \rightarrow x^1 \left( x^1, x^2, x^3 \right), \\
\theta &\rightarrow \theta (x, y, z), \text{ or } x^2 \rightarrow x^2 \left( x^1, x^2, x^3 \right), \\
\varphi &\rightarrow \varphi (x, y, z), \text{ or } x^3 \rightarrow x^3 \left( x^1, x^2, x^3 \right),
\end{align*}
\]

or

\[
\begin{align*}
x &\rightarrow x (r, \theta, \varphi), \text{ or } x^1 \rightarrow x^1 \left( x^1, x^2, x^3 \right), \\
y &\rightarrow y (r, \theta, \varphi), \text{ or } x^2 \rightarrow x^2 \left( x^1, x^2, x^3 \right), \\
z &\rightarrow z (r, \theta, \varphi), \text{ or } x^3 \rightarrow x^3 \left( x^1, x^2, x^3 \right).
\end{align*}
\]

Suppose we have a function, \(g (x, y, z)\) or \(g (r, \theta, \varphi)\), which can be expressed by \(g \left( x^1, x^2, x^3 \right)\) and \(g \left( x^1, x^2, x^3 \right)\), respectively, then one can write [Theoretical Physics I],

\[
\begin{align*}
\frac{\partial g}{\partial x^1} &= \frac{\partial g}{\partial x^1} \frac{\partial x^1}{\partial x^1} + \frac{\partial g}{\partial x^2} \frac{\partial x^2}{\partial x^1} + \frac{\partial g}{\partial x^3} \frac{\partial x^3}{\partial x^1}, \\
\frac{\partial g}{\partial x^2} &= \frac{\partial g}{\partial x^1} \frac{\partial x^1}{\partial x^2} + \frac{\partial g}{\partial x^2} \frac{\partial x^2}{\partial x^2} + \frac{\partial g}{\partial x^3} \frac{\partial x^3}{\partial x^2}, \\
\frac{\partial g}{\partial x^3} &= \frac{\partial g}{\partial x^1} \frac{\partial x^1}{\partial x^3} + \frac{\partial g}{\partial x^2} \frac{\partial x^2}{\partial x^3} + \frac{\partial g}{\partial x^3} \frac{\partial x^3}{\partial x^3}.
\end{align*}
\]

Using matrices [Theoretical Physics I] this can be put in the form

\[
\begin{bmatrix}
\frac{\partial g}{\partial x^1} \\
\frac{\partial g}{\partial x^2} \\
\frac{\partial g}{\partial x^3}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^1} & \frac{\partial x^3}{\partial x^1} \\
\frac{\partial x^1}{\partial x^2} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^2} \\
\frac{\partial x^1}{\partial x^3} & \frac{\partial x^2}{\partial x^3} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g}{\partial x^1} \\
\frac{\partial g}{\partial x^2} \\
\frac{\partial g}{\partial x^3}
\end{bmatrix}.
\]
2.3. \textit{COORDINATE TRANSFORMATIONS AND SUMMATION CONVENTION}

For the inverse case, following a similar procedure, we can write

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \frac{\partial g_1}{\partial x^3} \\
\frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \frac{\partial g_2}{\partial x^3} \\
\frac{\partial g_3}{\partial x^1} & \frac{\partial g_3}{\partial x^2} & \frac{\partial g_3}{\partial x^3}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^1} & \frac{\partial x^3}{\partial x^1} \\
\frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^2} \\
\frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g}{\partial x^1} \\
\frac{\partial g}{\partial x^2} \\
\frac{\partial g}{\partial x^3}
\end{bmatrix}
\]

so that

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \frac{\partial g_1}{\partial x^3} \\
\frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \frac{\partial g_2}{\partial x^3} \\
\frac{\partial g_3}{\partial x^1} & \frac{\partial g_3}{\partial x^2} & \frac{\partial g_3}{\partial x^3}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x^1}{\partial g_1} & \frac{\partial x^2}{\partial g_1} & \frac{\partial x^3}{\partial g_1} \\
\frac{\partial x^1}{\partial g_2} & \frac{\partial x^2}{\partial g_2} & \frac{\partial x^3}{\partial g_2} \\
\frac{\partial x^1}{\partial g_3} & \frac{\partial x^2}{\partial g_3} & \frac{\partial x^3}{\partial g_3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g}{\partial x^1} \\
\frac{\partial g}{\partial x^2} \\
\frac{\partial g}{\partial x^3}
\end{bmatrix}
\]

and can be written as

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \frac{\partial g_1}{\partial x^3} \\
\frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \frac{\partial g_2}{\partial x^3} \\
\frac{\partial g_3}{\partial x^1} & \frac{\partial g_3}{\partial x^2} & \frac{\partial g_3}{\partial x^3}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^3} \\
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^3} \\
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g}{\partial x^1} \\
\frac{\partial g}{\partial x^2} \\
\frac{\partial g}{\partial x^3}
\end{bmatrix}
\]

There follows that

\[
\begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^1} & \frac{\partial x^3}{\partial x^1} \\
\frac{\partial x^1}{\partial x^2} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^2} \\
\frac{\partial x^1}{\partial x^3} & \frac{\partial x^2}{\partial x^3} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^2}{\partial x^1} & \frac{\partial x^3}{\partial x^1} \\
\frac{\partial x^1}{\partial x^2} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^3}{\partial x^2} \\
\frac{\partial x^1}{\partial x^3} & \frac{\partial x^2}{\partial x^3} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
= 1.
\]

This means the matrix

\[
A^{-1} = \begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial x^3} \\
\frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial x^3} \\
\frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\]

must be the inverse matrix for

\[
A = \begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial x^3} \\
\frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial x^3} \\
\frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\]

so that

\[
A^{-1}A = AA^{-1} = 1.
\]

We note that the transpose, \(A^T\), is given by

\[
A^T = \begin{bmatrix}
\frac{\partial x^1}{\partial x^1} & \frac{\partial x^1}{\partial x^2} & \frac{\partial x^1}{\partial x^3} \\
\frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \frac{\partial x^2}{\partial x^3} \\
\frac{\partial x^3}{\partial x^1} & \frac{\partial x^3}{\partial x^2} & \frac{\partial x^3}{\partial x^3}
\end{bmatrix}
\]

Similarly for the inverse matrix, the transpose matrix which we express as

\[
\frac{\partial x'^a}{\partial x^b} = \begin{bmatrix}
\frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^1}{\partial x^3} \\
\frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^2}{\partial x^3} \\
\frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3}
\end{bmatrix}
\]
is the transformation matrix that transforms the coordinates \((x^1, x^2, x^3)\) to \((x'^1, x'^2, x'^3)\).

For a Manifold of dimension \(N\), the transformation matrix is given by

\[
 \frac{\partial x'^a}{\partial x^b} = \begin{bmatrix}
 \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \cdots & \frac{\partial x'^1}{\partial x^N} \\
 \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \cdots & \frac{\partial x'^2}{\partial x^N} \\
 \vdots & \vdots & \ddots & \vdots \\
 \frac{\partial x'^N}{\partial x^1} & \frac{\partial x'^N}{\partial x^2} & \cdots & \frac{\partial x'^N}{\partial x^N}
\end{bmatrix}.
\]  

As we recall from [Theoretical physics I], a Matrix is invertible provided the determinant is different from zero. Therefore, the inverse transformation is possible provide the determinant of the transformation matrix which is known as the Jacobian, \(J\), is different from zero

\[
J = \det \left[ \frac{\partial x'^a}{\partial x^b} \right] = \begin{bmatrix}
 \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \cdots & \frac{\partial x'^1}{\partial x^N} \\
 \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \cdots & \frac{\partial x'^2}{\partial x^N} \\
 \vdots & \vdots & \ddots & \vdots \\
 \frac{\partial x'^N}{\partial x^1} & \frac{\partial x'^N}{\partial x^2} & \cdots & \frac{\partial x'^N}{\partial x^N}
\end{bmatrix} \neq 0
\]  

The inverse transformation Matrix can be written as

\[
\frac{\partial x^a}{\partial x'^b} = \begin{bmatrix}
 \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \cdots & \frac{\partial x^1}{\partial x'^N} \\
 \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^2}{\partial x'^N} \\
 \vdots & \vdots & \ddots & \vdots \\
 \frac{\partial x^N}{\partial x'^1} & \frac{\partial x^N}{\partial x'^2} & \cdots & \frac{\partial x^N}{\partial x'^N}
\end{bmatrix}
\]  

and the Jacobian \(J'\)

\[
J' = \det \left[ \frac{\partial x^a}{\partial x'^b} \right] = \begin{bmatrix}
 \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \cdots & \frac{\partial x^1}{\partial x'^N} \\
 \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \cdots & \frac{\partial x^2}{\partial x'^N} \\
 \vdots & \vdots & \ddots & \vdots \\
 \frac{\partial x^N}{\partial x'^1} & \frac{\partial x^N}{\partial x'^2} & \cdots & \frac{\partial x^N}{\partial x'^N}
\end{bmatrix}
\]  

We note that

\[
\frac{\partial x'^a}{\partial x^1} = \frac{\partial x'^a}{\partial x^1} \frac{\partial x^1}{\partial x^1} + \frac{\partial x'^a}{\partial x^2} \frac{\partial x^2}{\partial x^1} + \cdots + \frac{\partial x'^a}{\partial x^N} \frac{\partial x^N}{\partial x^1} = \sum_{k=1}^{N} \frac{\partial x'^a}{\partial x^k} \frac{\partial x^k}{\partial x^1}
\]  

(2.30)
2.3. COORDINATE TRANSFORMATIONS AND SUMMATION CONVENTION

Figure 2.6: Point P and Q on a surface

Noting that for independent coordinates
\[
\frac{\partial x^a}{\partial x^c} = \begin{cases} 
0, & a \neq c \\
1, & a = c
\end{cases}
\] (2.31)

we can generally write
\[
\sum_{b=1}^{N} \frac{\partial x^a}{\partial x^b} \frac{\partial x^b}{\partial x^c} = \delta_c^a.
\] (2.32)

Consider two points $P$ and $Q$ on a Manifold with dimension $N$. Suppose these points are separated by infinitesimal interval so that if the coordinates of $P$ is $x^a$ and that of $Q$ is $x^a + dx^a$, then one can write
\[
dx^a = \frac{\partial x^a}{\partial x^1} dx^1 + \frac{\partial x^a}{\partial x^2} dx^2 \ldots \frac{\partial x^a}{\partial x^N} dx^N = \sum_{b=1}^{N} \frac{\partial x^a}{\partial x^b} dx^b,
\] (2.33)

where the summation is evaluated at $P$. Similarly for the interval between $x^a$ and $x^a + dx^a$, in the none-primed coordinate system, we can write
\[
dx^a = \frac{\partial x^a}{\partial x'^1} dx'^1 + \frac{\partial x^a}{\partial x'^2} dx'^2 \ldots \frac{\partial x^a}{\partial x'^N} dx'^N = \sum_{b=1}^{N} \frac{\partial x^a}{\partial x'^b} dx'^b,
\] (2.34)

here also the summation is evaluated at $P$.

**Einstein’s summation convention**: whenever an index occurs twice in an expression, once as subscript and once as a superscript, imply a summation over the index. An index should not occur more than twice. For example, according to Einstein’s summation convention, the summations in Eq. (2.33) and (2.34) can be expressed
\[
dx^a = \sum_{b=1}^{N} \frac{\partial x^a}{\partial x^b} dx^b = g_{ab}(x) dx^b = \frac{\partial x^a}{\partial x'^b} dx'^b
\] (2.35)

and
\[
dx^a = \sum_{b=1}^{N} \frac{\partial x^a}{\partial x'^b} dx'^b = g_{ab}(x') dx'^b = \frac{\partial x^a}{\partial x'^b} dx'^b.
\] (2.36)
The index $a$, which is known as the \textit{free index}, can take any value from 1 to $N$. The index $b$, which is known as the \textit{dummy index} and it must be summed up from 1 to $N$.

### 2.4 The Riemannian geometry

The \textit{local geometry} of a Manifold: The local geometry of a manifold is determined by defining the invariant 'distance' or (as we saw in the Minkowski spacetime Manifold) the interval $ds$ between points $P$ with coordinate $x^a$ and $Q$ with coordinates $x^a + dx^a$. This distance can be assigned in general to be a well-behaved function $g(x^a, dx^a)$ of the coordinates $x^a$ and $dx^a$.

$$ds^2 = g(x^a, dx^a).$$  \hfill (2.37)

Let’s reconsider the worldline of the alien in the spaceship in the previous chapter. Imagine there is a surface defined by the two coordinates $(x, ct)$ represented as $(x^1, x^2)$. Let’s consider two points $P$ and $Q$, as shown in Fig. 2.7. The coordinates for the points are $(x^1, x^2)$ and $(x^1 + dx^1, x^2 + dx^2)$, respectively. Suppose this surface is defined by the function $s(x^1, x^2)$ The interval between these two points, $ds$, can be expressed as

$$ds = \frac{\partial s}{\partial x^1} dx^1 + \frac{\partial s}{\partial x^2} dx^2 \quad \Rightarrow \quad ds^2 = \left( \frac{\partial s}{\partial x^1} dx^1 + \frac{\partial s}{\partial x^2} dx^2 \right) \left( \frac{\partial s}{\partial x^1} dx^1 + \frac{\partial s}{\partial x^2} dx^2 \right)$$

$$\Rightarrow \quad ds^2 = \frac{\partial s}{\partial x^1} \frac{\partial s}{\partial x^1} dx^1 dx^1 + \frac{\partial s}{\partial x^2} \frac{\partial s}{\partial x^2} dx^2 dx^2 + \frac{\partial s}{\partial x^1} \frac{\partial s}{\partial x^2} dx^1 dx^2 + \frac{\partial s}{\partial x^2} \frac{\partial s}{\partial x^1} dx^2 dx^1$$

$$+ \frac{\partial s}{\partial x^2} \frac{\partial s}{\partial x^2} dx^2 dx^2 \quad \hfill (2.38)$$

![Figure 2.7: A curved surface in a 3D manifold.](image-url)
2.4. THE RIEMANNIAN GEOMETRY

This can be rewritten as

\[ ds^2 = g_{11}(x^1, x^2) \, dx^1 \, dx^1 + g_{12}(x^1, x^2) \, dx^1 \, dx^2 + g_{21}(x^1, x^2) \, dx^1 \, dx^2 \]

\[ + g_{22}(x^1, x^2) \, dx^2 \, dx^2 = \sum_{a=1}^{2} \sum_{b=1}^{2} g_{ab}(x) \, dx^a \, dx^b \]  \hspace{1cm} (2.39)

where we replaced

\[ g_{11} = \frac{\partial s}{\partial x^1} \frac{\partial s}{\partial x^1}, g_{12} = \frac{\partial s}{\partial x^1} \frac{\partial s}{\partial x^2}, g_{21} = \frac{\partial s}{\partial x^2} \frac{\partial s}{\partial x^1}, g_{22} = \frac{\partial s}{\partial x^2} \frac{\partial s}{\partial x^2} \]

Consider a curved surface in 3-D Euclidean space. We know that this surface can be defined by a function \( g \), that depends on \((x, y, z)\) in Cartesian, \((r, \theta, \varphi)\) in spherical, or \((r, \varphi, z)\) in cylindrical coordinates. Suppose we represent these coordinates by \( x_1, x_2, x_3 \), then we may write the function that defines the surface as

\[ s(x) = g(x) = g(x^1, x^2, x^3) \]  \hspace{1cm} (2.40)

Now let’s consider a point \( P \) on this surface that has coordinates \( x_1, x_2, x_3 \). Suppose we consider another point \( Q \) with coordinates \( x_1 + dx_1, \, x_2 + dx_2, \, x_3 + dx_3 \), we may define the surface between these two points as \( ds^2 \). This displacement is just the differential of Eq. (2.40)

\[ ds^2 = \left( \frac{\partial g}{\partial x^1} \, dx^1 + \frac{\partial g}{\partial x^2} \, dx^2 + \frac{\partial g}{\partial x^3} \, dx^3 \right) \cdot \left( \frac{\partial g}{\partial x^1} \, dx^1 + \frac{\partial g}{\partial x^2} \, dx^2 + \frac{\partial g}{\partial x^3} \, dx^3 \right) \]  \hspace{1cm} (2.41)

or using Einstein’s summation convention, the geometry of the surface between the two points

\[ ds^2 = (ds) \cdot (ds) = \left( \sum_{a=1}^{3} \frac{\partial g}{\partial x^a} \, dx^a \right) \cdot \left( \sum_{b=1}^{3} \frac{\partial g}{\partial x^b} \, dx^b \right) \]

\[ = \sum_{a=1}^{3} \sum_{b=1}^{3} \frac{\partial g}{\partial x^a} \frac{\partial g}{\partial x^b} \, dx^a \, dx^b. \]  \hspace{1cm} (2.42)

Again using Einstein’s summation convention and the notation

\[ g_{ab}(x) = \frac{\partial g}{\partial x^a} \frac{\partial g}{\partial x^b} \]  \hspace{1cm} (2.43)

we may write

\[ ds^2 = g_{ab}(x) \, dx^a \, dx^b. \]  \hspace{1cm} (2.44)

which is the metric equation that we defined earlier. Note that the metric tensor in this case is a \( 3 \times 3 \) matrix given by

\[ G = \begin{bmatrix} \frac{\partial g}{\partial x^1} & \frac{\partial g}{\partial x^2} & \frac{\partial g}{\partial x^3} \\ \frac{\partial g}{\partial x^2} & \frac{\partial g}{\partial x^1} & \frac{\partial g}{\partial x^3} \\ \frac{\partial g}{\partial x^3} & \frac{\partial g}{\partial x^3} & \frac{\partial g}{\partial x^1} \end{bmatrix} \]  \hspace{1cm} (2.45)
and we can easily see that this matrix is symmetric as

\[
\frac{\partial g}{\partial x^a} \frac{\partial g}{\partial x^b} = \frac{\partial g}{\partial x^b} \frac{\partial g}{\partial x^a}.
\]

Suppose the coordinates for an orthonormal set like the Cartesian, spherical, or cylindrical, we note that

\[
G = \begin{pmatrix}
\left(\frac{\partial g}{\partial x^1}\right)^2 & 0 & 0 \\
0 & \left(\frac{\partial g}{\partial x^2}\right)^2 & 0 \\
0 & 0 & \left(\frac{\partial g}{\partial x^3}\right)^2
\end{pmatrix}
\]

(2.46)

In general theory of relativity, we are interested in a Manifold where the interval \( ds \) can be described by the equation of the form

\[
ds^2 = \sum_{a=1}^{N} \sum_{b=1}^{N} g_{ab} (x) \, dx^a dx^b,
\]

(2.47)

or simply using the Einstein’s summation convention

\[
ds^2 = g_{ab} (x) \, dx^a dx^b.
\]

(2.48)

A geometry of a Manifold defined by Eq. (2.48) is known as the Riemannian geometry if \( ds^2 > 0 \). As we have seen in the case of Minkowski spacetime manifold the interval \( ds^2 \) can also be negative (spacelike) or zero (lightlike). In such cases the geometry is referred as pseudo Riemannian geometry and the manifold can be referred as pseudo Riemannian. The function \( g (x) \) is known as the metric function, where \( g_{ab} (x) \) represent the element of a metric tensor.

Transformation of the interval: applying the relation in Eq. (2.36), we can express

\[
dx^a = \frac{\partial x^a}{\partial x'^c} \, dx'^c, \quad dx^b = \frac{\partial x^b}{\partial x'^d} \, dx'^d
\]

(2.49)

so that the interval can be transformed as

\[
ds^2 = g_{ab} (x) \, \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} \, dx'^c dx'^d,
\]

(2.50)

or

\[
ds^2 = g'_{cd} (x') \, dx'^c dx'^d,
\]

(2.51)

where

\[
g'_{cd} (x') = g_{ab} (x) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}.
\]

(2.52)

Note that \( x = x (x') \).
2.5 Intrinsic and extrinsic geometry and the metric

A given geometry of dimension $M$ defined by the metric equation

$$ds^2 = g_{ab}(x) dx^a dx^b.$$ \hfill (2.53)

and embedded in a higher dimension manifold of dimension $N$, ($N > M$) is said to be:

(a) *Intrinsic*: when the geometry remains unchanged as viewed in the higher dimensional manifold.

(b) *Extrinsic*: when the geometry is different as viewed in the higher dimensional manifold.

**Example 2.2 Extrinsic geometry**: One simple example of extrinsic geometry is 2-D cylindrical geometry. In order to see that let’s consider a plane geometry in a 3-D Euclidean Manifold. Let’s this plane depends on $(x^1, x^2)$.

We recall from theoretical physics I generally a plane:

**Equation of a plane**: If $\vec{N} = ai + bj + ck$ is normal (perpendicular) to a plane, then the scalar product of the vector $\vec{N}$ and the vector $\vec{r} - \vec{r}_0$

$$\vec{r} - \vec{r}_0 = (x - x_0) \hat{x} + (y - y_0) \hat{y} + (z - z_0) \hat{z}$$ \hfill (2.54)

is zero,

$$\vec{N} \cdot (\vec{r} - \vec{r}_0) = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$ \hfill (2.55)

This defines the equation of the plane. It can be rewritten as

$$ax + by + cz = d.$$ \hfill (2.56)
where
\[ d = ax_0 + by_0 + cz_0. \] (2.57)

Using the notation \((x^1, x^2, x^3)\) for \((x, y, z)\), we may write
\[ ax^1 + bx^2 + cx^3 = d, \] (2.58)

where
\[ d = ax_0^1 + bx_0^2 + cx_0^3. \] (2.59)

For a plane that depends on only \((x^1, x^2)\) we may write the interval between

\((x^1, x^2)\) and \((x^1 + dx^1, x^2 + dx^2)\) as
\[ ds^2 = (dx^1)^2 + (dx^2)^2. \] (2.60)

Now let’s consider the 2-D cylindrical surface which we can construct using our 2-D plane. Suppose the cylinder has radius, \(a\), with its axis along the \(z\)-axis (which we call it \(x^3\)-axis). Using cylindrical coordinates a point on the surface of the cylinder can be described by \((a, \varphi, z)\) or using our coordinates notations \((a, x^2, x^3)\), we can define the surface by the function
\[ g(a, \varphi, z) = g(a, x^2, x^3). \] (2.61)

we may write the interval between two points on this surface, point \(P\) and \(Q\) with coordinates \((x^1, x^2, x^3)\) and point \(Q\) with coordinates \((x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)\) just using geometrical visualization, as
\[ ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \] (2.62)

For the cylinder with radius \(a\) shown in Fig. 2.9
\[ x^1 = a \cos(x^2), x^2 = a \sin(x^2), x^3 = x^3 \]
we find
\[ ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = a^2 (dx^2)^2 + (dx^3)^2. \] (2.63)
This is a 2-D surface embedded in a 3-D manifold. It has cylindrically curved geometry when it is viewed in this 3-D Euclidean manifold. But you can actually obtain this geometry from the plane geometry from Eq. (2.40) by simply substituting

\[ x^1 = ax^2, \quad x^2 = x'^3 \]

Such kind of geometry is not intrinsic and it is called extrinsic. Its curvature is extrinsic and is a result of the way it is embedded in the three dimensional space.

**Example 2.3 Intrinsic geometry:** One simple example of intrinsic geometry is 2-D spherical geometry embedded in a 3-D Euclidean Manifold. In Fig. 2.5 we see a 3-D infinitesimal volume. Assume a sphere with radius \( a \).

Then the surface defined by a pair of points on this sphere separated by a distance \( ds \) can be expressed as

\[ ds^2 = (a d\theta)^2 + (a \sin(\theta) d\varphi)^2 = a^2(\theta)^2 + a^2 \sin^2(\theta)(d\varphi)^2. \quad (2.64) \]
or using the notation \((x^2, x^3)\) for \((\theta, \varphi)\), we can write
\[
ds^2 = a^2 (dx^2)^2 + a^2 \sin^2(x^2) (dx^3)^2. \tag{2.65}
\]

This is a 2-D surface embedded in a 3-D manifold. You can not obtain this geometry from the plane geometry like the 2-D cylindrical geometry. Such kind of geometry is **intrinsic**. This means the geometry of a sphere is intrinsically curved because we can not transform Eq. (2.65) to the Euclidean form
\[
ds^2 = (dx^1)^2 + (dx^2)^2 \tag{2.66}
\]
over the whole surface by any coordinate transformation. Note that this can be done locally but not for the whole spherical surface.

**Example 2.4** Find the metric for a two-dimensional sphere of radius, \(a\), embedded in a 3-D Euclidean space both in Cartesian coordinates \((x^1, x^2, x^3)\). Refer to Fig. 2.5

![Spherical Geometry](image)

**Solution:** We recall that the line element in a 3-D Euclidean space is given by
\[
ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{2.67}
\]

For a two dimensional spherical geometry with radius \(a\), we have
\[
(x^1)^2 + (x^2)^2 + (x^3)^2 = a^2, \tag{2.68}
\]

Using (2.68), we can write
\[
x^3 = \sqrt{a^2 - (x^1)^2 - (x^2)^2}, \tag{2.69}
\]
so that
\[ dx^3 = \frac{x^1 dx^1 + x^2 dx^2}{\sqrt{a^2 - (x^1)^2 - (x^2)^2}}. \]

Then for a 2-D sphere embedded in a 3-D Euclidean space, the metric is given by
\[ ds^2 = (dx^1)^2 + (dx^2)^2 + \left( \frac{x^1 dx^1 + x^2 dx^2}{\sqrt{a^2 - (x^1)^2 - (x^2)^2}} \right)^2. \tag{2.70} \]

If we consider a point in the neighborhood of the pole, we may set
\[ x^1 = x^2 \approx 0 \]
and the metric in Eq. (2.70) reduces to the Euclidean form
\[ ds^2 = (dx^1)^2 + (dx^2)^2. \tag{2.71} \]

Let’s use the coordinates \((x^0, x^1, x^2)\), defined by the transformation
\[ x^1 = x^0 \cos (x^2), x^2 = x^0 \sin (x^2), x^3 = x^3 \]
For any point on the surface of the sphere, we have
\[ (x^1)^2 + (x^2)^2 + (x^3)^2 = a^2 \]
\[ \Rightarrow x^3 = \sqrt{a^2 - (x^1)^2 - (x^2)^2} = \sqrt{a^2 - (x^0)^2}, \tag{2.72} \]

Note that the origin is set at the north pole of the sphere at point as shown in Fig. 2.5. Then the interval can be written as
\[ ds^2 = (dx^0 \cos (x^2) - x^0 \sin (x^2) dx^2)^2 + (dx^0 \sin (x^2) + x^0 \cos (x^2) dx^2)^2 + \left( \frac{x^0 dx^0}{\sqrt{a^2 - (x^0)^2}} \right)^2 \tag{2.73} \]
so that after a little algebra, we find
\[ ds^2 = (dx^0)^2 + (x^1)^2 (dx^2)^2 + \left( \frac{x^0 dx^0}{\sqrt{a^2 - (x^0)^2}} \right)^2 \tag{2.74} \]
which simplifies into
\[ ds^2 = \frac{a^2 (dx^1)^2}{a^2 - (x^1)^2} + (x^1)^2 (dx^2)^2 \tag{2.75} \]
or
\[ ds^2 = g_{11} (dx^1)^2 + g_{22} (dx^2)^2 \]  
(2.76)
where
\[ g_{11} = \frac{a^2}{a^2 - (x^1)^2}, g_{22} = (x^1)^2 \]  
(2.77)
are the none zero elements of the metric tensor. We will see the use of these elements of the metric tensor in the next section to determine length and area of a 2-D sphere in a 3-D Euclidean manifold.

**Example 2.5** Determine the metric for a three-dimensional sphere of radius \( a \) embedded in a 4-D Euclidean space: We can write the interval
\[ ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \]  
(2.78)
For a three dimensional spherical geometry with radius \( a \) we have
\[ (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2 \]
\[ \Rightarrow x^4 = \sqrt{a^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2)} \]  
(2.79)
so that
\[ dx^4 = -\frac{x^1 dx^1 + x^2 dx^2 + x^3 dx^3}{\sqrt{a^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2)}}. \]
Then the metric can be expressed as
\[ ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{(x^1 dx^1 + x^2 dx^2 + x^3 dx^3)^2}{a^2 - ((x^1)^2 + (x^2)^2 + (x^3)^2)}, \]  
(2.80)
We introduce the coordinates \((r, \theta, \phi)\) which we represent \((x^1, x^2, x^3)\).
and defined by the transformation
\[ x^1 = r \sin(\theta) \cos(\phi), x^2 = r \sin(\theta) \sin(\phi), x^3 = \cos(\theta), \]
or
\[ x^1 = x'^1 \sin(x^2) \cos(x^3), x^2 = \sin(x^2) \sin(x^3), x^3 = \cos(x^2). \]
Now referring to Fig.2.5 the "distance" squared between point \( P \) and \( Q \) can easily be determined using the Pythagorean theorem. First find the length of the hypotenuse of the green triangle. Suppose if we call this length \( ds_s \), we note that
\[ ds_s^2 = (r \sin(\theta) d\phi)^2 + (r \theta d\theta)^2. \]  
(2.81)
Then the distance between $P$ and $Q$ shown by the red line (hypotenuse side) can be expressed as

$$ds'^2 = dr^2 + ds^2 = dr^2 + r^2 \sin^2 (\theta) d\varphi^2 + r^2 d\theta^2$$

(2.82)

Note that I referred this distance as $ds'$ because it represents distance

$$ds'^2 = dx^2 + dy^2 + dz^2$$

We are considering a 3-D sphere in a 4-D Euclidean manifold, where the interval between point $P$ and $Q$ is given by

$$ds^2 = ds'^2 + (dx^4)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

(2.83)

with the constraint

$$x^4 = \sqrt{a^2 - \left( (x^1)^2 + (x^2)^2 + (x^3)^2 \right)}.$$  

which we may write, in terms of the polar coordinates, as

$$x^4 = \sqrt{a^2 - r^2}.$$  

(2.84)

so that

$$dx^4 = -\frac{r dr}{\sqrt{a^2 - r^2}}.$$  

(2.85)

Then the metric for a 3-D sphere in a 4-D Euclidean manifold is given by

$$ds^2 = ds'^2 + dx^4 = dr^2 + r^2 \sin^2 (\theta) d\varphi^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{a^2 - r^2},$$

(2.86)
which can be rewritten as
\[ ds^2 = \frac{a^2}{a^2 - r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \]  
(2.87)
or
\[ ds^2 = g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + g_{33} (dx^3)^2 , \]
(2.88)
where we used the notation \((x^1, x^2, x^3)\) for the coordinates \((r, \theta, \varphi)\) and identify the non-zero elements of the metric tensor
\[ g_{11} = \frac{a^2}{a^2 - r^2}, g_{22} = r^2, g_{33} = r^2 \sin^2(\theta) \]
We will see the use of these elements of the metric tensor in the next section to determine length, area, and volume of a 3-D sphere in a 4-D Euclidean manifold.

### 2.6 Length, area, and volume

**Length:** Suppose two points \(P\) and \(Q\) on a manifold of dimension \(N\) are connected by some curve. The length of the curve connecting these two points is given by
\[
L_{PQ} = \int_P^Q \sqrt{|dx^2|} = \int_P^Q \sqrt{|g_{ab}(x) dx^a dx^b|} \]
(2.89)
the absolute value is because of the fact that for pseudo-Riemannian manifolds it can be negative as is the case for spacelike in the Minkowski spacetime manifold. For \(x^a = x^a(u)\), we recall write
\[
dx^a = \frac{dx^a}{du} du, dx^b = \frac{dx^b}{du} du
\]
so that
\[
L_{PQ} = \int_P^Q \sqrt{\left| g_{ab}(u) \frac{dx^a}{du} \frac{dx^b}{du} \right|} du.
\]
(2.91)

**Area and volume:**

Generally the area can determined using
\[
A = \int \int ds_1 \times ds_2 = \int \int \sqrt{|g_{ab}(x) dx^a dx^b|} \times \sqrt{|g_{cd}(x) dx^c dx^d|}, \text{ for } c, d \neq a, b
\]
\[
= \int \int \sqrt{|g_{ab}(x) g_{cd}(x) dx^a dx^b dx^c dx^d|} = \int \int \sqrt{|g_{ab}(x) g_{cd}(x) \frac{dx^a}{dx^b} \frac{dx^c}{dx^d} (dx^b dx^d)^2|}
\]
\[
\Rightarrow A = \int \int \sqrt{|g_{ab}(x) g_{cd}(x) \frac{dx^a}{dx^b} \frac{dx^c}{dx^d}|} dx^b dx^d.
\]
(2.92)
For an orthogonal set of coordinates, \( x^p \)

\[
\frac{dx^p}{dx^q} = \delta^{pq} = \begin{cases} 
1, & p = q \\
0, & p \neq q
\end{cases}
\] (2.93)

and the area becomes

\[
A = \iint \sqrt{|g_{ab}(x) g_{cd}(x) \delta^{ab} \delta^{cd}|} \, dx^b \, dx^d = \iint \sqrt{|g_{bb}(x) g_{dd}(x)|} \, dx^b \, dx^d,
\]

note that, \( d \neq b \). As an example let’s consider a 2D hypersurface embedded in a 3D manifold. Then the surface area is given by

\[
A = \iint \sqrt{|g_{ab}(x) g_{cd}(x)|} \, dx^1 \, dx^2 + \iint \sqrt{|g_{ab}(x) g_{cd}(x)|} \, dx^2 \, dx^1
\]

\[
+ \iint \sqrt{|g_{ab}(x) g_{cd}(x)|} \, dx^3 \, dx^d
\]

\[
= \iint \sqrt{|g_{22}(x) g_{11}(x)|} \, dx^1 \, dx^2 + \iint \sqrt{|g_{11}(x) g_{22}(x)|} \, dx^2 \, dx^1
\]

\[
+ \iint \sqrt{|g_{33}(x) g_{11}(x)|} \, dx^1 \, dx^3 + \iint \sqrt{|g_{22}(x) g_{33}(x)|} \, dx^2 \, dx^3
\]

For a hypersurface, like the 2D sphere in the 3D manifold, the surface is defined by the equation

\[
x^3 = \text{constant} \Rightarrow dx^3 = 0.
\]

and the expression for the area reduces to

\[
A = \iint \sqrt{|g_{22}(x) g_{11}(x)|} \, dx^2 \, dx^1 + \iint \sqrt{|g_{11}(x) g_{22}(x)|} \, dx^1 \, dx^2
\]

which can simply be written as

\[
A = \iint \sqrt{|g_{22}(x) g_{11}(x)|} \, dx^2 \, dx^1
\]

where one can absorb the factor 2 into the metric elements. For such coordinates the metric is diagonal

\[
ds^2 = g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + g_{33} (dx^3)^2 + \cdots + g_{NN} (dx^N)^2.
\] (2.94)

and the infinitesimal area \( dA \) of the surface defined by

\[
x^a = \text{constant},
\] (2.95)

for, \( a = 3, 4 \ldots N \), is given by
\[ dA = \sqrt{|g_{11}g_{22}|} dx^1 dx^2 \]  
\[ (2.96) \]

and for 3-D volume in the \((x^1, x^2, x^3)\) defined by

\[ x^a = \text{constant} \]  
\[ (2.97) \]

for \(a = 4, 5, \ldots, N\), the infinitesimall volume is given by

\[ dV = \sqrt{|g_{11}g_{22}g_{33}|} dx^1 dx^2 dx^3 \]  
\[ (2.98) \]

**Example 2.5** For the two-dimensional sphere of radius \(a\) embedded in a 3-D Euclidean space consider a surface defined by a radius, \(\rho = R\). For this surface find

(a) The distance, \(D\), from the origin to the perimeter on this surface along a line of constant \(\varphi\) \(\text{(i.e. } x^2 = \text{cons})\) The origin is at the north pole as shown in Fig. 2.6.

(b) The circumference of the circle with radius, \(\rho = R\) \(\text{(i.e. } x^1 = R = \text{cons})\).

(c) The area of the spherical surface enclosed by the perimeter with radius, \(\rho = R\), \(\text{(i.e. the surface shaded green in Fig. 2.6)}\).

**Solution:**

(a) We recall that the metric for a 2-D sphere in a 3-D Euclidean manifold is given by

\[ ds^2 = g_{11} (dx^1)^2 + g_{22} (dx^2)^2, \]  
\[ (2.99) \]

where we used the notation \(\{x^1, x^2\}\) for the coordinates \((\rho, \varphi)\) and identify the none zero elements of the metric tensor

\[ g_{11} = \frac{a^2}{a^2 - \rho^2}, g_{22} = \rho^2. \]  
\[ (2.100) \]
2.6. LENGTH, AREA, AND VOLUME

We recall the length of a curve between two points, \( P \) and \( Q \), on a manifold in terms of the metric tensor is given by

\[
L_{PQ} = \int_P^Q \sqrt{|g_{ab}(x)| \, dx^a dx^b}.
\]  

(2.101)

For curve on the surface of the sphere shown in red in Fig. 2.6, the coordinates for point \( P \) is \( x^0 = 0, x^2 = \text{constant} \) and for point \( Q \) \( (x^1 = \rho = R, x^2 = \phi = \text{constant}) \) which leads to

\[
dx^2 = d\phi = 0.
\]

Then the length becomes

\[
D = \int_P^Q \sqrt{g_{11}(x^1)} \, dx^1 \, dx^2 = \int_0^\rho \frac{a}{\sqrt{a^2 - \rho^2}} \, d\rho = a \sin^{-1} \left[ \frac{R}{a} \right].
\]  

(2.102)

(b) Along the circumference (the curve shown in pink Fig. 2.6), we know that \( x^0 = \rho = \text{constant} \) which leads to

\[
dx^0 = d\rho = 0
\]

Thus the expression

\[
L_{PQ} = \int_P^Q \sqrt{|g_{ab}(x^1)| \, dx^0 dx^a}
\]

for the circumference becomes

\[
C = \int_0^{2\pi} \sqrt{|g_{22}(x^2)|} \, (dx^2)^2 = \int_0^{2\pi} \sqrt{|g_{22}(x^1 = \rho = R)|} \, (dx^2)
\]

\[
\Rightarrow C = \int_0^{2\pi} Rdx^2 = 2\pi R
\]

(2.105)
(c) The surface is defined by \( x^3 = \text{constant} \). Then according to Eq. (2.96), the infinitesimal area is given by

\[
dA = \sqrt{|g_{11}g_{22}|} dx^1 dx^2.
\]  

(2.106)

Then the surface area becomes

\[
A = \int \int \sqrt{\frac{a^2}{a^2 - \rho^2} \rho^2} \, d\rho d\phi = \int \int \frac{a}{\sqrt{a^2 - \rho^2}} \rho d\rho d\phi.
\]

(2.107)

To cover the area (colored Aqua) shown in Fig 2.6, we should have for the limits of integrations \([0, R]\) for \( \rho \) and \([0, 2\pi]\) for \( \varphi \)

\[
A = \int_0^R \int_0^{2\pi} \frac{a}{\sqrt{a^2 - \rho^2}} \rho d\rho d\phi = 2\pi a^2 \left[ 1 - \sqrt{1 - \frac{R^2}{a^2}} \right].
\]

(2.108)

**Homework:** In Example 2.5, express the circumference (part b) and the area (part c) in terms of the distance \( D \) (part a) and determine \( D \) for the maximum circumference and area. Find the maximum circumference and area of this sphere. Is your answer is consistent with what you know about the maximum circumference and surface area of a sphere with radius \( a \).

**Example 2.6** For the three-dimensional sphere embedded in a 4-D Euclidean space by considering a 2-D sphere of coordinate radius \( r = R \), find

(a) the distance from its center to the surface of this sphere along constant \( \theta \) and constant \( \phi \).

(b) the circumference across the equator
2.6. LENGTH, AREA, AND VOLUME

(c) the area of the spherical surface.

(d) The volume bounded by the spherical surface.

Solution:

(a) We recall the metric for a 3-D sphere embedded in a 4-D Euclidean space is given by

$$ds^2 = g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + g_{33} (dx^3)^2,$$  \hfill (2.109)

where we used the notation \((x^1, x^2, x^3)\) for the coordinates \((r, \theta, \varphi)\) and identify the non-zero elements of the metric tensor

$$g_{11} = \frac{a^2}{a^2 - r^2}, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2(\theta)$$ \hfill (2.110)

or in terms of \((x^1, x^2, x^3)\)

$$g_{11} = \frac{a^2}{a^2 - (x^1)^2}, \quad g_{22} = (x^1)^2, \quad g_{33} = (x^1)^2 \sin^2(x^2).$$ \hfill (2.111)

For \(x^2 = \theta = \text{constant}\) and \(x^3 = \phi = \text{constant}\), we have

$$dx^2 = dx^3 = 0.$$ \hfill (2.112)

so that the distance \(D\)

$$L_{PQ} = \int_P^Q \sqrt{|g_{ab}(x^a) dx^a dx^b|}$$ \hfill (2.113)

becomes

$$L_{PQ} = \int_P^Q \sqrt{|g_{11}(x^1) dx^1 dx^1|}$$ \hfill (2.114)

or

$$D = \int_0^R \sqrt{\frac{a^2}{a^2 - r^2} dr^2} = \int_0^R \frac{a}{\sqrt{a^2 - r^2}} dr = a \sin^{-1} \left[ \frac{R}{a} \right].$$ \hfill (2.115)

which is the same as the result we obtained in the previous example.

(b) Across the equator, we have \(x^1 = r = R, \ x^2 = \theta = \pi/2\) and obviously \(dx^1 = 0\) and \(dx^2 = 0\). Then the length

$$L_{PQ} = \int_P^Q \sqrt{|g_{ab}(x^a) dx^a dx^b|},$$ \hfill (2.116)

for the circumference, becomes

$$C = \int \sqrt{|g_{33}(x^3) (dx^3)^2|} = \int_0^{2\pi} \sqrt{|g_{33}(x^1 = r = R, x^2 = \pi/2)|} |dx^3|$$

$$\Rightarrow C = \int_0^{2\pi} \sqrt{R^2 \sin^2(\pi/2)} |dx^3| = \int_0^{2\pi} R dx^3 = 2\pi R$$ \hfill (2.117)
The spherical surface is defined by $x^1 = R = \text{constant}$. Therefore, in view of Eq. (2.96), the infinitesimal area on this surface should be expressed as

$$dA = \sqrt{|g_{22}g_{33}|} dx^2 dx^3.$$  \hspace{1cm} (2.118)

Noting that for $x^1 = r = R$

$$g_{22} = r^2 \bigg|_{r=R} = R^2$$
$$g_{33} = r^2 \sin^2 (\theta) \bigg|_{r=R} = R^2 \sin^2 (\theta)$$  \hspace{1cm} (2.119)

and the limit of integration for $x^2$ is $(0, \pi)$ and for $x^3$ is $(0, 2\pi)$, the surface area would become

$$A = \int_0^\pi \int_0^{2\pi} \sqrt{|R^4 \sin^2 (\theta)|} d\theta d\phi = \int_0^\pi \int_0^{2\pi} R^2 \sin (\theta) \ d\theta d\phi = 4\pi R^2.$$  \hspace{1cm} (2.120)

In this case we are considering a 3-D sphere embedded in a 4-D Euclidean space. For this space the volume is defined by $x^4 = \text{constant}$. Therefore, applying the relation in Eq. (2.98), an infinitesimal volume in this 4-D space is given by

$$dV = \sqrt{|g_{11}g_{22}g_{33}|} dx^1 dx^2 dx^3.$$  \hspace{1cm} (2.121)

Thus using

$$g_{11} = \frac{a^2}{a^2 - r^2}, g_{22} = r^2, g_{33} = r^2 \sin^2 (\theta)$$  \hspace{1cm} (2.122)

the volume bounded by the 2-D spherical surface of radius $x^1 = r = R$ becomes

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} \sqrt{\frac{a^2 r^4 \sin^2 (\theta)}{a^2 - r^2}} \ dr d\theta d\phi$$
$$V = a \int_0^R \frac{r^2 \, dr}{\sqrt{a^2 - r^2}} \int_0^\pi \int_0^{2\pi} \sin (\theta) \ d\theta d\phi$$  \hspace{1cm} (2.123)

or

$$V = a \int_0^R \frac{(x^1)^2 \ dx^1}{\sqrt{a^2 - (x^1)^2}} \int_0^\pi \int_0^{2\pi} \sin (x^2) \ dx^2 dx^3,$$

Let’s evaluate the integral

$$I = \int \frac{r^2 \, dr}{\sqrt{a^2 - r^2}}$$  \hspace{1cm} (2.124)

Introducing the transformation defined by

$$r = a \sin (\theta) \Rightarrow dr = a \cos (\theta) \ d\theta$$
we have

\[ I = \int \frac{r^2 dr}{\sqrt{a^2 - r^2}} = \int \frac{a^2 \sin^2 (\theta) a \cos (\theta) d\theta}{\sqrt{a^2 - a^2 \sin^2 (\theta)}} = a^2 \int \sin^2 (\theta) d\theta \]

\[ = \frac{a^2}{2} \int (1 - \cos (2\theta)) d\theta = \frac{a^2}{2} \left( \theta - \frac{\sin (2\theta)}{2} \right) \]

\[ \Rightarrow I = \int \frac{r^2 dr}{\sqrt{a^2 - r^2}} = \frac{a^2}{2} \left[ \theta - \sin (\theta) \cos (\theta) \right] \quad (2.125) \]

so that using

\[ r = a \sin (\theta) \Rightarrow \sin (\theta) = \begin{cases} \frac{R}{a}, & \text{for } r = R \\ 0, & \text{for } r = 0 \end{cases} \quad (2.126) \]

and

\[ \cos (\theta) = \sqrt{1 - \sin^2 (\theta)} = \begin{cases} \sqrt{1 - \left( \frac{R}{a} \right)^2}, & \text{for } r = R \\ 0, & \text{for } r = 0 \end{cases} \quad (2.127) \]

one finds

\[ \int_0^R \frac{r^2 dr}{\sqrt{a^2 - r^2}} = \frac{a^2}{2} \left[ \sin^{-1} \left( \frac{R}{a} \right) - \frac{R}{a} \sqrt{1 - \left( \frac{R}{a} \right)^2} \right] \quad (2.128) \]

Then the volume becomes

\[ V = 2\pi a^3 \left\{ \sin^{-1} \left( \frac{R}{a} \right) - \frac{R}{a} \sqrt{1 - \left( \frac{R}{a} \right)^2} \right\} \quad (2.129) \]

One must be able to recover the 3-D Euclidean space for \( a \to \infty \). This means \( \frac{R}{a} << 1 \) the result for the volume of a sphere with radius, \( R \) must be that of the volume of a sphere with radius \( R \) in 3D Euclidean space \( (V = \frac{4}{3} \pi R^3) \). One can easily find from Eq. (2.129) using the approximations for \( \frac{R}{a} << 1 \),

\[ \sin^{-1} \left( \frac{R}{a} \right) \approx \frac{R}{a} - \frac{1}{3} \left( \frac{R}{a} \right)^3 \cdot \sqrt{1 - \left( \frac{R}{a} \right)^2} = 1 - \frac{1}{2} \left( \frac{R}{a} \right)^2 \quad (2.130) \]

that gives

\[ V = 2\pi a^3 \left\{ \frac{R}{a} - \frac{1}{3} \left( \frac{R}{a} \right)^3 - \frac{R}{a} \left( 1 - \frac{1}{2} \left( \frac{R}{a} \right)^2 \right) \right\} \]

\[ = 2\pi a^3 \left\{ -\frac{1}{3} \left( \frac{R}{a} \right)^3 + \frac{1}{2} \left( \frac{R}{a} \right)^3 \right\} \Rightarrow V = \frac{4}{3} \pi R^3 \quad (2.131) \]
CHAPTER 2. MANIFOLDS

Homework:

(a) Express the circumference, the area, and the volume in terms of $D$ in the previous example and show that all have maximum values at

$$D = \frac{\pi a}{2} \quad (2.132)$$

(b) Show that the total volume of this space is finite and is equal to

$$V = 2\pi^2 a^3. \quad (2.133)$$

Homework: Determine the metric for a three-dimensional sphere with imaginary radius $a = ib$ embedded in a 4-D Euclidean space: and by considering a sphere defined by $r = R$ find

(a) Show that circumference, $C$, and the area, $A$ are still $C = 2\pi R$ and $A = 4\pi R^2$.

(b) The distance $D$ from the center of the sphere to the surface is

$$D = b \sinh^{-1}(R/b) \quad (2.134)$$

and in this case show that $A$ and $V$ of the sphere are monotonically increasing functions

2.7 Local Cartesian coordinates and tangent space

Generally $ds^2$, can be positive, negative, or zero as we saw in pseudo-Riemannian spaces, like the Minkowsky spacetime. For now we shall consider what normally refer as Riemannian where the metric

$$ds^2 = g_{cd}(x) dx^c dx^d, \quad (2.135)$$

is positive. It is not possible, in general, to find a coordinate transformation that transforms the metric into Euclidean form

$$ds^2 = (dx^1)^2 + (dx^2)^2 + ... + (dx^N)^2 = \delta_{ab} dx^a dx^b, \quad (2.136)$$

for all points on the manifold. However, it is possible to find coordinates $x'^a$ such that at the point $P$ the new metric functions $g'_{ab}(x)$ satisfy the conditions

$$g'_{ab}(x'^a_P) = \delta_{ab}, \quad (2.137)$$

$$\frac{\partial g'_{ab}(x)}{\partial x'^c} \bigg|_{x'^a_P} = 0. \quad (2.138)$$
Thus in the neighborhood of point \( P \), we have
\[
g'_{ab}(x') = \delta_{ab} + O \left[ (x' - x'_p)^2 \right]
\]

We recall from Mathematical methods for any function, \( g(x) \), that is differentiable for all values of \( x \) in the specified domain,
\[
\frac{d^n g(x)}{dx^n} \text{ exists for all } n \geq 0 \text{ and } x \in \mathbb{R},
\]
one can write the series expansion about \( x_p \) in the domain (Taylor series)
\[
g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n g(x')}{dx^n} \bigg|_{x=x_p} (x - x_p)^n. \tag{2.139}
\]

Considering only up to the second order terms,
\[
g(x) = g(x) + \left. \frac{dg(x')}{dx'} \right|_{x'=x_p} (x - x_p) + \frac{1}{2!} \left. \frac{dg(x')}{dx'^2} \right|_{x'=x_p} (x - x_p)^2 + \ldots \tag{2.140}
\]

For a function of two variables \( g(x) = g(x^1, x^2) \), this becomes
\[
g(x^1, x^2) = g(x^1_p, x^2_p) + \left. \frac{\partial g(x^1, x^2)}{\partial x^1} \right|_{x^1=x^1_p, x^2=x^2_p} (x^1 - x^1_p)
+ \frac{\partial g(x^1, x^2)}{\partial x^2} \bigg|_{x^1=x^1_p, x^2=x^2_p} (x^2 - x^2_p)^2
+ \frac{1}{2!} \left\{ \frac{\partial^2 g(x^1, x^2)}{\partial x^1 \partial x^2} \right|_{x^1=x^1_p, x^2=x^2_p} (x^1 - x^1_p)^2
+ 2 \frac{\partial^2 g(x^1, x^2)}{\partial x^1 \partial x^2} \bigg|_{x^1=x^1_p, x^2=x^2_p} (x^1 - x^1_p)(x^2 - x^2_p)
+ \frac{\partial^2 g(x^1, x^2)}{\partial (x^2)^2} \bigg|_{x^1=x^1_p, x^2=x^2_p} (x^2 - x^2_p)^2 \right\} + \ldots \tag{2.141}
\]

From this expression you can imagine how it gets nasty for a function of \( N \) variables, like the metric, \( g_{ab}(x) = g_{ab}(x^1, x^2, x^3, \ldots x^N) \).

**Example 2.8** Let’s reconsider the metric for 2D sphere embedded in a 3D manifold in Cartesian coordinates
\[
ds^2 = \left( dx^1 \right)^2 + \left( dx^2 \right)^2 + \left( \sqrt{\frac{x^1 dx^1 + x^2 dx^2}{\sqrt{a^2 - (x^1)^2 + (x^2)^2}}} \right)^2. \tag{2.142}
\]

which can be put in the form
\[
ds^2 = \left[ 1 + \frac{(x^1)^2}{a^2 - (x^1)^2 - (x^2)^2} \right] \left( dx^1 \right)^2 + \left[ 1 + \frac{(x^2)^2}{a^2 - (x^1)^2 - (x^2)^2} \right] \left( dx^2 \right)^2
+ \frac{2 x^1 x^2}{a^2 - (x^1)^2 - (x^2)^2} dx^1 dx^2. \tag{2.143}
\]
where \((x^1, x^2, x^3)\) corresponds to the usual Cartesian coordinates \((x, y, z)\).

If one picks a point \(P\) with coordinates \((x^1_p, x^2_p)\) so that one may introduce the transformation defined by

\[
x^1 = x^1 - x^1_p, x^2 = x^2 - x^2_p \Rightarrow dx^1 = dx^1, dx^2 = dx^2
\]

and express the metric as

\[
ds^2 = g_{11} \left( dx^1 \right)^2 + g_{22} \left( dx^2 \right)^2 + g_{12} dx^1 dx^2.
\]

where

\[
g_{11} (x^1, x^2) = 1 + \frac{\left( x^1 - x^1_p \right)^2}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2},
\]

\[
g_{22} (x^1, x^2) = 1 + \frac{\left( x^2 - x^2_p \right)^2}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2}
\]

\[
g_{12} (x^1, x^2) = \frac{2 \left( x^1 - x^1_p \right) \left( x^2 - x^2_p \right)^2}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2}
\]

Show that the metric is Locally Cartesian at point \(P\).

**Solution:** For a locally Cartesian, one must be able to show that at point \(P\)

\[
g'_{ab} (x^a_p) = \frac{\partial g'_{ab} (x^i)}{\partial x^c} \bigg|_{x^i_p} = 0.
\]

We note that for the 2D sphere in the 3D manifold, using the metric we find

\[
g_{11} (x^1_p, x^2_p) = g_{22} (x^1_p, x^2_p) = 1, g_{12} (x^1_p, x^2_p) = 0
\]

and

\[
\left. \frac{\partial g_{11} (x^1, x^2)}{\partial x^1} \right|_{x^1_p, x^2_p} = \frac{\partial}{\partial x^1} \left[ \frac{\left( x^1 - x^1_p \right)^2}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2} \right]
\]

\[
= \left[ \frac{2 \left( x^1 - x^1_p \right)}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2} + \frac{2 \left( x^1 - x^1_p \right)^3}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2} \right] \bigg|_{x^1_p, x^2_p} = 0
\]

Similarly one can easily show that

\[
\left. \frac{\partial g_{11} (x^1, x^2)}{\partial x^2} \right|_{x^1_p, x^2_p} = \left. \frac{\partial}{\partial x^2} \left[ \frac{\left( x^1 - x^1_p \right)^2}{a^2 - \left( x^1 - x^1_p \right)^2 - \left( x^2 - x^2_p \right)^2} \right] \right|_{x^1_p, x^2_p} = 0
\]
2.7. LOCAL CARTESIAN COORDINATES AND TANGENT SPACE

The line element is Euclidean. That means the line element is

\[ ds = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^N)^2 \]

this space is called the tangent space, \( T_p \). You can see in Fig.2.10, three different two-dimensional Tangent spaces (shown in red with green rectangular grids) at three different points on a 2D surface embedded in a 3D manifold. Fig.2.10 In each of these tangent spaces in Fig. 2.10, note that

\[ ds^2 = (dx^1)^2 + (dx^2)^2. \]  

Tangent Space to Manifolds: for an arbitrary point \( P \) in an \( N \)-dimensional Riemannian manifold we can find a space that consist of coordinates \( (x^a) \) such that in the Neighborhood of \( P \) the line element is Euclidean. That means the line element is

\[ ds^2 = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^N)^2 \]

this space is called the tangent space, \( T_p \). You can see in Fig.2.10, three different two-dimensional Tangent spaces (shown in red with green rectangular grids) at three different points on a 2D surface embedded in a 3D manifold. Fig.2.10 In each of these tangent spaces in Fig. 2.10, note that

\[ ds^2 = (dx^1)^2 + (dx^2)^2. \]  

as you can see from the rectangular shape of the grids. We can show this quantitatively using the function that defines the surface. The surface shown in Fig. 2.10 is defined by the function

\[ x^3 (x^1, x^2) = \sin^2(x^1) + \cos(x^2) \]  

which one may write as

\[ g (x^1, x^2, x^3) = \sin^2(x^1) + \cos(x^2) - x^3 = 0 \]
so that

\[
dg = \frac{\partial g}{\partial x^1} dx^1 + \frac{\partial g}{\partial x^2} dx^2 + \frac{\partial g}{\partial x^3} dx^3 = 0
\]

\[
\Rightarrow dg = \left( \frac{\partial g}{\partial x^1} \hat{x} + \frac{\partial g}{\partial x^2} \hat{y} + \frac{\partial g}{\partial x^3} \hat{z} \right) \cdot (dx^1 \hat{x} + dx^2 \hat{y} + dx^3 \hat{z}) = 0
\]

\[
dg = \vec{A} (x^1, x^2, x^3) \cdot d\vec{r} (x^1, x^2, x^3) = 0, \quad (2.151)
\]

where

\[
\vec{A} (x^1, x^2, x^3) = \frac{\partial g}{\partial x^1} \hat{x} + \frac{\partial g}{\partial x^2} \hat{y} + \frac{\partial g}{\partial x^3} \hat{z}
\]

\[
\frac{\partial g}{\partial x^1} = 2 \sin(x^1) \cos(x^1), \quad \frac{\partial g}{\partial x^2} = - \sin(x^2), \quad \frac{\partial g}{\partial x^3} = -1 \quad (2.152)
\]

Note Eq. (2.151) shows that the vector \( \vec{A} (x^1, x^2, x^3) \) is normal to the surface at the point with coordinates \((x^1, x^2, x^3)\). Now let’s pick a point, \( P \), on the surface with coordinates \( \vec{r}_p = (x^1_p, x^2_p, x^3_p) \) and another neighboring point \( Q \) with coordinates \( \vec{r} = (x^1, x^2, x^3) \) that are at the same plane that is tangent to the surface at point \( P \), then we have for the vector on this plane given by

\[
\Delta \vec{r} = \vec{r} - \vec{r}_p = (x^1 - x^1_p) \hat{x} + (x^2 - x^2_p) \hat{y} + (x^3 - x^3_p) \hat{z} \quad (2.153)
\]

and the vector normal to this tangent plane at point \( P \) is given by

\[
\vec{A} (x^1_p, x^2_p, x^3_p) = \left[ \frac{\partial g}{\partial x^1} \hat{x} + \frac{\partial g}{\partial x^2} \hat{y} + \frac{\partial g}{\partial x^3} \hat{z} \right]_{x^1_p, x^2_p, x^3_p} = m_1 \hat{x} + m_2 \hat{y} - \hat{z} \quad (2.154)
\]
where
\[ m_1 = 2 \sin(x_1^p) \cos(x_1^p), \quad m_2 = -\sin(x_2^p) \quad (2.155) \]

The equation of the tangent plane at this point is determined by
\[
\Delta g = \hat{A}(x^1, x^2, x^3) \cdot \Delta r(x^1, x^2, x^3)
= m_1 (x^1 - x_1^p) + m_2 (x^2 - x_2^p) - (x^3 - x_3^p) = 0
\Rightarrow x^3(x^1, x^2) = m_1(x^1 - x_1^p) + m_2(x^2 - x_2^p) + \sin^2(x_1^p) + \cos(x_2^p) \quad (2.156)
\]
or
\[
x^3(x^1, x^2) = 2 \sin(x_1^p) \cos(x_1^p) (x^1 - x_1^p) - \sin(x_2^p)(x^2 - x_2^p) + x_3^p. \quad (2.157)
\]

Introducing the coordinate transformation defined by
\[
x'^1 = x^1 - x_1^p, x'^2 = x^2 - x_2^p, x'^3 = x_3^p
\Rightarrow dx'^1 = dx'^1, dx'^2 = dx'^2, dx'^3 = 0 \quad (2.158)
\]
then the metric in the tangent space becomes
\[
(ds)^2 = (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2 = (dx'^1)^2 + (dx'^2)^2 \quad (2.159)
\]

## 2.8 The signature of a manifold

Let’s consider an arbitrary point \( P \) in the pseudo-Riemannian manifolds described by the coordinates, \( x^a \), and make a transformation to local Cartesian coordinates, \( x'^a \), at this point. Suppose this transformation, which satisfies the conditions in Eq. (2.137) and (2.138), is given by
\[
g'_{cd}(x') = g_{ab}(x) \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}. \quad (2.160)
\]

Then at point, \( P \), if the coordinates are \( x'_p = x'^1_p, x'^2_p, x'^3_p, \ldots x'^N_p \),
\[
g'_{cd}(x_p) = g_{ab}(x_p) \left. \left( \frac{\partial x^a}{\partial x'^c} \right) \right|_{x'_p} \left. \left( \frac{\partial x^b}{\partial x'^d} \right) \right|_{x'_p}, \quad (2.161)
\]
and if the transformation leads to Local Cartesian, we must have
\[
g'_{cd}(x_p) = \delta_{cd} \lambda_e \left. \left( \frac{\partial g'_{ab}(x')}{\partial x'^c} \right) \right|_{x'_p, x'_e} = 0, \quad (2.162)
\]
for pseudo-Riemannian manifolds in general. In fact \( \lambda_e = 1 \) for Local Cartesian in a Riemannian manifolds. Therefore, the transformation metric
\[
g'_{cd}(x') = g_{ab}(x') \left. \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} \right|_{x'_p} \quad (2.163)
\]
CHAPTER 2. MANIFOLDS

for local Cartesian at point $P$, one can write

$$g'_{cd}(x_p) = \delta_{cd} \gamma_c = g_{ab}(x') \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} |_{x_p} \tag{2.164}$$

$$\Rightarrow g'_{cd}(x_p) = \frac{\partial x^a}{\partial x'^c} |_{x_p} g_{ab}(x_p) \frac{\partial x^b}{\partial x'^d} |_{x_p}. \tag{2.165}$$

This can be put using matrices as

$$G' = X^T GX. \tag{2.166}$$

One can easily show this by considering, $N = 2$, and the matrices

$$G' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, G = \begin{bmatrix} g_{11}(x^1_p, x^2_p) & g_{12}(x^1_p, x^2_p) \\ g_{21}(x^1_p, x^2_p) & g_{22}(x^1_p, x^2_p) \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{\partial x^1(x^1, x^2)}{\partial x^1} |_{x^1_p, x^2_p} & \frac{\partial x^1(x^1, x^2)}{\partial x^2} |_{x^1_p, x^2_p} \\ \frac{\partial x^2(x^1, x^2)}{\partial x^1} |_{x^1_p, x^2_p} & \frac{\partial x^2(x^1, x^2)}{\partial x^2} |_{x^1_p, x^2_p} \end{bmatrix},$$

$$X^T = \begin{bmatrix} \frac{\partial x^1(x^1, x^2)}{\partial x^1} |_{x^1_p, x^2_p} & \frac{\partial x^2(x^1, x^2)}{\partial x^1} |_{x^1_p, x^2_p} \\ \frac{\partial x^1(x^1, x^2)}{\partial x^2} |_{x^1_p, x^2_p} & \frac{\partial x^2(x^1, x^2)}{\partial x^2} |_{x^1_p, x^2_p} \end{bmatrix}. \tag{2.167}$$

Since the matrix $G'$ is a diagonal matrix, the transformation is a similarity transformation (Mathematical Methods II). This means $X$ forms the eigenvector matrix and $G'$ forms the eigenvalue matrix and

$$X^T = X^{-1} \Rightarrow G' = X^{-1} GX. \tag{2.168}$$

Using the inverse transformation matrix, $X'$, and the corresponding inverse matrix, $X'^{-1}$, one must recover the matrix $G$,

$$G = X'^{-1} G' X' = X'^{-1} X^{-1} GX X' \tag{2.169}$$

Therefore we must have

$$XX' = I \Rightarrow X' = X^{-1}. \tag{2.170}$$

This can be true if the transformation from $x^a \rightarrow x'^a$ is linear. The transformation matrix, $X$, with element $X^a_b$, at point $P$ each element must be a constant and we can write

$$x'^a = X^a_b x^b. \tag{2.171}$$
2.8. THE SIGNATURE OF A MANIFOLD

Since the metric tensor is symmetric it can be diagonalized by a similarity transformation provided we chose the columns of $X$ to be the normalized eigenvectors of the matrix $G$. This means

$$G' = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_N
\end{bmatrix},$$

(2.172)

where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of the matrix $G$. The metric

$$ds^2 = g_{cd}(x') dx'^c dx'^d,$$

(2.173)

is positive for strictly Riemannian. This means $g_{cd}(x)$ must be positive definite and the eigenvalues, $\lambda_a$, for $G$ are also positive definite. On the other hand for Pseudo Riemannian since the metric can be negative the eigenvalues can be negative. Now if we scale the coordinates $x'^a$ by these eigenvalues (i.e. $x'^a \rightarrow x'^a/\sqrt{|\lambda_a|}$), for the metric tensor, $G'$, we can easily show

$$G' = \begin{bmatrix}
\pm 1 & 0 & \ldots & 0 \\
0 & \pm 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \pm 1
\end{bmatrix}.$$

(2.174)

Thus, at any arbitrary point, $P$, in a pseudo-Riemannian manifold, it is always possible to find a coordinate system $x'^a$ such that in the neighborhood of $P$ we have

$$g'_{ab}(x') = \eta_{ab} + O \left[(x' - x'_p)^2\right],$$

(2.175)

where $[\eta_{ab}] = diag(\pm 1, \pm 1, \ldots, \pm 1)$. The number of positive entries $(N_+)$ minus the number of negative entries $(N_-)$ in $[\eta_{ab}]$ is called the signature of the manifold. For example for the Minkowski spacetime manifold where the metric is given by

$$ds^2 = d(ct)^2 - dx^2 - dy^2 - dz^2,$$

(2.176)

where

$$\eta = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

(2.177)

the signature is $-2$.

N-dimensional volume without a constraint: In an N-dimensional (pseudo) Riemannian manifold with orthogonal coordinates system where the metric tensor is diagonal, the full N-dimensional volume element $d^N V$ is

$$d^N V = \sqrt{|g|} dx^1 dx^2 dx^3 \ldots dx^N$$

(2.178)
where $|g|$ is the determinant of the matrix

$$G = [g_{ab}] = \begin{bmatrix}
g_{11} & 0 & \cdots & 0 \\
0 & g_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & g_{NN}
\end{bmatrix} \quad (2.179)$$
Chapter 3

Vector Calculus on manifolds

3.1 The tangent vector

The tangent vector, \( \vec{t} \), at point \( p \) on a Manifold is the vector that lies in the tangent space, \( T_p \), at that point, \( p_1 \), and is given by

\[
\vec{t} = \lim_{\delta u \to 0} \frac{\delta s}{\delta u} = \lim_{\delta u \to 0} \frac{s(u + \delta u) - s(u)}{\delta u},
\]

where \( \delta s \) is the infinitesimal separation vector between the point \( P \) with coordinate \((u, s(u))\) and some nearby point \( Q \) with coordinate \((u + \delta u, s(u + \delta u))\) on the curve, \( C \), in the manifold corresponding (see Fig. ??).
3.2 The basis vectors

At each point $P$ on a manifold we can define a set of linearly independent basis vectors, $\hat{e}_a(x)$, for the tangent space, $T_p$. The number of the basis vectors is equal to the dimension of $T_p$. Any vector field at point $P$, $\vec{v}(x)$, can then be expressed as linear combination of these basis vectors

$$\vec{v}(x) = v^a(x) \hat{e}_a(x),$$  \hspace{1cm} (3.2)

where $v^a(x)$ are known as the contravariant components of the vector field, $\vec{v}(x)$, in the basis $\hat{e}_a$. For any set of basis vectors we can define another set of basis vectors known as the dual basis vectors, $\hat{e}^a(x)$, defined by

$$\hat{e}^a(x) \cdot \hat{e}_b(x) = \delta^a_b.$$  \hspace{1cm} (3.3)

The dual basis vectors, $\hat{e}^a$, and the basis vectors, $\hat{e}_b$, form a reciprocal system of vectors. The local vector field $\vec{v}(x)$ can also be expressed in terms of the dual basis vectors as

$$\vec{v}(x) = v_a(x) \hat{e}^a(x),$$  \hspace{1cm} (3.4)

where $v_a(x)$ are known as the covariant components of the vector field, $\vec{v}(x)$, in the dual basis vectors, $\hat{e}^a(x)$. The contravariant and covariant components of the vector field can be determined using Eq. (3.3)

$$\vec{v}(x) \cdot \hat{e}^b(x) = v^a(x) \hat{e}_a(x) \cdot \hat{e}^b(x) = v^a(x) \delta^a_b = v_b(x).$$  \hspace{1cm} (3.5)

Similarly

$$v_a(x) = \vec{v}(x) \cdot \hat{e}_a(x).$$  \hspace{1cm} (3.6)

The coordinate basis vectors: in any particular coordinate system $x^a$, we can define at every point $P$ of the manifold a set of $N$ coordinate basis vectors

$$\hat{e}_a = \lim_{\delta x^a \to 0} \frac{\delta \hat{s}}{\delta x^a},$$  \hspace{1cm} (3.7)

where $\delta \hat{s}$ is the infinitesimal separation vector between point $P$ and some nearby point $Q$ with coordinate separation $\delta x^a$ from $P$. For example, $\hat{e}_a$ is the tangent vector to the $x^a$ coordinate curve at the point $P$. As an example we reconsider the 2D (blue) surface embedded in a 3D Manifold shown in Fig. 3.2.

We recall that this surface is defined by

$$x^3(x^1, x^2) = \sin^2(x^1) + \cos(x^2)$$  \hspace{1cm} (3.8)

where $x^1$, $x^2$, and $x^3$ are the usual Cartesian coordinates $(x, y)$. The tangent plane at point $P$ is given by

$$x^3(x^1, x^2) = 2 \sin(x^1_p) \cos(x^1_p) (x^1 - x^1_p) - \sin(x^2_p) (x^2 - x^2_p) + x^3_p.$$  \hspace{1cm} (3.9)

where $(x^1_p, x^2_p, x^3_p)$ are the coordinates for point $P$. We note that a point on the surface can be defined by a vector

$$\hat{s} = x^1 \hat{x} + x^2 \hat{y} + (\sin^2(x^1) + \cos(x^2)) \hat{z}$$  \hspace{1cm} (3.10)
so that

\[ \delta \mathbf{s} = \delta x^1 \hat{x} + \delta x^2 \hat{y} + \left( 2 \sin(x^1) \cos(x^1) \delta x^1 - \sin(x^2) \delta x^2 \right) \hat{z} \]  

(3.11)

The two basis vectors, \( \hat{\mathbf{e}}_1 \) and \( \hat{\mathbf{e}}_2 \), at point \( P \), in Fig. 3.2 are given by

\[
\hat{\mathbf{e}}_1 = \lim_{\delta x^1 \to 0} \frac{\delta \mathbf{s}}{\delta x^1} = \lim_{\delta x^1 \to 0} \left[ \hat{x} + \frac{\partial \mathbf{s}}{\partial x^1} \hat{y} + \left( 2 \sin(x^1) \cos(x^1) \frac{\partial x^1}{\partial x^1} - \sin(x^2) \frac{\partial x^2}{\partial x^1} \right) \hat{z} \right] (3.12)
\]

\[
\hat{\mathbf{e}}_2 = \lim_{\delta x^2 \to 0} \frac{\delta \mathbf{s}}{\delta x^2} = \lim_{\delta x^2 \to 0} \left[ \frac{\partial x^1}{\partial x^2} \hat{x} + \hat{y} + \left( 2 \sin(x^1) \cos(x^1) \frac{\partial x^1}{\partial x^2} - \sin(x^2) \frac{\partial x^2}{\partial x^2} \right) \hat{z} \right] (3.13)
\]

and noting that

\[ \frac{\partial x^1}{\partial x^2} = \frac{\partial x^2}{\partial x^2} = 0 \]

we find

\[ \hat{\mathbf{e}}_1 = \hat{x} + 2 \sin(x^1) \cos(x^1) \hat{z}, \hat{\mathbf{e}}_2 = \hat{y} - \sin(x^2) \hat{z} \]  

(3.14)

### 3.3 The metric function and coordinate transformations

**Infinitesimal vector separation:** Consider two points \( P \) and \( Q \) on a manifold with coordinates \( x^a \) and \( x^a + dx^a \), where \( dx^a \) is none zero for all \( a \), then the infinitesimal vector separation between these two points is given by

\[ d\mathbf{s} = \hat{e}_a(x) \, dx^a \]  

(3.15)
The metric function—the covariant components: The equation that determines the elements of the metric tensor in the metric
\[ ds^2 = g_{ab} (x) dx^a dx^b \] (3.16)
can be obtained from the inner product of the infinitesimal vector separation. We note that
\[
\begin{align*}
    ds^2 &= \hat{s} \cdot \hat{s} = \hat{e}_a (x) dx^a \cdot \hat{e}_b (x) dx^b = \hat{e}_a (x) \cdot \hat{e}_b (x) dx^a dx^b \\
    \Rightarrow ds^2 &= g_{ab} (x) dx^a dx^b,
\end{align*}
\]
where
\[ g_{ab} (x) = \hat{e}_a (x) \cdot \hat{e}_b (x) \]
is the metric function.

Example 3.1 Let’s reconsider the 2D sphere of radius, \( a \), in a 3D Euclidean space. For an origin set at the north pole of the sphere, a point \( P \) is described by the vector
\[
\hat{s} = \rho \cos (\varphi) \hat{x} + \rho \sin (\varphi) \hat{y} + \sqrt{a^2 - \rho^2} \hat{z}
\]

(a) Find the basis vectors \( \hat{e}_1' \) and \( \hat{e}_2' \) in the tangent space at point \( P \).
3.3. THE METRIC FUNCTION AND COORDINATE TRANSFORMATIONS

(b) Re-derive the metric elements for a 2D sphere from the basis vectors.

Solution:

(a) We note that the tangent vector connecting point \( P \) with coordinate \((\rho, \varphi)\) with its neighboring point \( Q \) with coordinates \((\rho + \delta \rho, \varphi + \delta \varphi)\) (See Fig. 3.4) is expressible as

\[
\delta \mathbf{s} = \delta (\rho \cos (\varphi)) \mathbf{i} + \delta (\rho \sin (\varphi)) \mathbf{j} + \delta \left( \sqrt{a^2 - \rho^2} \right) \mathbf{k}
\]

\[
= \left[ \cos (\varphi) \delta \rho - \rho \sin (\varphi) \delta \varphi \right] \mathbf{i} + \left[ \sin (\varphi) \delta \rho + \rho \cos (\varphi) \delta \varphi \right] \mathbf{j} - \frac{\rho d\rho}{\sqrt{a^2 - \rho^2}} \mathbf{k}
\]

(3.18)

\[
= \left[ \cos (\varphi) \mathbf{i} + \sin (\varphi) \mathbf{j} - \frac{\rho}{\sqrt{a^2 - \rho^2}} \mathbf{k} \right] \delta \rho + \left[ -\rho \sin (\varphi) \mathbf{i} + \rho \cos (\varphi) \mathbf{j} \right] \delta \varphi
\]

(3.19)

There follows that

\[
\hat{e}_\rho = \lim_{\delta \rho \to 0} \frac{\delta \mathbf{s}}{\delta \rho}
\]

\[
= \left[ \cos (\varphi) \mathbf{i} + \sin (\varphi) \mathbf{j} - \frac{\rho}{\sqrt{a^2 - \rho^2}} \mathbf{k} \right] + \left[ -\rho \sin (\varphi) \mathbf{i} + \rho \cos (\varphi) \mathbf{j} \right] \frac{\delta \varphi}{\delta \rho}
\]

(3.20)

and

\[
\hat{e}_\varphi = \lim_{\delta \varphi \to 0} \frac{\delta \mathbf{s}}{\delta \varphi}
\]

\[
= \left[ \cos (\varphi) \mathbf{i} + \sin (\varphi) \mathbf{j} - \frac{\rho}{\sqrt{a^2 - \rho^2}} \mathbf{k} \right] \frac{\delta \rho}{\delta \varphi} + \left[ -\rho \sin (\varphi) \mathbf{i} + \rho \cos (\varphi) \mathbf{j} \right]
\]

(3.21)

Noting that

\[
\frac{\delta \rho}{\delta \varphi} = \frac{\delta \rho}{\delta \varphi} = 0
\]

(3.22)

we find

\[
\hat{e}_\rho = \cos (\varphi) \mathbf{i} + \sin (\varphi) \mathbf{j} - \frac{\rho}{\sqrt{a^2 - \rho^2}} \mathbf{k}
\]

(3.23)

and

\[
\hat{e}_\varphi = -\rho \sin (\varphi) \mathbf{i} + \rho \cos (\varphi) \mathbf{j}
\]

(3.24)
Figure 3.4: The basis vectors \((\hat{e}_\rho, \hat{e}_\varphi)\) at a point \(P\) in the tangent space of a 2D sphere embedded in a 3D Euclidean space.

(b) The metric elements can be determined using the basis vectors

\[
g_{ab}(x) = \hat{e}_a(x) \cdot \hat{e}_b(x),
\]

and one finds

\[
g_{11} = g_{\rho\rho} = \hat{e}_\rho \cdot \hat{e}_\rho = 1 - \frac{\rho^2}{a^2 - \rho^2} = \frac{a^2}{a^2 - \rho^2},
\]

\[
g_{22} = g_{\varphi\varphi} = \hat{e}_\varphi \cdot \hat{e}_\varphi = \rho^2 g_{12} = g_{21} = 0
\]

Homework: Consider the 3D sphere of radius, \(a\), in a 4D Euclidean space. A point \(P\) on this 3D sphere is described by the vector

\[
\vec{s} = r \sin(\theta) \cos(\varphi) \hat{x} + r \sin(\theta) \sin(\varphi) \hat{y} + r \cos(\theta) \hat{z} + \sqrt{a^2 - \rho^2} \hat{w}
\]

(a) Find the basis vectors \(\vec{e}_r, \vec{e}_\theta,\) and \(\vec{e}_\varphi\) in the tangent space at point \(P\).

(b) Re-derive the metric elements for a 3D sphere from the basis vectors.

The metric function-the covariant components: noting that the infinitesimal distance between point \(P\) and \(Q\) can be expressed using the infinitesimal vector separation

\[
d\vec{s} = \vec{e}_a(x) dx^a
\]
we have
\[ \hat{e}^b(x) \cdot ds = \hat{e}^b(x) \cdot \hat{e}_a(x) \, dx^a = \delta^b_a \, dx^a = dx^b. \]  
(3.29)

Expressing the infinitessimal vector separation using the dual coordinates basis vectors as
\[ ds = \hat{e}^a(x) \, dx^a \]  
(3.30)

we also find
\[ \hat{e}_b(x) \cdot ds = \hat{e}_b(x) \cdot \hat{e}^a(x) \, dx^a = \delta^b_a \, dx^a = dx^b. \]  
(3.31)

Thus the metric
\[ ds^2 = ds \cdot ds = \hat{e}^a(x) \, dx^a \cdot \hat{e}^b(x) \, dx^b = g^{ab}(x) \, dx^a \, dx^b, \]  
(3.32)

where the controvariant components of the metric tensor can be defined as
\[ g^{ab}(x) = \hat{e}^a(x) \cdot \hat{e}^b(x) \]  
(3.33)

**Orthonormal basis vector:** at a point on a manifold an orthonormal basis vectors are defined by
\[ \hat{e}_a(x) \cdot \hat{e}_b(x) = \eta_{ab}, \]  
(3.34)

where
\[ [\eta_{ab}] = \begin{pmatrix} \pm1 & 0 & \cdots & 0 \\ 0 & \pm1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm1 \end{pmatrix}. \]  
(3.35)

or in short \([\eta_{ab}] = diag(\pm1, \pm1, \ldots, \pm1)\) is the Cartesian line element of the tangent space, \(T_p\), and depends on the signature of the pseudo-Riemannian manifold.

**Basis vectors and coordinate transformations:** Suppose we make a coordinate transformation from \(x^a\) where the basis vectors are \(\hat{e}_a\) to a new another coordinates \(x'^a\) we want to find how the new basis vectors are transformed into, \(\hat{e}'_a\). If we consider point \(P\) and another point \(Q\) at an infinitesimal distance away from point \(P\). The infinitesimal displacement, \(ds\), between these points is independent of the coordinate transformation and we must have
\[ ds = \hat{e}_a dx^a = \hat{e}'_c dx'^c. \]  
(3.36)

Using
\[ dx^a = \frac{\partial x^a}{\partial x'^b} dx'^b, \]  
(3.37)

we have
\[ ds = \hat{e}_a dx^a = \frac{\partial x^a}{\partial x'^b} \hat{e}_a dx'^b = \hat{e}'_c dx'^c. \]  
(3.38)

Noting that
\[ \hat{e}_b \cdot \hat{e}^b = 1 \]
we have
\[
\frac{\partial x^a}{\partial x'^b} \hat{e}_a dx^b = (\hat{e}'_b \cdot \hat{e}'_c) \hat{e}'_d dx^c = (\hat{e}'_c \cdot \hat{e}'_d) \hat{e}'_b dx^d = \delta'_b \delta'_c dx^c \tag{3.39}
\]
so that
\[
\hat{e}'_b = \frac{\partial x^a}{\partial x'^b} \hat{e}_a. \tag{3.40}
\]

**Homework: Show that for the dual basis vector**
\[
\hat{e}^e = \frac{\partial x'^a}{\partial x^c} \hat{e}^e. \tag{3.41}
\]

**Components of a vector in coordinate transformations:** In coordinate transformations the vector components are different but the vector itself is unchanged. Suppose the vector, \( \vec{v} \), is a vector at point \( P \) in the \( x^a \) coordinate system and \( \vec{v}' \) is the vector in the \( x'^a \) coordinates at the same point on the manifold. These vectors may be expressed in terms of the basis vectors in the two coordinates differently
\[
\vec{v} = v^a \hat{e}_a, \quad \vec{v}' = v'^a \hat{e}'_a \tag{3.42}
\]
or in terms of the dual basis vectors
\[
\vec{v} = v_a \hat{e}^a, \quad \vec{v}' = v'^b \hat{e}'^b. \tag{3.43}
\]
But the vector is the same since it describes a geometrical entity that is independent of the coordinate system. Therefore, we must have
\[
\vec{v} = v^a \hat{e}_a = v'^a \hat{e}'_a, \tag{3.44}
\]
so that taking the inner product of \( \vec{v} \) and \( \hat{e}'_b \), we can write
\[
v^a \hat{e}'_b \cdot \hat{e}_a = v'^a \hat{e}'_b \cdot \hat{e}'_d \Rightarrow v'^a \delta'_a = v^a \hat{e}'_b \cdot \hat{e}_a \Rightarrow v'^b = v^a \hat{e}'_b \cdot \hat{e}_a. \tag{3.45}
\]
Applying the relation in Eq. (3.42), one can write
\[
\hat{e}'_b = \frac{\partial x'^b}{\partial x^c} \hat{e}^c, \tag{3.46}
\]
so that
\[
v'^b = v^a \frac{\partial x'^b}{\partial x^c} \hat{e}^c \cdot \hat{e}_a = v^a \frac{\partial x'^b}{\partial x^c} \delta'_a = \frac{\partial x'^b}{\partial x^a} v^a. \tag{3.47}
\]

**Homework: Show that for the covariant components of a vector transformed by the equation**
\[
v'_b = \frac{\partial x^a}{\partial x'^b} v_a. \tag{3.48}
\]
3.3. THE METRIC FUNCTION AND COORDINATE TRANSFORMATIONS

3.3.1 Raising and lowering vector indices

The scalar product: The scalar product of two vectors at a point, $P$, on a manifold

$$\vec{v} = v^a \hat{e}_a. \tag{3.50}$$

and

$$\vec{w} = w^b \hat{e}_b. \tag{3.51}$$

is given by

$$\vec{v} \cdot \vec{w} = v^a \hat{e}_a \cdot w^b \hat{e}_b = g_{ab} v^a w^b. \tag{3.52}$$

where

$$g_{ab} = \hat{e}_a \cdot \hat{e}_b. \tag{3.53}$$

is the covariant components of the metric tensor. We can also use the dual basis vectors to express the vectors $\vec{v}(x)$ and $\vec{w}(x)$

$$\vec{v} = v_a \hat{e}^a. \tag{3.54}$$

and

$$\vec{w} = w_b \hat{e}^b. \tag{3.55}$$

so that the inner product becomes

$$\vec{v} \cdot \vec{w} = v_a \hat{e}^a \cdot w_b \hat{e}^b = g^{ab} v_a w_b. \tag{3.56}$$

where

$$g^{ab} = \hat{e}^a \cdot \hat{e}^b. \tag{3.57}$$

We can use the covariant and contravariant components of the vectors to determine the inner products

$$\vec{v} \cdot \vec{w} = v^a \hat{e}_a \cdot w_b \hat{e}^b = \hat{e}_a \cdot \hat{e}^b v^a w_b = \delta_a^b v^a w_b = v^a w_a \tag{3.58}$$

or

$$\vec{v} \cdot \vec{w} = v_a \hat{e}^a \cdot w_b \hat{e}_b = \hat{e}^a \cdot \hat{e}_b v_a w_b = \delta_b^a v_a w_b = v_a w^a. \tag{3.59}$$

Whichever way we determine the inner products we must get the same values. Thus from Eqs. (3.52) and (3.58), we find

$$\vec{v} \cdot \vec{w} = g_{ab} v^a w^b = v^a w_a \Rightarrow g_{ab} w^b = w_a. \tag{3.60}$$

Similarly, from Eqs. (3.56) and (3.59), we find

$$\vec{v} \cdot \vec{w} = g^{ab} v_a w_b = v_a w^a \Rightarrow g^{ab} w_b = w^a. \tag{3.61}$$

From Eq. (3.60), we note that the covariant form of the metric tensor can be used to lower an index and from Eq. (3.61) we also see that the contravariant form of the metric tensor can be used to raise an index. Applying Eqs. (3.60) and (3.61), we can express the basis vectors

$$\hat{e}_a = g_{ad} \hat{e}^d \tag{3.62}$$
and
\[ \hat{e}^c = g^{cb} \hat{e}_b. \]  
(3.63)

Then noting that the inner product
\[ \hat{e}_a \cdot \hat{e}^c = \delta^c_c \Rightarrow g_{ad} \hat{e}^d \cdot g^{cb} \hat{e}_b = g_{ad} g^{cb} \delta^d_c = \delta^a_c \]
\[ \Rightarrow g_{ad} g^{cb} \hat{e}^d \cdot \hat{e}_b = g_{ad} g^{cb} \delta^d_b = \delta^a_c \]
\[ \Rightarrow g_{ab} g^{cb} = g^{cb} g_{ab} = \delta^a_c. \]  
(3.64)

This means the metric tensor \([g^{ab}]\) with the contravariant components, \(g^{ab}\), is the inverse matrix of the metric tensor \([g_{ab}]\) with the covariant elements, \(g_{ab}\).

Thus
\[ G \tilde{G} = \tilde{G} G = I, \]
where \(G = [g_{ab}]\) is the metric tensor and \(\tilde{G} = [g^{ab}]\) is its inverse.

### 3.4 The inner product and null vectors

The scalar product of two vectors at a point on a manifold which can be expressed in four different ways
\[ g_{ab} v^a w^b = v_a w^a = g^{ab} v_a w_b = v^a w_a \]  
(3.65)

Suppose we take the scalar product of vector, \(\tilde{v}\), with itself, we have
\[ g_{ab} v^a v^b = v^{ab} v_a = v_a v^a, \]  
(3.66)

and it can be zero without the vector being actually be a zero vector. We can see this if we recall the pseudo-Riemannian manifold metric
\[ ds^2 = g_{ab} (x) dx^a dx^b \]
which could be zero or negative. To accommodate such kind of vectors we define the length of a vector, \(\tilde{v}\), as
\[ v = \sqrt{[g_{ab} v^a v^b]} = \sqrt{|g^{ab} v_a v_b|} = \sqrt{|v^a v_a|} = \sqrt{|v_a v^a|}. \]  
(3.67)

As is the case in pseudo-Riemannian manifold, the length of a vector can be zero without the vector being actually be a zero vector (i.e. \(v_a \neq 0\)). Vectors with length (magnitude) zero with none zero components is known as **null vectors**.

**The cosine angle between vectors**: The angle between two non-null vectors at a point on a manifold is defined by:
\[ \cos (\theta) = \frac{v^a w_a}{\sqrt{|v_b v^b|} \sqrt{|w_a w^a|}} \]  
(3.68)

In the pseudo-Riemannian manifold, Eq. (3.68) can lead to \(|\cos (\theta)| > 1\).
3.5. THE AFFINE CONNECTIONS

Orthogonal vectors: two vectors

\[ \vec{v} = v^a \hat{e}_a. \]  \hspace{1cm} (3.69)

and

\[ \vec{w} = w^b \hat{e}_b. \]  \hspace{1cm} (3.70)

are said to be orthogonal when

\[ g_{ab} v^a w^b = g^{ab} v_a w_b = v^a w_a = v_a w^a = 0. \]  \hspace{1cm} (3.71)

3.5 The affine connections

It is important to know how vectors change as the coordinate or the parameter that defines the coordinate changes. For example, in Mankowski space time we may be interested in the 4D momentum, \( \vec{P} \), how it changes with time (the proper time, \( \tau \)) so that one can explain the condition for conservation of momentum in general relativity. In such cases for the 4D momentum expressed in terms of its contravariant components as

\[ \vec{P} = p^a \hat{e}_a \]  \hspace{1cm} (3.72)

one must be able to determine

\[ \frac{d\vec{P}}{d\tau} = \frac{d}{d\tau} (p^a \hat{e}_a) = \hat{e}_a \frac{dp^a}{d\tau} + p^a \hat{e}_a \frac{d\hat{e}_a}{d\tau} \]  \hspace{1cm} (3.73)

For the coordinates \( x^a = x^a(\tau) \), we have

\[ \frac{d\hat{e}_a}{d\tau} = \frac{\partial \hat{e}_a}{\partial x^b} \frac{dx^b}{d\tau} \]  \hspace{1cm} (3.74)

so that

\[ \frac{d\vec{P}}{d\tau} = \hat{e}_a \frac{dp^a}{d\tau} + p^a \frac{dx^b}{d\tau} \frac{\partial \hat{e}_a}{\partial x^b} \]  \hspace{1cm} (3.75)

In order to better understand the origin of the affine connections we shall reconsider the 2D sphere embedded in a 3D manifold. We saw that the tangent space is a plane with basis vectors defined by

\[ \hat{e}_\rho = \cos(\varphi) \hat{x} + \sin(\varphi) \hat{y} - \frac{\rho}{\sqrt{a^2 - \rho^2}} \hat{z}. \]  \hspace{1cm} (3.76)

and

\[ \hat{e}_\varphi = -\rho \sin(\varphi) \hat{x} + \rho \cos(\varphi) \hat{y}. \]  \hspace{1cm} (3.77)

There follows that

\[ \frac{\partial \hat{e}_\rho}{\partial \varphi} = \frac{\partial \hat{e}_\varphi}{\partial \rho} = -\sin(\varphi) \hat{x} + \cos(\varphi) \hat{y} = \frac{1}{\rho} \hat{e}_\varphi = f_{\rho \varphi}(\rho, \varphi) \hat{e}_\varphi. \]  \hspace{1cm} (3.78)
CHAPTER 3. VECTOR CALCULUS ON MANIFOLDS

where

\[
f_{\rho\varphi}(\rho, \varphi) = \frac{1}{\rho},
\]

is a function that connects the change in the basis vectors with respect to the coordinates to the basis vectors. Here we note that we are still in the tangent space. However, if we switch the variables for the derivatives, we find

\[
\frac{\partial \mathbf{e}_\rho}{\partial \rho} = \left[ \frac{1}{\sqrt{a^2 - \rho^2}} + \frac{\rho^2}{(a^2 - \rho^2)^{3/2}} \right] \hat{z} = -\frac{a^2}{(a^2 - \rho^2)^{3/2}} \hat{z},
\]

\[
\frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\rho \cos(\varphi) \hat{x} - \rho \sin(\varphi) \hat{y} = -\rho (\cos(\varphi) \hat{x} + \sin(\varphi) \hat{y}).
\]

that we can not tell whether it belongs to the tangent space or not. In order to find out that we introduce a basis vector normal to the tangent space in terms of the basis vector in the tangent space as

\[
\hat{e}_\perp = \hat{e}_\rho \times \hat{e}_\varphi.
\]

This normal basis vector is found to be

\[
\hat{e}_\perp = \frac{\rho^2}{\sqrt{a^2 - \rho^2}} (\cos(\varphi) \hat{x} + \sin(\varphi) \hat{y}) + \rho \hat{z}.
\]

Combining this relation with the basis vector, \( \hat{e}_\rho \),

\[
\hat{e}_\rho = \cos(\varphi) \hat{x} + \sin(\varphi) \hat{y} - \frac{\rho}{\sqrt{a^2 - \rho^2}} \hat{z}.
\]

one find

\[
\hat{z} = \frac{a^2 - \rho^2}{a^2} \left( \frac{1}{\rho} \hat{e}_\perp - \frac{\rho}{\sqrt{a^2 - \rho^2}} \hat{e}_\rho \right)
\]

and

\[
\cos(\varphi) \hat{x} + \sin(\varphi) \hat{y} = \frac{1}{a} \sqrt{1 - \frac{\rho^2}{a^2}} \hat{e}_\perp - \frac{\rho^2}{a^2} \hat{e}_\rho.
\]

Using these relations, we can then write

\[
\frac{\partial \mathbf{e}_\rho}{\partial \rho} = -\frac{\rho}{\sqrt{a^2 - \rho^2}} \hat{e}_\rho + \frac{1}{\rho} \hat{e}_\perp, \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\frac{\rho}{a} \sqrt{1 - \frac{\rho^2}{a^2}} \hat{e}_\perp + \frac{\rho^2}{a^2} \hat{e}_\rho.
\]

or

\[
\frac{\partial \mathbf{e}_\rho}{\partial \rho} = [f_{\rho\rho}(\rho, \varphi)]_\parallel \hat{e}_\rho + [f_{\rho\rho}(\rho, \varphi)]_\perp \hat{e}_\perp, \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = [f_{\varphi\varphi}(\rho, \varphi)]_\parallel \hat{e}_\rho + [f_{\varphi\varphi}(\rho, \varphi)]_\perp \hat{e}_\perp.
\]

where

\[
[f_{\rho\rho}(\rho, \varphi)]_\parallel = -\frac{\rho}{\sqrt{a^2 - \rho^2}} [f_{\rho\rho}(\rho, \varphi)]_\perp = \frac{1}{\rho}
\]

\[
[f_{\varphi\varphi}(\rho, \varphi)]_\parallel = -\frac{\rho}{a} \sqrt{1 - \frac{\rho^2}{a^2}} [f_{\varphi\varphi}(\rho, \varphi)]_\perp = \frac{\rho^3}{a^3}
\]
These results show that the derivatives for the basis vectors can lead to components that do not belong to the tangent space. Let’s see where these components are located for a specific point for a 2D sphere with radius, \(a = 2\) units. The point we shall consider has coordinates \((x = 1, y = 0, z = \sqrt{3})\) which is on the surface of the 2D sphere. At this point the two basis vectors \((\hat{e}_\rho, \hat{e}_\varphi)\) and the vector normal to the tangent space, are shown in Fig. 3.5.

\[
\frac{\partial \hat{e}_\rho}{\partial \rho} = -\frac{1}{\sqrt{3}} \hat{e}_\rho + \hat{e}_\perp, \quad \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\frac{\sqrt{3}}{4} \hat{e}_\perp + \frac{1}{4} \hat{e}_\rho.
\] (3.86)

Therefore, generally for none spherical geometry or none Euclidean manifold, the change in the basis vectors with respect to the coordinate at a point on a manifold (derivative of the basis vector at that point) can have both normal and tangential components to the tangent space. However, since we are confined to the tangent space, for example in the 2D sphere of radius \(a\), we are confined to the tangent space that is a plane and the normal component does not belong to the tangent space at that particular point. Therefore, we shall consider only the projection parallel to the tangent space at point \(P\),

\[
\frac{\partial \hat{e}_a}{\partial x^c} = \left( \lim_{\delta x^c \to 0} \frac{\delta \hat{e}_a}{\delta x^c} \right)_{\| T_p}. \] (3.87)

Suppose we represent the coefficients resulting from the derivative of the basis vectors that are components that belong to the tangent space at the point (e.g. \([f_{ac}(\rho, \varphi)]_{\| T_p}\)) by \(\Gamma_{ac}^b\), then we can write

\[
\frac{\partial \hat{e}_a}{\partial x^c} = \Gamma_{ac}^1 \hat{e}_1 + \Gamma_{ac}^2 \hat{e}_2 + \Gamma_{ac}^3 \hat{e}_3 \ldots + \Gamma_{ac}^N \hat{e}_N = \Gamma_{ac}^b \hat{e}_b.
\] (3.88)
CHAPTER 3. VECTOR CALCULUS ON MANIFOLDS

where \( N \) is the dimension of the tangent space. The \( N^3 \) coefficients \( \Gamma^b_{ac} \) are known collectively as the affine connection or in older textbooks, the Christoffel symbol (of the second kind) at point \( P \).

**Homework:**

1. Find all the elements for the affine connection, \( \Gamma^b_{ac} \), for a point on a 2D sphere embedded in a 3D Euclidean space. Note that in the expressions

\[
\frac{\partial \hat{e}_a}{\partial \theta} = \Gamma^\theta_{\theta \theta} \hat{e}_\theta + \Gamma^\phi_{\theta \phi} \hat{e}_\phi, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = \Gamma^\theta_{\phi \theta} \hat{e}_\theta + \Gamma^\phi_{\phi \phi} \hat{e}_\phi,
\]

you are going to determine

\[
\Gamma^\theta_{\theta \theta}, \Gamma^\phi_{\theta \phi}, \Gamma^\phi_{\phi \theta}, \Gamma^\theta_{\phi \theta}, \Gamma^\phi_{\phi \phi}, \text{ and } \Gamma^\phi_{\phi \phi}.
\]

Also in this case the origin is at the center of the sphere.

2. Find all the elements for the affine connection, \( \Gamma^b_{ac} \), for a point on a 3D sphere embedded in a 4D Euclidean space. Note that in the expressions

\[
\frac{\partial \hat{e}_d}{\partial x^c} = \Gamma^b_{ac} \hat{e}^d \cdot \hat{e}_b,
\]

applying the properties of the dual basis vectors

\[
\hat{e}^d \cdot \hat{e}_b = \delta^d_b
\]

we find

\[
\frac{\partial \hat{e}_a}{\partial x^c} = \Gamma^b_{ac} \delta^d_b \Rightarrow \Gamma^b_{ac} = \hat{e}^d \cdot \frac{\partial \hat{e}_a}{\partial x^c}
\]

Since \( d \) is a dummy index, we can write

\[
\Gamma^b_{ac} = \hat{e}^d \cdot \frac{\partial \hat{e}_a}{\partial x^c} = \hat{e}^b \cdot \partial_c \hat{e}_a
\]

From now on we will use the notation \( \partial_c \hat{e}_a \),

\[
\partial_c \hat{e}_a = \frac{\partial \hat{e}_a}{\partial x^c}.
\]

Differentiating Eq. (3.92) with respect to the coordinate, \( x^c \), and applying the notation, we find

\[
\hat{e}^a \cdot \partial_c \hat{e}_b + \hat{e}_b \cdot \partial_c \hat{e}^a = \partial_c \delta^b_a = 0 \Rightarrow \hat{e}_b \cdot \partial_c \hat{e}^a = -\hat{e}^a \cdot \partial_c \hat{e}_b
\]

Using the definition of the derivative of the basis vectors and our notation

\[
\partial_c \hat{e}_a = \frac{\partial \hat{e}_a}{\partial x^c} = \Gamma^b_{ac} \hat{e}_b \Rightarrow \hat{e}^d \cdot \partial_c \hat{e}_a = \Gamma^b_{ac} \hat{e}_b \cdot \hat{e}^d = \Gamma^b_{ac} \delta^d_b = \Gamma^d_{ac}
\]

(3.97)
applying this relation,
\[ \hat{e}_b \cdot \partial_c \hat{e}^a = - \hat{e}^a \cdot \partial_c \hat{e}_b \Rightarrow \hat{e}_b \cdot \partial_c \hat{e}^a = - \Gamma_{bc}^a, \]  
(3.99)
and noting that
\[ \hat{e}_b \cdot \hat{e}^b = 1 \]  
(3.100)
we find
\[ \hat{e}_b \cdot \partial_c \hat{e}^a = - \Gamma_{bc}^a \hat{e}_b \cdot (\hat{e}^b) \Rightarrow \partial_c \hat{e}^a = - \Gamma_{bc}^a \hat{e}^b \]  
(3.101)

The affine connection under coordinate transformation: Suppose we make the coordinate transformation \( x^a \rightarrow x'^a \), for the affine connection
\[ \Gamma_{ac}^b = \hat{e}^b \cdot \frac{\partial \hat{e}_a}{\partial x'^c}. \]  
(3.102)
we have
\[ \Gamma_{ac}^{b'} = \hat{e}^{b'} \cdot \frac{\partial \hat{e}'_a}{\partial x'^c}. \]  
(3.103)
so that applying the relations
\[ \hat{e}'_a = \frac{\partial x^f}{\partial x'^a} \hat{e}_f, \hat{e}'^b = \frac{\partial x'^b}{\partial x^d} \hat{e}^d. \]  
(3.104)
we may write
\[
\Gamma_{ac}^{b'} = \frac{\partial x'^b}{\partial x'^c} \hat{e}'^d \cdot \frac{\partial}{\partial x'^c} \left( \frac{\partial x^f}{\partial x'^a} \hat{e}_f \right) \\
= \frac{\partial x'^b}{\partial x'^d} \hat{e}'^d \cdot \left[ \frac{\partial \hat{e}'_f}{\partial x'^c} \frac{\partial x^f}{\partial x'^a} + \hat{e}'_f \frac{\partial^2 x^f}{\partial x'^a \partial x'^d} \right] \\
= \frac{\partial x'^b}{\partial x'^d} \frac{\partial x^f}{\partial x'^a} \hat{e}'^d \cdot \frac{\partial \hat{e}'_f}{\partial x'^c} + \frac{\partial^2 x^f}{\partial x'^c \partial x'^a} \frac{\partial x'^b}{\partial x'^d} \hat{e}'^d \cdot \hat{e}_f. \]  
(3.105)
Using the chain rule one can write
\[ \frac{\partial \hat{e}'_f}{\partial x'^c} = \frac{\partial \hat{e}'_f}{\partial x^9} \frac{\partial x^9}{\partial x'^c} \]
so that Eq. (3.105) becomes
\[ \Gamma_{ac}^{b'} = \frac{\partial x'^b}{\partial x'^d} \frac{\partial x^f}{\partial x'^a} \hat{e}'^d \cdot \frac{\partial \hat{e}'_f}{\partial x'^c} + \frac{\partial^2 x^f}{\partial x'^c \partial x'^a} \frac{\partial x'^b}{\partial x'^d} \hat{e}'^d \cdot \hat{e}_f. \]  
(3.106)
Now applying
\[ \hat{e}'^d \cdot \frac{\partial \hat{e}'_f}{\partial x'^g} = \Gamma_{fg}^d \hat{e}'^d \cdot \hat{e}_f = \delta^d_f \]
Eq. (3.106) becomes
\[ \Gamma_{ac}^{b'} = \frac{\partial x'^b}{\partial x'^d} \frac{\partial x^f}{\partial x'^a} \frac{\partial x^9}{\partial x'^c} \Gamma_{fg}^d + \frac{\partial x'^b}{\partial x'^d} \frac{\partial^2 x^d}{\partial x'^c \partial x'^a} \]  
(3.107)
Homework: If we swap the derivatives with respect to \( x \) and \( x' \) in the expression

\[
\frac{\partial x'^b}{\partial x^d} \frac{\partial^2 x'^d}{\partial x'^c \partial x'^a}
\]

we will find

\[
\frac{\partial x'^b}{\partial x^d} \frac{\partial^2 x'^d}{\partial x'^c \partial x'^a} = - \frac{\partial x^d}{\partial x'^a} \frac{\partial x^f}{\partial x'^e} \frac{\partial^2 x'^b}{\partial x'^c \partial x'^f}
\]

show that we arrive at an alternative expression for the affine connect under coordinates transformations

\[
\Gamma'^b_{ac} = \frac{\partial x'^b}{\partial x^d} \frac{\partial x^d}{\partial x'^a} \Gamma'^d_{ac} - \frac{\partial x^d}{\partial x'^a} \frac{\partial x^f}{\partial x'^e} \frac{\partial^2 x'^b}{\partial x'^c \partial x'^f}
\]

(3.108)

**The affine connection and the metric:** we recall the affine connection

\[
\Gamma'^b_{ac} = \frac{\partial e'^b}{\partial x^d} \frac{\partial e^d}{\partial x'^a} = \frac{\partial e'^b}{\partial x^d} \frac{\partial e^d}{\partial x'^a}
\]

(3.109)

in a similar manner we can also write

\[
\Gamma'^b_{ca} = \frac{\partial e'^b}{\partial x^d} \frac{\partial e^d}{\partial x'^a}
\]

(3.110)

The difference between Eq. (3.109) and (3.110), \( T'^b_{ac} \)

\[
T'^b_{ac} = \Gamma'^b_{ac} - \Gamma'^b_{ca}
\]

(3.111)

is known as the torsion tensor. We will consider a torsionless manifolds, for which

\[
T'^b_{ac} = 0 \Rightarrow \Gamma'^b_{ac} = \Gamma'^b_{ca}.
\]

(3.112)

We will determine the relationship between the affine connection and the metric for a torsionless manifold. We recall the metric

\[
g_{ab} = \hat{e}_a \cdot \hat{e}_b
\]

so that

\[
\frac{\partial g_{ab}}{\partial x'^c} = \partial_c g_{ab} = \partial_c (\hat{e}_a \cdot \hat{e}_b) = \hat{e}_a \cdot \partial_c \hat{e}_b + \hat{e}_b \cdot \partial_c \hat{e}_a.
\]

(3.113)

Applying the relation

\[
\partial_c \hat{e}_b = \Gamma'^d_{bc} \hat{e}_d
\]

(3.114)

we can rewrite Eq. (3.113) as

\[
\partial_c g_{ab} = \Gamma'^d_{bc} \hat{e}_d + \hat{e}_a \cdot \partial_c \hat{e}_b + \hat{e}_b \cdot \partial_c \hat{e}_a.
\]

(3.115)

Similarly

\[
\partial_b g_{ca} = \hat{e}_b \cdot \partial_c \hat{e}_a = \hat{e}_b \cdot \partial_c \hat{e}_a + \hat{e}_a \cdot \partial_b \hat{e}_c
\]

\[
\Rightarrow \partial_b g_{ca} = \hat{e}_b \cdot \partial_c \hat{e}_a + \hat{e}_a \cdot \partial_b \hat{e}_c
\]

\[
\Rightarrow \partial_b g_{cd} = \Gamma'^d_{bc} \hat{e}_d + \hat{e}_a \cdot \partial_b \hat{e}_c
\]

\[
\Rightarrow \partial_b g_{ca} = \Gamma'^d_{bc} g_{ad} + \Gamma'^d_{cb} g_{ad}
\]

(3.116)
and
\[ \partial_a g_{bc} = \Gamma^d_{ba} g_{dc} + \Gamma^d_{ca} g_{bd} \]  
(3.117)

Now combining Eqs. (3.115)- (3.117), we can write
\[ \partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc} = \Gamma^d_{ac} g_{bd} + \Gamma^d_{bd} g_{ac} + \Gamma^d_{cb} g_{ad} - \Gamma^d_{ba} g_{dc} - \Gamma^d_{ca} g_{bd} \]  
(3.118)

Recalling that we will be interested in a torsionless manifold where \( \Gamma^b_{ac} = \Gamma^b_{ca} \), Eq. (3.118)
\[ \partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc} = 2 \Gamma^d_{bc} g_{ad} \]  
(3.119)

Multiplying Eq. (3.120) by \( g^{pa} \),
\[ g^{pa} (\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}) = 2 \Gamma^p_{bc} g^{pa} g_{ad} \]  
(3.120)

From Eq. (3.64), we have \( g^{ab} g_{bc} = \delta^a_c \)
so that Eq. (3.64) becomes
\[ g^{pa} (\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}) = 2 \Gamma^d_{bc} g^{pd} \]  
(3.121)

Since the summation is over \( d \), replacing \( d \) by \( p \), we find
\[ g^{pa} (\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}) = 2 \Gamma^p_{bc} \]  
(3.122)

Now relabeling the index \( a \) by \( d \)
\[ g^{pd} (\partial_c g_{db} + \partial_b g_{cd} - \partial_d g_{bc}) = 2 \Gamma^p_{bc} \]  
(3.123)

and then the index \( p \) by \( a \), we can write
\[ \Gamma^a_{bc} = \frac{g^{pd}}{2} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \]  
(3.124)

The right hand side of Eq. (3.124) is known as the metric connection and is often represented by \( \{ \Gamma^a_{bc} \} \)

**Useful Formulae:**
\[ \Gamma_{abc} = g_{ad} \Gamma^d_{bc} \]  
(3.125)

Multiplying by \( g^{fa} \)
\[ g^{fa} \Gamma_{abc} = g^{fa} g_{ad} \Gamma^d_{bc} = \delta^f_d \Gamma^d_{bc} = \Gamma^f_{bc} \]
\[ \Rightarrow \Gamma^f_{bc} = g^{fa} \Gamma_{abc} \]  
(3.126)

Applying the relation in Eq (3.124), we can express Eq. (3.125) as
\[ \Gamma^f_{bc} = \frac{g^{fd}}{2} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \]  
(3.127)
Relabeling the index \( d \) by \( a \) in Eq. (3.127), we have

\[
\Gamma^f_{bc} = \frac{g^f_{a}}{2} (\partial_b g_{ca} + \partial_c g_{ab} - \partial_a g_{bc})
\]  

(3.128)

so that equating Eqs. (3.126) and (3.128), we find

\[
\Gamma_{abc} = \frac{1}{2} (\partial_b g_{ca} + \partial_c g_{ab} - \partial_a g_{bc})
\]  

(3.129)

The quantity \( \Gamma_{abc} \) is traditionally known as a Christoffel symbol of the first kind.

Noting that

\[
\Gamma_{bac} = \frac{1}{2} (\partial_a g_{cb} + \partial_c g_{ba} - \partial_b g_{ac})
\]  

(3.130)

we can write

\[
\Gamma_{abc} + \Gamma_{bac} = \frac{1}{2} (\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac})
\]  

(3.131)

taking into account the symmetry of the metric tensor, from Eq. (3.131), we find

\[
\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}
\]  

(3.132)

Eq. (3.132) allows us to express the partial derivative of the metric components in terms of the connection coefficients.

We recall that the determinant of the matrix \( G \) can be expressed as

\[
\det G = g_{ab} \left( -1 \right)^{a+b} M_{ab} = g_{ab} \{ \text{cof} [G]_{ab} \},
\]  

(3.133)

where

\[
[\text{cof} [G]]_{ab} = (-1)^{a+b} M_{ab}
\]  

(3.134)

is the the cofactor matrix to \( G \) which is determined from the minor, \( M_{ab} \) and it is a constant matrix. We recall that the minor of matrix \( G \) denoted by \( M_{ab} \) is the determinant of the matrix formed from matrix \( G \) by removing the \( a^{th} \) row and \( b^{th} \) column. Let \( \det G = g \), so that

\[
g = g_{ab} \text{cof} [G]_{ab} \Rightarrow \partial_c g = \text{cof} [G]_{ab} \partial_c g_{ab}.
\]  

(3.135)

Noting that the cofactor matrix can be expressed as

\[
g^{ab} g = g^{ab} g_{ab} \text{cof} [G]_{ab} \Rightarrow \text{cof} [G]_{ab} = g^{ab} g
\]  

(3.136)

we may write

\[
\partial_c g = \text{cof} [G]_{ab} \partial_c g_{ab} = g g^{ab} \partial_c g_{ab}.
\]  

(3.137)

Now using the relation (3.132), we can write

\[
\partial_c g = g g^{ab} \partial_c g_{ab} = g g^{ab} (\Gamma_{abc} + \Gamma_{bac})
\]  

(3.138)

We recall

\[
\Gamma_{abc} = g_{ad} \Gamma_{d hc}^f
\]  

(3.139)
so that
\[ \partial_c g = g^{ab} (\Gamma_{abc} + \Gamma_{bac}) = g^{ab} (g_{ad} \Gamma^d_{bc} + g_{bd} \Gamma^d_{ac}) \]
\[ = g (g^{ab} g_{ad} \Gamma^d_{bc} + g^{ab} g_{bd} \Gamma^d_{ac}) = g \left( \delta^b_d \Gamma^d_{bc} + \delta^a_d \Gamma^d_{ac} \right) \]
\[ = g (\Gamma^a_{bc} + \Gamma^a_{ac}). \] (3.140)

Taking into account that \(a\) and \(b\) are dummy indices, we can replace \(b\) by \(a\) so that
\[ \partial_c g = 2g \Gamma^a_{ac}. \] (3.141)

This can be rearranged as
\[ \Gamma^a_{ac} = \frac{1}{2g} \partial_c g = \frac{1}{2} \partial_c \ln |g| = \partial_c \ln \sqrt{|g|}. \] (3.142)

Now for the sake of convenience if we replace \(c\) by \(b\), we may write the above equation as
\[ \Gamma^a_{ab} = \partial_b \ln \sqrt{|g|}. \] (3.143)

The modulus is for the case where the manifold is pseudo-Riemannian where the metric elements can be negative.

### 3.6 Local geodesic and Cartesian coordinates

Let’s consider a manifold with coordinates system, \(x^a\), and a point \(P\) on this manifold with coordinates, \(x^a_P\). Let’s now define a new system of coordinates, \(x^0\), in terms of \(x^a_P\), and the coordinates \(x^a\) as
\[ x^a = x^a - x^a_P + \frac{1}{2} \Gamma^a_{bc} (P) (x^b - x^b_P) (x^c - x^c_P). \] (3.144)

We must know that under this transformation how the affine connection transformed. We recall that under coordinate transformation \(x^a \rightarrow x'^a\), the affine connection is transformed according to
\[ \Gamma'^{ab}_{ac} = \frac{\partial x'^b}{\partial x^c} \frac{\partial x^d}{\partial x'^a} \Gamma^d_{ef} - \frac{\partial x^d}{\partial x'^a} \frac{\partial x^f}{\partial x'^e} \frac{\partial^2 x'^b}{\partial x^c \partial x'^d}. \] (3.145)

In order to determine this at point \(P\), we need to differentiate Eq. (3.144)
\[ \frac{\partial x'^a}{\partial x^d} = \frac{\partial x^a}{\partial x^d} + \frac{1}{2} \Gamma^a_{bc} (P) \frac{\partial}{\partial x^d} \left\{ (x^b - x^b_P) (x^c - x^c_P) \right\} \]
\[ = \frac{\partial x^a}{\partial x^d} + \frac{1}{2} \Gamma^a_{bc} (P) \left\{ (x^c - x^c_P) \frac{\partial x^b}{\partial x^d} + (x^b - x^b_P) \frac{\partial x^c}{\partial x^d} \right\}, \] (3.146)

where we used the fact that \(\Gamma^a_{bc} (P)\), \(x^a_P\), \(x^b_P\), and \(x^c_P\) are constant at point \(P\). The coordinates \(x^a\) are independent coordinates,
\[ \frac{\partial x^a}{\partial x^d} = \delta^a_d, \frac{\partial x^b}{\partial x^a} = \delta^b_a, \frac{\partial x^c}{\partial x^a} = \delta^c_a \] (3.147)
so that
\[
\frac{\partial x'^a}{\partial x'^d} = \delta_d^a + \frac{1}{2} \Gamma_{dc}^a (P) \left\{ (x^c - x'^c_P) \delta_d^b + (x^b - x'^b_P) \delta_d^d \right\} \tag{3.148}
\]
which results in
\[
\frac{\partial x'^a}{\partial x'^d} = \delta_d^a + \frac{1}{2} \Gamma_{dc}^a (P) (x^c - x'^c_P) + \Gamma_{bd}^a (P) (x^b - x'^b_P) \tag{3.149}
\]
The summation indices are dummy indices and therefore we can replace \( b \) by \( c \) in the second term so that
\[
\frac{\partial x'^a}{\partial x'^d} = \delta_d^a + \frac{1}{2} \left[ \Gamma_{dc}^a (P) (x^c - x'^c_P) + \Gamma_{cd}^a (P) (x^c - x'^c_P) \right] \tag{3.150}
\]
Since for a torsionless manifold
\[
\Gamma_{dc}^a (P) = \Gamma_{cd}^a (P),
\]
we can write
\[
\frac{\partial x'^a}{\partial x'^d} = \delta_d^a + \frac{1}{2} \Gamma_{dc}^a (P) (x^c - x'^c_P) \tag{3.151}
\]
We are interested in the affine connection at point \( P \) so that when we are evaluating Eq. (3.151) at \( x^c = x'^c_P \), we find
\[
\frac{\partial x'^a}{\partial x'^d} \bigg|_P = \delta_d^a \tag{3.152}
\]
Similarly, the inverse is also given by
\[
\frac{\partial x^a}{\partial x^d} \bigg|_P = \delta_d^a \tag{3.153}
\]
Differentiating Eq. (3.151) with respect to \( x^c \), we have
\[
\frac{\partial x'^a}{\partial x^c \partial x^d} = \frac{\partial}{\partial x^c} \delta_d^a + \Gamma_{dc}^a (P) \frac{\partial}{\partial x^c} (x^c - x'^c_P) \Rightarrow \frac{\partial x'^a}{\partial x^c \partial x^d} = \Gamma_{dc}^a (P) \delta_c^e = \Gamma_{dc}^a (P) \tag{3.154}
\]
Using the results in Eqs. (3.152)-(3.154), the transformation equation for the affine connection
\[
\Gamma_{ac}^{\beta b} = \frac{\partial x'^b}{\partial x^d} \frac{\partial x'^f}{\partial x^c} \frac{\partial x'^g}{\partial x^f} f_g^d - \frac{\partial x'^d}{\partial x^a} \frac{\partial x'^f}{\partial x^c} \frac{\partial^2 x'^b}{\partial x^d \partial x^f} \tag{3.155}
\]
at point \( P \) becomes
\[
\Gamma_{ac}^{\beta b} (P) = \delta^b_d \delta^f_a \delta^g_d (P) - \delta^d_a \delta^f_c \Gamma_{df}^b (P) \tag{3.156}
\]
3.7. THE GRADIENT, THE DIVERGENCE, THE CURL ON A MANIFOLD

which simplified into

\[
\Gamma^b_{ac}(P) = \delta^b_d \delta^d_a \Gamma^d_{fc}(P) - \delta^d_a \Gamma^d_{fc}(P) = \delta^b_d \Gamma^d_{ac}(P) - \Gamma^b_{ac}(P) \quad (3.157)
\]

\[
\Rightarrow \Gamma^b_{ac}(P) = \Gamma^b_{ac}(P) - \Gamma^b_{ac}(P) = 0.
\]

The result in Eq. (3.157) shows that for the coordinate transformation defined by Eq. (3.144), the affine connection becomes zero. Such coordinates where the affine connection becomes zero at a point \( P \) on a manifold is known as \textit{local geodesic coordinates} about \( P \).

In chapter 2 we have shown that the conditions for local Cartesian coordinates at a given point \( P \) in a pseudo-Riemmanian manifold are

\[
g^{ab}(x^0) = \eta_{ab}, \quad (3.158)
\]

\[
\frac{\partial g^a_b(x')}{\partial x^c}(P) = 0 \quad (3.159)
\]

where \( \eta_{ab} = \text{diag}(\pm 1, \pm 1, \ldots, \pm 1) \). We have also learned that the number of positive entries \( (N_+) \) minus the number of negative entries \( (N_-) \) in \( \eta_{ab} \) is called the \textit{signature} of the manifold. For geodesic coordinates Eq. (3.159) can easily be shown applying the equation that relates the connection with the metric in Eq. (3.115). For the \( x'^a \) coordinates, Eq. (3.115) can be written as

\[
\partial_c g^a_b = \Gamma^d_{ac} g^d_b + \Gamma^d_{bd} g^d_a \quad (3.160)
\]

When this equation is evaluated at point \( P \)

\[
\partial_c g^a_b = \left. \frac{\partial g^a_b(x')}{\partial x^c} \right|_P = \Gamma^d_{ac}(P) g^d_b(P) + \Gamma^d_{bd}(P) g^d_a(P) \quad (3.161)
\]

and for a geodesic coordinates the connection is zero at point \( P \), and therefore

\[
\left. \frac{\partial g^a_b(x')}{\partial x^c} \right|_P = 0.
\]

It is important to note that for geodesic coordinates the metric does not necessarily satisfy Eq. (3.158). But we can find coordinates \( x'^a \) that satisfy Eq. (3.158) by making a linear transformation to the \( x'^a \)

\[
x'^a = X x^a \quad (3.162)
\]

where \( X^a_b \) are constants.

3.7 The gradient, the divergence, the curl on a manifold

Before we see how the a gradient of a scalar, the divergence or the curl of a vector is determined on a manifold, first we need to know how a vector is differentiated. Consider a vector, \( \vec{v} \), in terms of its controvariant components

\[
\vec{v} = v^a \hat{e}_a, \quad (3.163)
\]

\[
\Delta \frac{\partial g^a_b(x')}{\partial x^c} \quad (3.164)
\]

and for a geodesic coordinates the connection is zero at point \( P \), and therefore

\[
\left. \frac{\partial g^a_b(x')}{\partial x^c} \right|_P = 0.
\]

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\[
\Delta \frac{\partial g^a_b(x')}{\partial x^c} \quad (3.164)
\]
CHAPTER 3. VECTOR CALCULUS ON MANIFOLDS

where \( \hat{e}_a \) are the coordinate basis vectors. The derivative of this vector with respect to the coordinate, \( x^b \), can be expressed as

\[
\frac{\partial \mathbf{v}}{\partial x^b} = \partial_b \mathbf{v} = \partial_b (v^a \hat{e}_a) = \hat{e}_a \partial_b (v^a) + v^a \partial_b (\hat{e}_a)
\]  (3.164)

We recall

\[
\partial_b (\hat{e}_a) = \Gamma^c_{ab} \hat{e}_c
\]

so that

\[
\partial_b \mathbf{v} = (\partial_b v^a) \hat{e}_a + \Gamma^c_{ab} v^a \hat{e}_c
\]  (3.165)

Switching the places for the indices \( c \) and \( a \) one can write

\[
\Gamma^c_{ab} v^a \hat{e}_c = a_{cb} v^c \hat{e}_a
\]  (3.166)

so that

\[
\partial_b \mathbf{v} = (\partial_b v^a) \hat{e}_a + v^c \Gamma^a_{cb} \hat{e}_c = (\partial_b v^a + \Gamma^a_{cb} v^c) \hat{e}_a
\]  (3.167)

The quantity in the bracket which is represented as

\[
\bar{v}_b v^a = \partial_b v^a + a_{cb} v^c
\]  (3.168)

is known as the covariant derivative of the vector components. Thus the derivative of a vector can be expressed as

\[
\partial_b \mathbf{v} = (\bar{v}_b v^a) \hat{e}_a
\]  (3.169)

For geodesic coordinates where the affine connection vanishes,

\[
\Gamma^a_{cb} = 0
\]

the covariant derivative reduces to

\[
\bar{v}_b v^a = \partial_b v^a
\]  (3.170)

which is just the ordinary derivative that we are very familiar with!

**Homework:** Suppose the vector is expressed in terms of its covariant components

\[
\bar{v} = v_a \hat{e}^a
\]  (3.171)

*show that*

\[
\bar{v}_b v^a = \partial_b v^a - \Gamma^c_{ab} v^c
\]  (3.172)

**The covariant derivative of a scalar function:** for a scalar function \( \phi \) the covariant derivative is

\[
\nabla_b \phi = \partial_b \phi
\]  (3.173)

**The gradient:** The gradient of a scalar function \( \phi \) is given by

\[
\nabla \phi = (\partial_a \phi) \hat{e}^a
\]  (3.174)
3.7. THE GRADIENT, THE DIVERGENCE, THE CURL ON A MANIFOLD

The divergence: The divergence of a vector field expressed in terms of its covariant components

$$\vec{v} = v^a \hat{e}_a$$

is given by

$$\nabla \cdot \vec{v} = \nabla_a v^a$$

(3.175)

Using the relation we obtained

$$\nabla_b v^a = \partial_b v^a + \Gamma^a_{bc} v^c$$

(3.176)

for $b = a$ and replacing $c$ by $b$, we find

$$\nabla \cdot \vec{v} = \nabla_a v^a = \partial_a v^a + \Gamma^a_{ab} v^b.$$ (3.177)

Using the relation

$$\Gamma^a_{ab} = \partial_b \ln \sqrt{|g|} = \frac{1}{\sqrt{|g|}} \partial_a \sqrt{|g|}$$

(3.178)

we can write

$$\nabla \cdot \vec{v} = \partial_a v^a + v^a \frac{1}{\sqrt{|g|}} \partial_a \sqrt{|g|} = \frac{1}{\sqrt{|g|}} \partial_a \left[ v^a \sqrt{|g|} \right]$$

(3.179)

The Laplacian: We recall that in the Euclidean space the Laplacian of the scalar function, $\phi$, is given by

$$\nabla^2 \phi = \nabla \cdot \nabla \phi.$$ (3.180)

Applying Eq. (3.174) we can write

$$\nabla^2 \phi = \nabla \cdot \left[ (\partial_a \phi) \hat{e}^a \right].$$ (3.181)

But in order to apply the relation we derive for the divergence in Eq. (3.180), we need the vector

$$\vec{v} = (\partial_a \phi) \hat{e}^a = v_a \hat{e}_a.$$ (3.182)

in terms of its contravariant components. We have seen that the index can be raised or lowered using the metric tensor. In this case we want to raise it, so we have

$$g^{ab} v_b = v^a$$

and we can express the vector $\vec{v}$ as

$$\vec{v} = v^a \hat{e}_a = g^{ab} (\partial_b \phi) \hat{e}_a.$$ (3.183)

Then the Laplacian becomes

$$\nabla^2 \phi = \nabla \cdot \left[ g^{ab} (\partial_b \phi) \hat{e}_a \right].$$ (3.184)

Now applying the relation

$$\nabla \cdot \vec{v} = \frac{1}{\sqrt{|g|}} \partial_a \left[ v^a \sqrt{|g|} \right]$$

(3.185)
we may write the Laplacian
\[ \nabla^2 \phi = \nabla_a \nabla^a \phi = \frac{1}{\sqrt{|g|}} \partial_a \left[ \sqrt{|g|} g^{ab} \partial_b \phi \right]. \quad (3.187) \]

The Laplacian symbol \( \nabla^2 \) is used in the usual 3-D Euclidean space or in an N-D manifold. In 4-D spacetime manifold, as you will see (or have seen) in the relativistic electrodynamics in *Theoretical Physics IV (Electricity & Magnetism II)*, \( \nabla^2 \) is replaced by \( \Box^2 \) known as the d‘Alembertian operator.

**Curl:** The curl is defined as a rank-2 antisymmetric tensor with components
\[ (\text{curl} \, \vec{v})_{ab} = \nabla_a v_b - \nabla_b v_a. \quad (3.188) \]

Using the relation
\[ \nabla_b v_a = \partial_b v_a - \Gamma^c_{ab} v^c \]
we can express the curl as
\[ (\text{curl} \, \vec{v})_{ab} = \partial_a v_b - \Gamma^c_{ab} v^c - \partial_b v_a + \Gamma^c_{ab} v^c = \partial_a v_b - \partial_b v_a. \quad (3.190) \]

### 3.8 Intrinsic derivative of a vector along a curve

We will encounter vector fields that does depend on a curve instead of the entire or some region of the manifold. In such cases the curve may be defined by the coordinates \( x^a \) that depends on some parameter, \( u \). Let’s consider a vector, \( \vec{v} \), expressed in terms of its controvariant components. Since the coordinates on this curve depends on the parameter, \( u \), this vector can be expressed in terms of this parameter as
\[ \vec{v} (u) = v^a (u) \hat{e}_a (u). \quad (3.191) \]

The derivative of this vector along this curve is given by
\[
\frac{d}{du} \vec{v} (u) = \frac{d}{du} [v^a (u) \hat{e}_a (u)] = v^a (u) \frac{d\hat{e}_a (u)}{du} + \hat{e}_a (u) \frac{dv^a (u)}{du} \quad (3.192)
\]
\[
\Rightarrow \quad \frac{d}{du} \vec{v} (u) = v^a (u) \frac{d\hat{e}_a (u)}{du} \frac{dx^b}{du} + \hat{e}_a (u) \frac{dv^a (u)}{du} \quad (3.193)
\]

Using the relation
\[ \frac{d\hat{e}_a}{dx^b} = \Gamma^d_{ab} \hat{e}_d \]
we have
\[ \frac{d}{du} \vec{v} = v^a \frac{dx^b}{du} \Gamma^d_{ab} \hat{e}_d + \hat{e}_a \frac{dv^a}{du} \quad (3.195) \]
and replacing the index \( a \) by \( c \) in the first term
\[ \frac{d}{du} \vec{v} = v^c \frac{dx^b}{du} \Gamma^d_{cb} \hat{e}_d + \hat{e}_a \frac{dv^a}{du} \quad (3.196) \]
3.9 PARALLEL TRANSPORT

and \( d \) by \( a \)

\[
\frac{d}{du} \bar{v}(u) = v^c \frac{dx^b}{du} \Gamma_{cb}^a \hat{e}_a + \hat{e}_a \frac{dv^a}{du} = \left( \frac{dv^a}{du} + \Gamma_{cb}^a \frac{dx^b}{du} \right) \hat{e}_a, \quad (3.197)
\]

which we put in the form

\[
\frac{d}{du} \bar{v}(u) = \frac{Dv^a}{Du} \hat{e}_a, \quad (3.198)
\]

where

\[
\frac{Dv^a}{Du} = \frac{dv^a}{du} + \Gamma_{cb}^a \frac{dx^b}{du} \quad (3.199)
\]

is called the intrinsic (or absolute) derivative of the component \( v^a \). Substituting

\[
\frac{dv^a}{du} = \frac{dv^a}{dx^b} \frac{dx^b}{du} \quad (3.200)
\]

into Eq. (3.199), we find

\[
\frac{Dv^a}{Du} = \frac{\partial v^a}{\partial x^b} \frac{dx^b}{du} + \Gamma_{cb}^a \frac{dx^b}{du} = \frac{\partial v^a}{\partial x^b} \frac{dx^b}{du} + \Gamma_{cb}^a \frac{dx^b}{du} = \left( \partial_b v^a + \Gamma_{cb}^a \right) \frac{dx^b}{du} \Rightarrow \frac{Dv^a}{Du} = (\nabla_b v^a) \frac{dx^b}{du}, \quad (3.201)
\]

where we used the relation in Eq. (3.168).

Homework: Suppose the vector, \( \bar{v} \), depends on the parameter \( u \) on a curve defined by \( x^a(u) \) is expressed in terms of its covariant components

\[
\bar{v} = v_a(u) \hat{e}^a(u). \quad (3.202)
\]

Show that the intrinsic derivative of this vector is given by

\[
\frac{Dv_a}{Du} = \frac{dv_a}{du} - \Gamma_{ac}^b v_b \frac{dx^c}{du} \quad (3.203)
\]

3.9 Parallel transport

In order to understand the idea of parallel transport of a vector on a manifold let’s consider motion of a particle in space. Suppose the position of the particle depends on time, \( t \), then the displacement is parametrized by time \( t \), \( \bar{D}(t) \). The velocity of the particle is given by

\[
\bar{v} = \frac{d\bar{D}}{dt} \quad (3.204)
\]

and the acceleration by

\[
\bar{a} = \frac{d\bar{v}}{dt}. \quad (3.205)
\]
CHAPTER 3. VECTOR CALCULUS ON MANIFOLDS

Suppose you plot the displacement of the particle at different times, then you would get generally a curve. The particle would have a constant velocity through this curve provided its acceleration is zero.

\[ \ddot{a} = \frac{d\dot{v}}{dt} = 0. \]  

(3.206)

This means the particle travels along this curve with a constant velocity. The velocity would have the same magnitude and direction. The velocity vector remains parallel at each point on the curve describing the displacement of the particle as a function of time.

On a curve \( C \) on a manifold (See Fig.3.5), a parallel transport of a vector

\[ \vec{v} = v_a(u) \hat{e}^a(u) \]  

(3.207)

is when the intrinsic derivative of this vector is zero

\[ \frac{Dv_a}{Du} = \frac{dv_a}{du} - \Gamma^b_{ac} v_b \frac{dx^c}{du} = 0 \]  

(3.208)

3.10 Null curves, non-null curves, and affine parameter

We recall that the tangent vector \( t \) at point \( p \) on a Manifold is the vector that lies in the tangent space \( T_p \) at that point and is given by

\[ \hat{t} = \lim_{\delta u \to 0} \frac{\partial \vec{s}}{\delta u} = \frac{d\vec{s}}{du}, \]  

(3.209)

where \( \partial \vec{s} \) is the infinitesimal separation vector between the point \( P \) and some nearby point \( Q \) on the curve on the manifold corresponding to the parameter value \( u + \delta u \).
In a given coordinate system $x^a$ with basis vectors $\hat{e}_a$, we can express the infinitesimal separation vector $d\vec{s}$ as

$$d\vec{s} = \frac{dx^a}{du} \hat{e}_a$$

so that the tangent vector becomes

$$\vec{t} = \frac{dx^a}{du} \hat{e}_a$$

Recalling that in pseudo-Riemannian manifold the length of a vector $\vec{v}$ is given by

$$|v| = \sqrt{|g_{ab}v^av^b|} = \sqrt{|g^{ab}v_av_b|} = \sqrt{|v^av_b|} = \sqrt{|v_av_b|}$$

we have

$$|t| = \sqrt{|g_{ab}\alpha^a\chi^b|} = \sqrt{|g^{ab}\frac{dx^a}{du} \frac{dx^b}{du}|} = \sqrt{|g^{ab}dx^adx^b|}$$

we recall the metric or the interval (the distance squared along the curve on the manifold between the two points $P$ and $Q$) is

$$ds^2 = g^{ab}dx^adx^b$$

Eq. (3.213) becomes

$$|\vec{t}| = \left| \frac{d\vec{s}}{du} \right|$$

**The non-null vectors and the affine parameter:** for the non-null vectors $|\vec{t}| \neq 0$. This means according to Eq. (3.214) the distance $ds$ at all points on the curve must be different from zero and therefore it depends on the parameter $u$ at all points on the curve, $s = s(u)$. If parameter $u$ and the distance $s$ are related by

$$u = as + b$$
for \(a, b \neq 0\), the parameter \(u\) is called the \textit{affine parameter} on the curve.

\textbf{The null vectors:} if the tangent vector is a null vector, then

\[ |\vec{z}| = \left| \frac{d\vec{s}}{du} \right| = 0 \tag{3.216} \]

at all points on the curve and the distance, \(s\), does not depend on the parameter, \(u\), and we clearly can not use it as affine parameter since it does not satisfy the condition in Eq. (3.215). But it is possible to find a \textit{privileged family of affine parameter}.

\section*{3.11 The calculus of variation-(a review from Theoretical Physics I)}

\subsection*{3.11.1 Geodesic and stationary points}

\textit{Geodesic:} The curve along a surface which marks the shortest distance between two neighboring points. Finding geodesics is one of the problems which can be solved using the calculus of variation.

\textit{Stationary point:} A point with coordinates, \((x_0, f(x_0))\), on a curve defined by the function \(f(x)\) is said to be a stationary point when

\[ \frac{df(x)}{dx} \bigg|_{x=x_0} = 0. \tag{3.217} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{stationary_points.png}
\caption{Stationary points.}
\end{figure}

\subsection*{3.11.2 The geodesic in Euclidean space}

Consider two points in a \(x-y\) plane \(P_1\) and \(P_2\). Prove that the shortest distance between the two points is the distance measured along a straight line (i.e. show that the geodesic is given by an equation of a straight line, \(y(x) = mx + b\)).
3.11. THE CALCULUS OF VARIATION-(A REVIEW FROM THEORETICAL PHYSICS I)

Let’s consider two points on the x-y plane. Let $P_1$ be $(x_1, y_1)$ and $P_2$ be $(x_2, y_2)$. Then the distance between these points is given by the integral

$$L = \int_{(1)}^{(2)} ds,$$

where

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$  \hspace{1cm} (3.219)

We may rewrite this distance as

$$L = \int_{(1)}^{(2)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$  \hspace{1cm} (3.220)

Out of the infinitely many functions that can be used to connect the two points, we want to determine the one that would give the minimum distance. Let these function be denoted by $Y(x)$. From these infinite number of functions there is only one function that gives the minimum distance between the two points. If this function is $y(x)$, then we may write $Y(x)$ in terms of $y(x)$ as

$$Y(x, \epsilon) = y(x) + \epsilon \eta(x),$$  \hspace{1cm} (3.221)

where $\eta(x)$ is an arbitrary function which must satisfy the condition

$$\eta(x_1) = \eta(x_2) = 0$$  \hspace{1cm} (3.222)

so that at the two end points ($x = x_1 = x_2$), we find

$$Y(x, \epsilon) = y(x).$$  \hspace{1cm} (3.223)
Here $\epsilon$ is the constant of variation. It is this constant that determines by how much $Y(x)$ differs from $y(x)$. Now in terms of $Y(x)$, we may write

$$L(\epsilon) = \int_{(1)}^{(2)} \sqrt{1 + Y'^2} \, dx,$$

where

$$Y' = \frac{dY(x, \epsilon)}{dx}.$$  

(3.225)

We are interested in the path that gives the minimum distance between the two points (i.e. the geodesic). The necessary condition for the distance, $L(\epsilon)$, to be minimum is that the length function, $L(\epsilon)$, must have a stationary point at $(\epsilon = 0, L(\epsilon = 0))$. This requires

$$\left. \frac{dL(\epsilon)}{d\epsilon} \right|_{\epsilon = 0} = 0,$$

(3.226)

which leads to

$$\left. \frac{dL(\epsilon)}{d\epsilon} \right|_{\epsilon = 0} = \left[ \int_{(1)}^{(2)} \left( -\frac{1}{2} \right) \frac{1}{\sqrt{1 + Y'^2}} \left( 2Y' \right) \left( \frac{dY'(x, \epsilon)}{d\epsilon} \right) \, dx \right]_{\epsilon = 0} = 0. \tag{3.227}$$

Using

$$Y(x, \epsilon) = y(x) + \epsilon \eta(x)$$

we may write

$$\frac{dY}{dx} = \frac{dy}{dx} + \epsilon \frac{d\eta}{dx} \Rightarrow Y'(x, \epsilon) = y'(x) + \epsilon \eta'(x)$$

(3.229)

so that

$$\frac{dY'(x)}{d\epsilon} = \frac{d}{d\epsilon}[y'(x) + \epsilon \eta'(x)] = \eta'(x).$$

(3.230)

There follows that

$$Y'(x, \epsilon)|_{\epsilon = 0} = y'(x).$$

(3.231)

and

$$\left. \frac{dY'(x)}{d\epsilon} \right|_{\epsilon = 0} = \eta'(x)$$

(3.232)

In view of these results, one finds for the stationary point

$$\left. \frac{dL(\epsilon)}{d\epsilon} \right|_{\epsilon = 0} = \left[ \int_{(1)}^{(2)} \left( -\frac{1}{2} \right) \frac{1}{\sqrt{1 + Y'^2}} \left( 2y' \right) \left( \frac{dY'(x)}{d\epsilon} \right) \, dx \right]_{\epsilon = 0} = \left[ \int_{(1)}^{(2)} \frac{y'(x) \eta'(x)}{\sqrt{1 + y'^2(x)}} \, dx \right] = 0. \tag{3.233}$$

Using integration by parts

$$\int u dv = uv - \int v du \tag{3.234}$$
for
\[ \eta'(x) = dv \Rightarrow v = \eta(x), \] (3.235)
that gives
\[ u = \frac{y'}{\sqrt{1 + y'^2}} \Rightarrow du = \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) dx, \] (3.236)
we may write the integral as
\[ \int_{y(1)}^{y(2)} \frac{y'y'(x)}{\sqrt{1 + y'^2}} dx = \frac{y'}{\sqrt{1 + y'^2}} \eta(x) \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) dx = 0. \] (3.237)
Due to the conditions
\[ \eta(x_1) = \eta(x_2) = 0 \] (3.238)
the first term in the above expression becomes zero. Thus one can write
\[ \frac{dL(x)}{de} \bigg|_{e=0} = \int_{y(1)}^{y(2)} \frac{y'y'(x)}{\sqrt{1 + y'^2}} dx = -\int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) dx = 0. \] (3.239)
Since \( \eta(x) \) is an arbitrary function, for the integral to be zero, we must have
\[ \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) = 0 \Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = c, \] (3.240)
where \( c \) is a constant. Upon solving for \( y' \)
\[ y' = \left[ \frac{c^2}{1 - c^2} \right]^{1/2} = m. \] (3.241)
Note that we have introduced another constant in terms of the constant \( c \). There follows that
\[ \frac{dy}{dx} = m \Rightarrow y(x) = mx + b, \] (3.242)
which is equation of a straight line.

### 3.11.3 The general problem

In the previous section we saw the application of the calculus of variation to show that the shortest path connecting two points on a plane (the geodesic) is a straight line
\[ \frac{dy}{dx} = m \Rightarrow y(x) = mx + b. \] (3.243)
Next we shall consider the application of the calculus of variation to the general problem. In an Euclidean space a surface is defined by the function, \( F(x, y, z) \),
where it depends on the Cartesian coordinates $x, y$, and $z$. Instead of the Euclidean space let’s consider a surface defined by the function, $F \left(x, y(x) \cdot y'(x) = \frac{dy}{dx}\right)$.

This surface could, for example, be a surface on a phase space if we replace $x \to t$, $y(x) \to y(t)$, $y'(x) \to y'(t) = \frac{dy}{dt} = v_y(t) = \frac{p_y}{m}$

describing the dynamics of a particle mass, $m$, moving along the $y$-direction in terms of the parameters ($time = t$, $position = y(t)$, $velocity = v_y(t) \cdot \dot{y} = \frac{p_y}{m}$), where $p_y \dot{y}$ is the momentum. In classical mechanics, the dynamics of a particle is determined by an equation derived from Newton’s second law. As we shall see, this equation can be derived from a more general equation know as the Euler-Lagrange Equation

$$\frac{\partial}{\partial x} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0,$$

(3.244)

where $F = F(x, y(x), y'(x))$ is the function that defines the surface constructed by the set of points with coordinates, $(x, y(x), y'(x))$. The Euler-Lagrange Equation is derived by applying the calculus of variation. In general, in the problem that we want to solve applying the calculus of variation, we know the coordinates of two different points $(x_1, y(x_1), y'(x_1))$ and $(x_2, y(x_2), y'(x_2))$ on the surface defined by $F = F(x, y(x), y'(x))$. Form the infinitely many trajectories

that can connect these two points, there is only one trajectory on this surface that is the shortest (the Geodesic). Finding the Geodesic is the general problem that can be solved applying the calculus of variation.

The surface is defined by the function $F(x, y(x), y'(x))$. The distance between these two points determined by evaluating the integral

$$I = \int_{x_1}^{x_2} F(x, y(x), y'(x)) \, dx.$$

(3.245)
3.11. THE CALCULUS OF VARIATION-(A REVIEW FROM THEORETICAL PHYSICS I)

To determine the equation for the function $F$ that gives the shortest length joining the two points, let the function for any path connecting the two points be $Y(x)$. From these infinite number of functions there is only one function that gives the minimum distance between the two points. If this function is $y(x)$, then we may write $Y(x)$ in terms of $y(x)$ as

$$Y(x, \epsilon) = y(x) + \epsilon \eta(x),$$

where $\eta(x)$ is an arbitrary function which must satisfy the condition

$$\eta(x_1) = \eta(x_2) = 0$$

so that at the two points ($x = x_1 = x_2$), we find

$$Y(x, \epsilon) = y(x).$$

We also have

$$\frac{dY(x, \epsilon)}{d\epsilon} = \eta(x) \Rightarrow \frac{dY(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \eta(x),$$

and

$$\frac{dY}{dx} = \frac{dy}{dx} + \epsilon \frac{d\eta}{dx} \text{ or } Y'(x, \epsilon) = y'(x) + \epsilon \eta'(x) \Rightarrow Y'(x, \epsilon) \bigg|_{\epsilon=0} = y'(x).$$

There follows

$$\frac{dY'(\epsilon)}{d\epsilon} = \frac{d}{d\epsilon} [y'(x) + \epsilon \eta'(x)] = \eta'(x) \Rightarrow \frac{dY'(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \eta'(x).$$

For the Geodesic the integral

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y(x, \epsilon), Y'(x, \epsilon)) \, dx,$$

must be stationary, that means

$$\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \int_{x_1}^{x_2} \left. \frac{d}{d\epsilon} \left[ F(x, Y(x, \epsilon), Y'(x, \epsilon)) \right] \right|_{\epsilon=0} \, dx = 0.$$

Noting that

$$\left. \frac{d}{d\epsilon} \left[ F(x, Y(x, \epsilon), Y'(x, \epsilon)) \right] \right|_{\epsilon=0} = \frac{\partial F}{\partial Y} \left. \frac{dY(\epsilon)}{d\epsilon} \right|_{\epsilon=0} + \frac{\partial F}{\partial Y'} \left. \frac{dY'(\epsilon)}{d\epsilon} \right|_{\epsilon=0}$$

and substituting

$$Y(x, \epsilon) \bigg|_{\epsilon=0} = y(x), \quad \left. \frac{dY(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \eta(x), \quad Y'(x, \epsilon) \bigg|_{\epsilon=0} = y'(x),$$

$$\left. \frac{dY'(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \eta'(x),$$

(3.255)
we find
\[
\frac{d}{d\epsilon} [F(x, Y(x, \epsilon), Y'(x, \epsilon))] \Big|_{\epsilon=0} = \left[ \frac{\partial}{\partial y} F(x, y(x), y'(x)) \right] \eta(x) 
+ \left[ \frac{\partial}{\partial y'} F(x, y(x), y'(x)) \right] \eta'(x),
\]
(3.256)

Then the integral for the Geodesic line becomes
\[
\frac{dI(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial y} F(x, y(x), y'(x)) \right] \eta(x) \, dx + \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial y'} F(x, y(x), y'(x)) \right] \eta'(x) \, dx.
\]
(3.257)

Using integration by parts the second integral can be written as
\[
\int_{x_1}^{x_2} \left[ \frac{\partial}{\partial y'} F(x, y(x), y'(x)) \right] \eta'(x) \, dx = \eta(x_1) \frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial y'} \right] \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial y'} \right] \eta(x) \, dx
\]
(3.259)

recalling that
\[
\eta(x_1) = \eta(x_2) = 0
\]
(3.260)

we find
\[
\int_{x_1}^{x_2} \left[ \frac{\partial}{\partial y} F(x, y(x), y'(x)) \right] \eta'(x) \, dx = - \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial y'} \right] \eta(x) \, dx.
\]
(3.261)

Thus the stationary integral can be put in the form
\[
\frac{dI(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial y} F(x, y(x), y'(x)) \right] \eta(x) \, dx + \int_{x_1}^{x_2} \eta(x) \frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial y'} \right] \eta(x) \, dx = 0.
\]
(3.262)

or
\[
\frac{dI(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} \right] \eta(x) \, dx = 0.
\]
(3.263)

There follow that
\[
\frac{\partial}{\partial x} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0
\]
(3.264)

where \( F = F(x, y(x), y'(x)) \).
3.12 The geodesic on a manifold

We recall that a local geodesic at point $P$ is where the affine connection is zero

$$\Gamma^b_{ac}(P) = \Gamma^b_{ac}(P) - \Gamma^b_{ac}(P) = 0.$$  \tag{3.265}

Suppose we have a set of points in Euclidean space defining a geodesic curve, then at all of these points the affine connection must vanish.

$$\partial_c \hat{e}_b = \Gamma^f_{bc} \hat{e}_f = 0$$ \tag{3.266}

The basis vector with respect to the coordinates $x^c$ does not change along the geodesic curve. This means for the tangent vector

$$\dot{t} = \frac{dx^a}{du} \hat{e}_a$$ \tag{3.267}

to the curve at least the direction remains the same. In the Euclidean space this makes the curve to be a straight line where the tangent vectors has same direction along the line. Thus for Euclidean space the geodesic is a straight line.

For general curve defined by $x^a = x^a(u)$, on a manifold, if the curve is geodesic then the tangent curve must have the same direction at all points on the curve. This mean the change in the tangent vector with respect to the parameter $u$ is then only the magnitude that changes and it is given by

$$\frac{d\dot{t}}{du} = \lambda(u) \dot{t} = \lambda(u) \frac{dx^a}{du} \hat{e}_a,$$ \tag{3.268}

where $\lambda(u)$ is some function of $u$. Using our result for intrinsic derivative of a vector $\dot{v}(u)$

$$\frac{d}{du} \dot{v}(u) = \frac{Dv^a}{Du} \hat{e}_a = \left( \frac{dv^a}{du} + \Gamma^a_{bc} \frac{dx^b}{du} \right) \hat{e}_a,$$ \tag{3.269}

for the tangent vector intrinsic derivative, we have

$$\frac{d\dot{t}}{du} = \frac{Dt^a}{Du} \hat{e}_a = \left( \frac{dt^a}{du} + \Gamma^a_{bc} \frac{dx^b}{du} \right) \hat{e}_a,$$ \tag{3.270}

so that substituting this into Eq. (3.268), we find

$$\frac{dt^a}{du} + \Gamma^a_{bc} \frac{dx^b}{du} = \lambda(u) \frac{dx^a}{du}$$ \tag{3.271}

Noting that

$$\dot{t} = \frac{dx^a}{du} \hat{e}_a = t^a \hat{e}_a$$ \tag{3.272}

Eq. (3.271) can be written as

$$\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^c}{du} \frac{dx^b}{du} = \lambda(u) \frac{dx^a}{du}$$ \tag{3.273}
The result in Eq. (3.273) is valid for both non-null and null geodesics parameterized in terms of some general parameter \( u \). For **affine parameter** \( u \) where it is related to the distance \( s \) on the curve

\[
u = as + b
\]  

(3.274)

we have

\[
du = ads
\]  

(3.275)

and

\[
|\dot{t}| = \left| \frac{ds}{du} \right| = \frac{1}{a}
\]  

(3.276)

which is a tangent vector with a constant length that is independent of the parameter \( u \). This means

\[
\frac{d|\dot{t}|}{du} = \lambda(u) |\dot{t}| = 0 \Rightarrow \lambda(u) = 0
\]  

(3.277)

Therefore in general for a **affine parameter**, where \( \lambda(u) = 0 \), called the **affine parameter** the equation for the geodesic in Eq. (3.273) can be written as

\[
\frac{d^2x^a}{du^2} + \Gamma^a_{cb} \frac{dx^c}{du} \frac{dx^b}{du} = 0
\]  

(3.278)

Eq. (3.278) is a **parallel transport** for the tangent vector that we discussed in the previous section.

\[
\frac{Dt^a}{Du} = \frac{dt^a}{du} + \Gamma^a_{cb} \frac{dx^b}{du} = 0
\]  

(3.279)

which can be shown by replacing

\[
\frac{dx^a}{du} = t^a, \quad \frac{dx^c}{du} = t^c
\]  

(3.280)

in Eq. (3.278).

**Homework:**

*If we change the affine parameter \( u \) to \( u' \), the coordinates that define the geodesic curve would change from \( x^a(u) \) to \( x^a(u') \). Show that in terms of the new affine parameter, \( u' \), the geodesic in Eq. (3.278) becomes*

\[
\frac{d^2x^a}{du'^2} + \Gamma^a_{cb} \frac{dx^c}{du'} \frac{dx^b}{du'} = \left( \begin{array}{c} \frac{d^2u}{du'^2} \\ \frac{du}{du'} \end{array} \right) \frac{dx^a}{du'}
\]  

(3.281)

### 3.13 Stationary property of the non-null geodesic

Let’s consider the curve \( C \) in our manifold defined by the coordinates \( x^a(u) \). Suppose we have two points 1 and 2 on this curve and we are interested in the
3.13. STATIONARY PROPERTY OF THE NON-NULL GEODESIC

Length along this curve joining these two points, this length can be determined from

\[ L = \int_{1}^{2} \sqrt{g_{ab} dx^a dx^b} = \int_{1}^{2} \sqrt{g_{ab} \frac{dx^a}{du} \frac{dx^b}{du}} du \quad (3.282) \]

or using the notation

\[ \frac{dx^a}{du} = \dot{x}^a \quad (3.283) \]

this length can be expressed as

\[ L = \int_{1}^{2} \sqrt{g_{ab} \dot{x}^a \dot{x}^b} du = \int_{1}^{2} F du, \quad (3.284) \]

where

\[ F = \sqrt{g_{ab} \dot{x}^a \dot{x}^b} = s = \frac{ds}{du} \quad (3.285) \]

Using the principle of variation that you have learned in *Theoretical Physics Part I* and revised in the previous section the curve along a surface which marks the shortest distance between two neighboring points is the geodesic. Using the principle of variation it can be shown that the integral must be stationary for the geodesic and the function satisfies the Euler-Lagrange Equation:

\[ \frac{d}{du} \left( \frac{\partial F}{\partial \dot{x}^c} \right) - \frac{\partial F}{\partial x^c} = 0. \quad (3.286) \]

Using Eq. (3.285), we have

\[
\begin{align*}
\frac{\partial F}{\partial x^c} &= \frac{\partial}{\partial x^c} \left[ \sqrt{g_{ab} \dot{x}^a \dot{x}^b} \right] = \frac{\dot{x}^a \dot{x}^b \partial_c g_{ab}}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} = \frac{\dot{x}^a \dot{x}^b \partial_c g_{ab}}{2s} \\
\frac{\partial F}{\partial \dot{x}^c} &= \frac{\partial}{\partial \dot{x}^c} \left[ \sqrt{g_{ab} \dot{x}^a \dot{x}^b} \right] = \frac{g_{ab}}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} \left( \dot{x}^b \frac{\partial \dot{x}^a}{\partial \dot{x}^c} + \dot{x}^a \frac{\partial \dot{x}^b}{\partial \dot{x}^c} \right) \\
&= \frac{g_{ab} \dot{x}^b}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} \delta^a_c + \frac{g_{ab} \dot{x}^a}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} \delta^b_c = \frac{g_{ab} \dot{x}^b}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} + \frac{g_{ac} \dot{x}^a}{2 \sqrt{g_{ac} \dot{x}^a \dot{x}^c}} \\
\Rightarrow \frac{\partial F}{\partial \dot{x}^c} &= \frac{g_{ab} \dot{x}^b}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} + \frac{g_{ac} \dot{x}^a}{2 \sqrt{g_{ac} \dot{x}^a \dot{x}^c}} \\
&= \frac{g_{ab} \dot{x}^b}{2 \sqrt{g_{ab} \dot{x}^a \dot{x}^b}} + \frac{g_{ac} \dot{x}^a}{2 \sqrt{g_{ac} \dot{x}^a \dot{x}^c}} \quad (3.287)
\end{align*}
\]

If we replace \( b \) by \( a \) in the first term, we can write

\[
\frac{\partial F}{\partial \dot{x}^c} = \frac{g_{ac} \dot{x}^a}{2 \sqrt{g_{ac} \dot{x}^a \dot{x}^c}} + \frac{g_{ac} \dot{x}^a}{2 \sqrt{g_{ac} \dot{x}^a \dot{x}^c}} = \frac{g_{ac} \dot{x}^a}{\sqrt{g_{ac} \dot{x}^a \dot{x}^c}} = \frac{g_{ac} \dot{x}^a}{s}
\]

so that the Euler-Lagrange Equation becomes

\[
\frac{d}{du} \left( \frac{g_{ac} \dot{x}^a}{s} \right) - \frac{\dot{x}^a \dot{x}^b \partial_a g_{ab}}{2s} = 0. \quad (3.288)
\]
For the first terms we may write
\[
\frac{d}{du} \left( \frac{g_{ac} \dot{x}^a}{s} \right) = \frac{\dot{x}^a}{s} \frac{dg_{ac}}{du} + \frac{g_{ac} \ddot{x}^a}{s} \frac{d}{du} \left( \frac{1}{s} \right) = \frac{\dot{x}^a}{s} \frac{dg_{ac}}{du} + \frac{g_{ac} \ddot{x}^a}{s} \frac{d}{du} - \frac{g_{ac} \dot{x}^a}{s} \frac{d}{du} \left( \frac{1}{s^2} \right)
\]
and noting that
\[
\frac{dg_{ac}}{du} = \frac{\partial g_{ac}}{\partial x^b} \frac{dx^b}{du} = (\partial_b g_{ac}) \dot{x}^b
\]
we find
\[
\frac{d}{du} \left( \frac{g_{ac} \dot{x}^a}{s} \right) = \frac{1}{s} \left[ (\partial_b g_{ac}) \dot{x}^a \dot{x}^b + g_{ac} \ddot{x}^a - \frac{\dot{x}^a \dot{x}^b \partial_b g_{ab}}{2} \right] = 0.
\]
Now substituting Eq. (3.291) into Eq. (3.288), we find
\[
\frac{1}{s} \left[ (\partial_b g_{ac}) \dot{x}^a \dot{x}^b + g_{ac} \ddot{x}^a - \frac{\dot{x}^a \dot{x}^b \partial_b g_{ab}}{2} \right] = \frac{\ddot{x}^a \partial_b g_{ab}}{2} = \frac{g_{ac} \ddot{x}^a}{s}.
\]
Now applying the relation \(g_{d}^{de} g_{ac} = \delta^d_a\), we find
\[
\frac{\delta^d_a \dot{x}^a}{s} + \frac{1}{2} g^{de} [\partial_b g_{ac} + \partial_a g_{bc} - \partial_\alpha g_{\alpha b}] \dot{x}^a \dot{x}^b = \frac{(8)}{s} \delta^d_a \dot{x}^a.
\]
which simplifies into
\[
\dot{x}^d + \frac{1}{2} g^{de} [\partial_b g_{ac} + \partial_a g_{bc} - \partial_\alpha g_{\alpha b}] \dot{x}^a \dot{x}^b = \frac{(8)}{s} \dot{x}^d.
\]
Using the expression for the affine connection in terms of the metric

$$\Gamma^a_{bc} = \frac{1}{2} g^{dc} \left[ \partial_d g_{ac} + \partial_a g_{dc} - \partial_c g_{da} \right]$$

(3.301)

we find

$$\ddot{x}^d + \Gamma^a_{bc} \dot{x}^a \dot{x}^b = \left( \frac{\ddot{s}}{s} \right) \dot{x}^d.$$  

(3.302)

Comparing the result in Eq. (3.302) with Eq. (3.281)

$$\frac{d^2 x^a}{du'^2} + \Gamma^a_{bc} \frac{dx^c}{du'} \frac{dx^b}{du'} = \left( \frac{d^2 u}{du'^2} \right) \frac{dx^a}{du'}$$

(3.303)

we see that these equations are equivalent to one another.
Chapter 4

Tensor Calculus on manifolds

4.1 Tensors fields and rank of a tensor

In order to understand what a tensor is and what is its rank is, it is important to have a better understanding of a vector field, \( \vec{v} \), on a manifold. How do we define a vector field on a manifold. We have learned that a vector field at a given point, \( P \), on a manifold is defined by the tangent plane, \( T_P \), at that point on the manifold. This tangent plane is defined by the tangent vector, \( \vec{t} \). The tangent plane is defined by the basis vectors, \( \hat{e}_a \). We can denote the number produced by the action of the vector, \( \vec{t} \) on the vector, \( \vec{v} \), (the component of the vector field, \( \vec{v} \), on the tangent space) by the scalar product

\[
\vec{t} (\vec{v}) = \vec{t} \cdot \vec{v}.
\]

This maps the vector field, \( \vec{v} \), to the tangent space, \( T_p \). In this case one vector (\( \vec{v} \)) is linearly mapped into the tangent space by the tangent vector \( \vec{t} \) (i.e. \( \vec{t} \to \vec{t} (\vec{v}) \)). Therefore the tangent vector \( \vec{t} \) forms a first rank tensor \( t \).

A tensor: Based on the notion of a vector on a manifold, a tensor is defined by the precise set of operations applied to the a set of vectors to produce a scalar and the number of vectors in the set determines the rank of the tensor. If there are \( N \) number of vectors in the set, the tensor is \( N^{th} \) rank tensor (See the table below)

<table>
<thead>
<tr>
<th>Tensor</th>
<th>Operation</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t(\vec{u}, \vec{v}) )</td>
<td>( \vec{u} \cdot \vec{v} )</td>
<td>2</td>
</tr>
<tr>
<td>( t(\vec{u}, \vec{v}, \vec{w}) )</td>
<td>( \vec{u} \cdot \vec{v} \cdot \vec{w} )</td>
<td>3</td>
</tr>
<tr>
<td>( t(\vec{u}, \vec{v}, \vec{w}, \vec{x}) )</td>
<td>( \vec{u} \cdot \vec{v} \cdot \vec{w} \cdot \vec{x} )</td>
<td>4</td>
</tr>
<tr>
<td>( t(\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{y}) )</td>
<td>( \vec{u} \cdot \vec{v} \cdot \vec{w} \cdot \vec{x} \cdot \vec{y} )</td>
<td>5</td>
</tr>
</tbody>
</table>

From this table we can easily see that a scalar field can be classified as a zero ranked tensor field since it does not depend on a vector field.
A tensor is a linear map of the vectors into the real and therefore any ranked tensor is linear. This means, for example, for 1\textsuperscript{st} rank tensor, we must have

$$\tilde{t}(\alpha \bar{u} + \beta \bar{v}) = \tilde{t}(\alpha \bar{u}) + \tilde{t}(\beta \bar{v}) = \alpha \tilde{t}(\bar{u}) + \beta \tilde{t}(\bar{v}) \quad (4.2)$$

Homework: Show that 2\textsuperscript{nd} rank tensor is linear.

Components of a tensor: We recall that the tangent plane is defined in terms of the basis ($\vec{e}_a$) or dual basis vectors ($\vec{e}^b$). When vectors are expressed in terms of basis or dual basis vectors, we can determine the components of a tensor in different forms. But first let’s consider if the vectors, $\bar{v}$ and $\bar{u}$, are just the basis or the dual basis vectors. In this case we have

(a) 1\textsuperscript{st} rank tensor

$$t (\vec{e}_a) = \tilde{t} \cdot \vec{e}_a = t_a, \quad (4.3)$$
$$t (\vec{e}^b) = \tilde{t} \cdot \vec{e}^b = t^b. \quad (4.4)$$

(b) 2\textsuperscript{nd} rank tensor

$$t (\vec{e}_a, \vec{e}_b) = t^{ab}, \quad t (\vec{e}_a, \vec{e}^b) = t_{ab} \quad (4.5)$$
$$t (\vec{e}_a, \vec{e}^b) = t^b_a, \quad t (\vec{e}^a, \vec{e}^b) = t^a_b \quad (4.6)$$

Example 4.1 Let’s reconsider the 2D sphere in the 3D manifold For the vector on the tangent plane shown in Fig. 4.1

$$t (\vec{e}_a) = t_1 \vec{e}^1 + t_2 \vec{e}^2 \text{ or } t (\vec{e}^a) = t^1 \vec{e}_1 + t^2 \vec{e}_2$$
we have
\[ t_1 = t(e_a^1) \cdot e^a = (t^1 e_1 + t^2 e_2) \cdot e^a, \quad t_2 = t(e_a^2) \cdot e^a = (t^1 e_1 + t^2 e_2) \cdot e^a \]
\[ \Rightarrow t(e_a^2) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \]
which is a 1-st rank tensor. Now let’s consider a quantity defined by an operation on set that consist of two vectors as
\[ t(e_a^1, e_b^1) = (e^1 \cdot e_1) + (e^2 \cdot e_1) + (e^1 \cdot e_2) + (e^2 \cdot e_2) \]
\[ \Rightarrow t(e_a^1, e_b^1) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \]
which forms a 2-nd rank tensor. Using the dual basis vector, we can also express this 2-nd rank tensor as
\[ t(e_a^1, e_b^1) = (e^1 \cdot e_1) + (e^2 \cdot e_1) + (e^1 \cdot e_2) + (e^2 \cdot e_2) \]
\[ \Rightarrow t(e_a^1, e_b^1) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \]
or
\[ t(e_a^1, e_b^1) = (e^1 \cdot e_1) + (e^2 \cdot e_1) + (e^1 \cdot e_2) + (e^2 \cdot e_2) \]
\[ \Rightarrow t(e_a^1, e_b^1) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \]

Now we can apply the linearity of tensors to determine the components of a tensor for the general case where we have two vectors, \( \vec{u} \) and \( \vec{v} \), expressed in terms of the covariant or contravariant components
\[ \vec{u} = u_a e^a, \quad \vec{v} = v_b e^b, \quad (4.7) \]
\[ \vec{u} = u^a e_a, \quad \vec{v} = v^b e_b. \quad (4.8) \]

(a) 1\textsuperscript{st} rank tensor
\[ t(\vec{u}) = t(u_a e^a) = \frac{t}{t^a} u_a = t^a u_a, \quad (4.9) \]
\[ t(\vec{u}) = t(u^a e_a) = u^a t(e_a^a) = t_a u^a \quad (4.10) \]

(b) 2\textsuperscript{nd} rank tensor
\[ t(\vec{u}, \vec{v}) = t(u^a e_a, v_b e^b) = t(e_a^a, e_b^b) u^a v_b = t^b u^a v_b, \quad (4.11) \]
\[ t(\vec{u}, \vec{v}) = t(u_a e^a, v^b e_b) = t(e^a, \vec{e}_b) u_a v^b = t^b u_a v^b, \quad (4.12) \]
\[ t(\vec{u}, \vec{v}) = t(u^a e^a, v^b e_b) = t(e_a^a, \vec{e}_b) u^a v^b = t_{ab} u^a v^b. \quad (4.13) \]

Note that
\[ t(\vec{u}, \vec{v}) = t_{ab} u^a v^b = t^b u^a v_b = t^b u_a v^b \quad (4.14) \]
Symmetries of a tensor: A tensor can be symmetric or antisymmetric. For a second ranked tensor \( t(\vec{u}, \vec{v}) \)

\[
t(\vec{u}, \vec{v}) = \begin{cases} -t(\vec{v}, \vec{u}), & \text{Antisymmetric} \\ t(\vec{v}, \vec{u}), & \text{symmetric} \end{cases}
\]

Any tensor can be expressed as a sum of symmetric and antisymmetric tensor. Again if we consider a second rank tensor with elements \( t_{ab} \), we can express this elements as

\[
t_{ab} = \frac{1}{2} (t_{ab} + t_{ba}). \tag{4.15}
\]

Introducing the notations for the symmetric part

\[
t_{(ab)} = \frac{1}{2} (t_{ab} + t_{ba}) \tag{4.16}
\]

and the antisymmetric part

\[
t_{[ab]} = \frac{1}{2} (t_{ab} - t_{ba}) \tag{4.17}
\]

we can write

\[
t_{ab} = t_{(ab)} + t_{[ab]} \tag{4.18}
\]

\(N\)th ranked tensor: for an \(N\)th ranked tensor the symmetric and antisymmetric covariant components of the tensor, \( t_{a_1a_2...a_N} \) are given by

\[
t_{(a_1a_2...a_N)} = \frac{1}{N!} \text{(addition over all permutations of the indices } a_1a_2...a_N) \tag{4.19}
\]

and

\[
t_{[a_1a_2...a_N]} = \frac{1}{N!} \text{(Alternating subtraction and addition over all permutations of the indices } a_1a_2...a_N) \tag{4.20}
\]

\[
t_{[a_1a_2a_3]} = \frac{1}{3!} (t_{[a_1a_2a_3]} \tag{4.21}
\]

For example for 3rd rank tensor, we have

\[
t_{(a_1a_2a_3)} = \frac{1}{3!} (t_{a_1a_2a_3} + t_{a_2a_1a_3} + t_{a_1a_2a_3} + t_{a_3a_2a_1} + t_{a_1a_2a_3} + t_{a_1a_3a_2}) \tag{4.22}
\]

and

\[
t_{(a_1a_2a_3)} = \frac{1}{3!} (t_{a_1a_2a_3} - t_{a_2a_1a_3} + t_{a_1a_2a_3} - t_{a_3a_2a_1} + t_{a_1a_2a_3} - t_{a_1a_3a_2}) \tag{4.23}
\]

Particular subset of indices permutation: we have a different notations when the permutation is to particular subset of indices. This is described using a 4th
4.1. TENSORS FIELDS AND RANK OF A TENSOR

rank tensor.

\[
t_{(ab)cd} = \frac{1}{2} (t_{abcd} + t_{bacd})
\]

symmetric permutation to indices \(a\) and \(b\) only \hspace{1cm} (4.24)

\[
t_{[ab]cd} = \frac{1}{2} (t_{abcd} - t_{bacd})
\]

antisymmetric permutation to indices \(a\) and \(b\) only \hspace{1cm} (4.25)

\[
t_{a[b|c|d]} = \frac{1}{2} (t_{abcd} - t_{bdca})
\]

antisymmetric permutation to indices \(b\) and \(d\) only \hspace{1cm} (4.26)

Note that the symbol \(\|\) are used to exclude unwanted indices from the symmetrization () antisymmetrization [] implied.

**Example 4.2** Let’s reconsider the 3D sphere in the 4D manifold. We recall the tangent space at a point on this 3D sphere form three basis and dual basis vectors, \(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\) (or \(\tilde{e}_r, \tilde{e}_q, \tilde{e}_\phi\)) and \(\hat{e}^1, \hat{e}^2, \hat{e}^3\) (or \(\hat{e}^r, \hat{e}^q, \hat{e}^\phi\)). Write the expression for the symmetric, \(t_{(123)}\) and antisymmetric \(t_{[123]}\) components.

**Solution:** The expressions for these components are given by

\[
t_{(123)} = \frac{1}{3!} \left[ t_{(123)} + t_{(132)} + t_{(213)} + t_{(231)} + t_{(312)} + t_{(321)} \right]
\]

and

\[
t_{[123]} = \frac{1}{3!} \left[ t_{(123)} - t_{(132)} + t_{(312)} - t_{(321)} + t_{(231)} - t_{(213)} \right]
\]
Homework: Write the explicit form of the expressions for
(a) $t_{(312)}$
(b) $t_{[ab]_{(cd)}}$

### 4.2 The metric tensor revisited

A good example of a 2nd rank tensor is the metric tensor, $g$. Here more generally we define the metric tensor as a linear map of two vectors $\bar{u}$ and $\bar{v}$ into the number that is the inner product

$$g(\bar{u}, \bar{v}) = \bar{u} \cdot \bar{v} \tag{4.27}$$

We can easily see that the metric tensor is symmetric tensor as

$$g(\bar{u}, \bar{v}) = g(\bar{v}, \bar{u}) = \bar{v} \cdot \bar{u} \tag{4.28}$$

We recall that covariant and controvariant components of the metric tensor are given by

$$g_{ab} = g(\bar{e}_a, \bar{e}_b) = \bar{e}_a \cdot \bar{e}_b, g^{ab} = g(\bar{e}^a, \bar{e}^b) = \bar{e}^a \cdot \bar{e}^b \tag{4.29}$$

and the mixed components

$$g^b_a = g(\bar{e}_a, \bar{e}^b) = g(\bar{e}^a, \bar{e}_b) = \delta_b^a \tag{4.30}$$

*Raising and lowering tensor indices.* Consider the third rank tensor, $t$, expressed in terms of its covariant components

$$t_{abc} = t(\bar{e}_a \cdot \bar{e}_b \cdot \bar{e}_c) \tag{4.31}$$

or mixed components

$$t^c_{ab} = t(\bar{e}^c \cdot \bar{e}_a \cdot \bar{e}_b) \tag{4.32}$$

We recall from chapter 3 that the covariant form of the metric tensor can be used to lower and the controvariant to raise the index of a vector. This can be extended to raising and lowering indices of a higher ranked tensor. For example, we can lower the index of the 3-rd ranked tensor in Eq. (4.32) by multiplying it with the covariant form of the metric tensor, $g_{dc}$,

$$g_{dc}t^c_{ab} = g_{dc}t(\bar{e}^c \cdot \bar{e}_b \cdot \bar{e}_a) \tag{4.33}$$

noting that

$$g_{dc} = \bar{e}^d \cdot \bar{e}_c, \text{ and } \bar{e}^b \cdot \bar{e}^c = \delta_b^c$$

we have

$$g_{dc}t^c_{ab} = \bar{e}^d \cdot \bar{e}_c \cdot \bar{e}^c = \delta^c_{dc} = \bar{e}^d \cdot \bar{e}_b \cdot \bar{e}^a = t_{dba} \tag{4.34}$$

so that

$$t_{dba} = g_{dc}t^c_{ab} \tag{4.35}$$
4.3. MAPPING TENSORS INTO TENSORS

Since the indices are dummy variables and tensor $t_{abc}$ is symmetric, one can write

$$t_{abc} = g_{cd} t_{ab}$$

(4.36)

An alternative way of showing this is to use the linearity of operations involving tensors

$$g_{dc} t_{ab} = g_{dc} (e^a, e^b, e^c) = t (e^a, e^b, g_{dc} e^c) = t_{abd}$$

(4.37)

where we used

$$g_{dc} e^c = e^d$$

Noting that

Homework: Raise the tensor $t_{abc}$ to $t_{abc}^c$.

4.3 Mapping tensors into tensors

We have learned that tensor maps a set of vectors operation into real numbers. For example we saw that $1^{st}$ rank tensor $t(\tilde{u})$ maps the vector $\tilde{u}$ into real numbers in the tangent space.

$$t(\tilde{u}) = t(u^a \tilde{e}_a) = u^a t(\tilde{e}_a) = t_a u^a$$

(4.38)

and the second rank tensor $t(\tilde{u}, \tilde{v})$ maps the operation involving the vectors $\tilde{u}$ and $\tilde{v}$ into real numbers in the tangent space

$$t(\tilde{u}, \tilde{v}) = t(u^a \tilde{e}_a, v^b \tilde{e}_b) = t(\tilde{e}_a, \tilde{e}_b) u^a v^b = t_{ab} u^a v^b.$$ 

For example 2D sphere where the tangent space has two basis vectors $t(\tilde{e}_1, \tilde{e}_2)$ we

$$t(\tilde{u}, \tilde{v}) = t_{ab} u^a v^b$$

$$= t_{11} u^1 v^1 + t_{12} u^2 v^1 + t_{11} u^1 v^2 + t_{12} u^2 v^2 + t_{21} u^1 v^1 + t_{22} u^2 v^1 + t_{21} u^1 v^2 + t_{22} u^2 v^2$$

and using matrices one can put this in the form

$$t(\tilde{u}, \tilde{v}) = \begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}$$

The 2-nd rank tensor $t(\tilde{e}_a, \tilde{e}_b)$ maps the vector operation involving $\tilde{u}(\tilde{e}_a)$ and $\tilde{v}(\tilde{e}_a)$ into a real number in the tangent space. Now the question is, can we map a tensor into another tensor of a different rank. Consider a 3-rd rank tensor $t(\tilde{u}, \tilde{v}, \tilde{w})$ which maps the three vectors into real numbers. Let’s replace the two vectors $(\tilde{u}, \tilde{v})$ by the basis vectors $(\tilde{e}_a, \tilde{e}_b)$ in the tangent space,

$$t(\tilde{e}_a, \tilde{e}_b, \tilde{w}) = \tilde{e}_a \cdot \tilde{e}_b \cdot \tilde{w}. \quad (4.39)$$

Suppose we express the vector, $\tilde{w}$, in terms of its controvariant components

$$\tilde{w} = w^c \tilde{e}_c,$$ 

(4.40)
we find

\[ t\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = t\left(\varepsilon^a, \varepsilon^b, w^e \varepsilon^e\right) = t\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) w^e = t_{abc} w^e = s_{ab}. \quad (4.41) \]

which is a 2-nd rank tensor. We see that the 3-rd rank tensor \( t \) maps the vector, \( \vec{w} \), into a second rank tensor \( s \): As another example let’s consider the third rank tensor

\[ t\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = \varepsilon^a \varepsilon^b \varepsilon^c. \quad (4.42) \]

Suppose we express the vectors, \( \vec{v} \) and \( \vec{w} \), in terms of its contravariant components

\[ \vec{w} = w^e \varepsilon^e, \vec{v} = v^a \varepsilon^a, \quad (4.43) \]

then

\[ t\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = t\left(u^a \varepsilon^a, \varepsilon^b, w^c \varepsilon^c\right) = t\left(\varepsilon^a \cdot \varepsilon^b \cdot \varepsilon^c\right) u^a w^c = t_{abc} u^a w^c = v_b, \quad (4.44) \]

where we find a 1-st rank tensor, \( s_b \). This means the 3-rd rank tensor mapped the two vectors into 1-st rank tensor (i.e. a vector).

### 4.4 Elementary tensor operations

*Adding, subtracting, and multiplying by a scalar:*

\[ s_{ab} = \alpha s\left(\varepsilon^a, \varepsilon^b\right) = \alpha t\left(\varepsilon^a, \varepsilon^b\right) \pm \alpha r\left(\varepsilon^a, \varepsilon^b\right) = \alpha t_{ab} \pm \alpha r_{ab}. \quad (4.45) \]

*Outer product:* Consider two first rank tensors (i.e. two vectors) \( u\left(\vec{p}\right) \) and \( v\left(\vec{q}\right) \). The outer product or (tensor product) of these two tensors, which is denoted by \( u \otimes v \) is given by

\[ u \otimes v\left(\vec{p}, \vec{q}\right) = u\left(\vec{p}\right) v\left(\vec{q}\right), \quad (4.46) \]

which is a second rank tensor. Suppose \( \left(\vec{p}, \vec{q}\right) \rightarrow \left(\varepsilon^a, \varepsilon^b\right) \), then

\[ u \otimes v\left(\varepsilon^a, \varepsilon^b\right) = u\left(\varepsilon^a\right) v\left(\varepsilon^b\right) = u_a v_b. \quad (4.47) \]

Now let’s consider 2-nd rank tensor, \( t\left(\vec{p}, \vec{q}\right) \), and 1-st rank tensor (a vector), \( s\left(\vec{r}\right) \), the outer product of these two tensors give another tensor, \( h \), of a different rank given by

\[ t \otimes s\left(\vec{p}, \vec{q}, \vec{r}\right) = t\left(\vec{p}, \vec{q}\right) s\left(\vec{r}\right) = h. \quad (4.48) \]

The tensor \( h \) is a 3-rd rank tensor. Using the basis and dual basis vectors we may express the components as

\[ t \otimes s\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = t\left(\varepsilon^a, \varepsilon^b\right) s\left(\varepsilon^c\right) = h_{abc} \quad (4.49) \]

\[ t \otimes s\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = t\left(\varepsilon^a, \varepsilon^b\right) s\left(\varepsilon^c\right) = h_{abc}^c \quad (4.50) \]

\[ t \otimes s\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = t\left(\varepsilon^a, \varepsilon^b\right) s\left(\varepsilon^c\right) = h_{abc}^b \quad (4.51) \]

\[ t \otimes s\left(\varepsilon^a, \varepsilon^b, \varepsilon^c\right) = t\left(\varepsilon^a, \varepsilon^b\right) s\left(\varepsilon^c\right) = h_{abc}^a \quad (4.52) \]
Example 4.2 Let’s consider two vectors in the tangent space for a point in a 3D sphere embedded in 4D manifold given by

\[ \vec{p} = p^a \hat{e}_a, \vec{q} = q^b \hat{e}_b \]

where \( a, b = 1, 2, 3 \), and \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \) (or \( \hat{e}_r, \hat{e}_\theta, \) and \( \hat{e}_\phi \)). Find the components of the 2-nd rank tensor for

\[ u \otimes v (\vec{p}, \vec{q}) = u (\vec{p}) v (\vec{q}), \quad (4.53) \]

Sol: The components of this 2-nd rank tensor are given by

\[ u \otimes v (\vec{p}, \vec{q}) = p^a q^b \hat{e}_a \hat{e}_b \]

\[ = p^1 q^1 \hat{e}_1 \hat{e}_1 + p^1 q^2 \hat{e}_1 \hat{e}_2 + p^1 q^3 \hat{e}_1 \hat{e}_3 + p^2 q^1 \hat{e}_2 \hat{e}_1 + p^2 q^2 \hat{e}_2 \hat{e}_2 + p^2 q^3 \hat{e}_2 \hat{e}_3 + p^3 q^1 \hat{e}_3 \hat{e}_1 + p^3 q^2 \hat{e}_3 \hat{e}_2 + p^3 q^3 \hat{e}_3 \hat{e}_3 \]

and using a matrix this is expressed as

\[ u \otimes v (\vec{p}, \vec{q}) = \begin{bmatrix} p^1 q^1 & p^1 q^2 & p^1 q^3 \\ p^2 q^1 & p^2 q^2 & p^2 q^3 \\ p^3 q^1 & p^3 q^2 & p^3 q^3 \end{bmatrix} \]

\[ \text{NB: from what we learned in all mathematical or physics courses up to this point, what we know is that inner product of two 1-st rank tensor (two vectors) is commutative. However, that generalization does not apply to higher ranked tensor. So from now on we must keep in mind that inner product of tensors in general is not commutative including the 1-st rank tensor for the reason described in terms of tensor contraction. For example, if } t \text{ is a 2-nd rank tensor and } s \text{ is a first rank tensor, the inner product} \]

\[ t \cdot s = t^{ab} s_b \quad (4.54) \]

\[ \text{and} \]

\[ s \cdot t = t^{ab} s_a \quad (4.55) \]

\[ \text{are not necessarily the same.} \]

Like vectors (1-st rank tensor) tensors of higher rank are geometrical objects too: We already know that vectors are geometrical objects that can be made up from a linear combination of the basis vectors

\[ t = t_a \hat{e}^a = t^a \hat{e}_a. \quad (4.56) \]

The vector that defines a given geometry on a manifold, does not depend on how it is represented. The geometry that a vector defines remain the same geometry whatever representation we used to describe the vector. The same is true for
higher ranked tensors. As an example, let’s consider a 2-nd rank tensor with component $t_{ab}$ constructed from the outer product of two basis vectors

$$t = \tilde{e}_a \otimes \tilde{e}_b. \quad (4.57)$$

The controvariant components of $t$

$$t = \tilde{e}_a \otimes \tilde{e}_b (\tilde{e}^c, \tilde{e}^d) = \delta^c_a \delta^d_b. \quad (4.58)$$

Now suppose we have some general 2-nd rank tensor, $t$, with controvariant components, $t^{ab} (\tilde{e}_a \otimes \tilde{e}_b)$. The action of this second rank tensor on two basis vectors

$$t^{ab} (\tilde{e}_a \otimes \tilde{e}_b) (\tilde{e}^c, \tilde{e}^d) = t^{ab} \delta^c_a \delta^d_b = \delta^{cd}$$

Therefore what is true for example for 2-d rank tensor

$$t = t^{ab} (\tilde{e}_a \otimes \tilde{e}_b) = t^{ba} (\tilde{e}_a \otimes \tilde{e}_b) = t_{a}^{\, b} \tilde{e}_a \otimes \tilde{e}^b \quad (4.59)$$

is true for any ranked tensor.

### 4.5 Tensors and coordinate transformations

We recall the coordinates basis and the dual basis vectors, under the coordinate transformation $x^a \rightarrow x'^a$, are transformed as

$$\hat{e}^c = \frac{\partial x^a}{\partial x'^c} \hat{e}_a. \quad (4.61)$$

and

$$\hat{e}^a = \frac{\partial x'^a}{\partial x^c} \hat{e}_c. \quad (4.62)$$

**1-st rank tensor:** Suppose we have a 1-st rank tensor, $t$

$$t (\tilde{e}^a) = t_a \tilde{e}^a = t^a \tilde{e}_a. \quad (4.63)$$

in the $x'^a$ coordinate system is given by

$$t'^a = t'^a (\tilde{e}'^a) = \frac{\partial x'^a}{\partial x^c} t (\tilde{e}^c) = \frac{\partial x'^a}{\partial x^c} t_c. \quad (4.64)$$

$$t'_a = t'_a (\tilde{e}'_a) = \frac{\partial x^c}{\partial x'^a} t (\tilde{e}_c) = \frac{\partial x^c}{\partial x'^a} t_c. \quad (4.65)$$

**2-nd rank tensor:** Suppose we have a 2-nd rank tensor, $t (\tilde{e}^a, \tilde{e}^b), t (\tilde{e}'_a, \tilde{e}'_b)$, and $t (\tilde{e}'_a, \tilde{e}_b)$ in the $x'^a$ coordinate system is given by

$$t' (\tilde{e}'^a, \tilde{e}'^b) = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} t^{cd} \quad (4.66)$$

$$t' (\tilde{e}'_a, \tilde{e}'_b) = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} t_{cd} \quad (4.67)$$

$$t' (\tilde{e}'_a, \tilde{e}_b) = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x''^b} t_{cd} \quad (4.68)$$
4.6. TENSOR EQUATIONS AND THE QUOTIENT THEOREM

3-rd rank tensor: Suppose we have a mixed 3-rd rank tensor, \( t_{ab}^c \) in the \( x^a \) coordinate system is given by

\[
t (e'_a, e'_b, e'_c) = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \frac{\partial x^e}{\partial x'^f} t_{cd}^f
\]  

(4.69)

4.6 Tensor equations and the quotient theorem

A tensor equation which holds in one coordinate system must hold in another coordinate system. Suppose we have an equation that states two second rank tensors, \( t \) and \( s \) are equal in the \( x^a \) coordinate system. That means

\[
t_{cd} = s_{cd}
\]  

(4.70)

Multiplying both sides of this equation by \( \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \)

we have

\[
\frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} t_{cd} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} s_{cd}
\]  

(4.71)

so that using Eq. (4.68), we find

\[
t'_{ab} = s'_{ab}.
\]  

(4.72)

Well Eq. (4.72) shows that the equality holds under the coordinate transformation. However, the question is that these components (set of quantities) are actually form a tensor. The quotient theorem set’s the condition for a set of quantities actually represent a tensor component.

The quotient theorem: if a set of quantities when contracted with a tensor produces another tensor, then the original set of quantities are also a tensor.

Suppose in an \( N \) dimensional manifold you are given a 3-rd rank tensor, \( t \), and 1-st rank tensor, \( v \). The tensor, \( t \), has a set of \( N^3 \) quantities \( t_{a}^{b} s_{c}^{d} \) and the tensor \( v \) has \( N \) quantities of \( v^a \). Suppose we form a set of \( N^2 \) quantities by contracting the 4-th rank tensor formed by the outer product of these two tensors (i.e. \( s_{b}^{a} = t_{a}^{b} v^c \)). Under coordinate transformation \( x^a \rightarrow x'^a \), these set of elements, using the transformation relations,

\[
t'^{a} = \frac{\partial x'^a}{\partial x^c} t_{c}
\]  

(4.73)

\[
t'_{a} = \frac{\partial x'^a}{\partial x^c} t_{c}
\]  

(4.74)

in the new coordinate system are given by

\[
s_{b}^{a} = \frac{\partial x'^a}{\partial x^d} s_{d}^{b} \frac{\partial x^c}{\partial x'^b} s_{e}^{c} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} s_{e}^{d} = t_{b}^{a} v^c = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} t_{e}^{d} v^f.
\]  

(4.75)
In the relation

$$v'f = \frac{\partial x'^f}{\partial x^c} v^c$$  \hspace{1cm} (4.76)$$

by switching the the primes with the none primes, we can write

$$v'f = \frac{\partial x^f}{\partial x'^c} v'^c$$  \hspace{1cm} (4.77)$$

Substituting this expression into Eq. (4.75), we find

$$t^a_{bc} v^c = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} v^c \left( \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \right) = 0$$  \hspace{1cm} (4.78)$$

There follows that for an arbitrary vector components, $v^c$,

$$t^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} t^d_{ef}$$.  \hspace{1cm} (4.79)$$

We recall the covariant derivative from chapter 3

$$\nabla_b v^a = \frac{\partial v^a}{\partial x^b} + \Gamma^a_{b\gamma} v^\gamma$$  \hspace{1cm} (4.83)$$

so that the gradient of the first rank tensor can be expressed as

$$\nabla v = (\nabla_b v^a) e^a_a = \frac{\partial v^a}{\partial x^b} + \Gamma^a_{b\gamma} v^\gamma$$  \hspace{1cm} (4.84)$$

$\nabla_b v^a$ forms a mixed 2-nd rank tensor. We denote this second rank tensor by $\nabla v$. Noting that

$$\nabla = e^a_a, v = v^b e_b$$
4.7. COVARIANT DERIVATIVES OF A TENSOR

we can express
\[ \mathbf{v} = \mathcal{E}^a \partial_a \otimes v^b \mathcal{E}_b = \mathcal{E}^a \otimes \partial_a (v^b \mathcal{E}_b) = (\nabla_a v^b) \mathcal{E}^a \otimes \mathcal{E}_b \] (4.85)

Let’s consider the covariant derivative of a second rank tensor \( t^{ab} \) expressed in terms of its contravariant components \( t^{ab} \)
\[ \nabla_c t = \nabla_c t^{ab} \mathcal{E}_a \otimes \mathcal{E}_b \] (4.86)

Using the product rule, we have
\[ \partial_c t = (\partial_c t^{ab}) \mathcal{E}_a \otimes \mathcal{E}_b + t^{ab} (\partial_c \mathcal{E}_a) \otimes \mathcal{E}_b + t^{ab} \mathcal{E}_a \otimes (\partial_c \mathcal{E}_b) \] (4.87)

and recalling that
\[ \partial_c \mathcal{E}_b = \Gamma^f_{bc} \mathcal{E}_f \] (4.88)

one can write
\[ \partial_c t = (\partial_c t^{ab}) \mathcal{E}_a \otimes \mathcal{E}_b + t^{ab} \Gamma^f_{ca} \mathcal{E}_f \otimes \mathcal{E}_b + t^{ab} \mathcal{E}_a \otimes \Gamma^f_{cb} \mathcal{E}_f \] (4.89)

If we interchange the indices \( f \) and \( a \) in the second term and \( f \) by \( b \) in the third terms, we have
\[ \partial_c t = (\partial_c t^{ab}) (\mathcal{E}_a \otimes \mathcal{E}_b) + t^{ab} \Gamma^a_{cf} \mathcal{E}_a \otimes \mathcal{E}_b + t^{ab} \Gamma^b_{cf} \mathcal{E}_a \otimes \mathcal{E}_b \] (4.90)

which can be rewritten as
\[ \partial_c t = \left[(\partial_c t^{ab} + \Gamma^a_{cf} t^{fb} + \Gamma^b_{cf} t^{af})\right] (\mathcal{E}_a \otimes \mathcal{E}_b) = (\nabla_c t^{ab}) \mathcal{E}_a \otimes \mathcal{E}_b \] (4.92)

where
\[ \nabla_c t^{ab} = \partial_c t^{ab} + \Gamma^a_{cd} t^{db} + \Gamma^b_{cd} t^{ad} \] (4.93)

is the covariant derivative.

Homework: Show that for the covariant derivatives of the mixed and covariant component of a 2-nd rank tensor \( t \) are given by
\[ \nabla_c t^b = \partial_c t^b + \Gamma^a_{cd} t^d - \Gamma^d_{bc} t^a \] (4.94)
\[ \nabla_c t_{ab} = \partial_c t_{ab} - \Gamma^d_{ac} t_{db} - \Gamma^d_{bc} t_{ad} \] (4.95)

Useful relation
\[ \partial_c \mathcal{E}^a = -\Gamma^a_{bc} \mathcal{E}^b \] (4.96)

Homework: Show that the covariant derivative of the metric tensor is zero
\[ \nabla g = 0 \]

Suppose we represent the metric tensor in terms of its contravariant components, \( g^{ab} \), then you must show that the covariant derivative expressed as
\[ \nabla_c g^{ab} = \partial_c g^{ab} + \Gamma^a_{cd} g^{db} + \Gamma^b_{cd} g^{ad} = 0 \] (4.97)
Useful relations, for example, the affine connection and the metric are related by
\[ \Gamma^f_{bc} = \frac{g^{fd}}{2} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \] (4.98)

Application of the property of the metric tensor: Suppose we have a 2-nd rank tensor, \( t \), for which we want to find the covariant derivative from its components, for example, in mixed form. That means we want to find \( \nabla_c t^{ab} \) from \( t^a_d \). We can express the covariant components of this tensor using contraction as
\[ t^{ab} = g^{bd} t^a_d. \]

Note that the metric tensor is symmetric (\( g^{bd} = g^{db} \)). Then
\[ \nabla_c t^{ab} = \nabla_c (g^{bd} t^a_d) = \left( \nabla_c g^{bd} \right) t^a_d + g^{bd} \nabla_c t^a_d \] (4.99)
since
\[ \nabla_c g^{bd} = 0 \]
we can express
\[ \nabla_c t^{ab} = g^{bd} \nabla_c t^a_d. \] (4.100)

### 4.8 Intrinsic derivative

Like vectors (1-rank tensor), tensor of rank 2 or more can depend on a submanifold instead of the entire manifold. For example, a given tensor \( t \) can depend on a curve \( C \) on the manifold that defined by some parameter \( u \). This curve may be defined by the equation \( x^a(u) \). The tensor, \( t \), expressed in terms of its covariant components, then in terms of this parameter that the curve, \( C \), depends on can be expressed as
\[ t(u) = t^{ab} \hat{e}_a(u) \otimes \hat{e}_b(u). \]

The intrinsic derivative will then be
\[ \frac{dt(u)}{du} = \frac{dt^{ab}}{du} \hat{e}_a(u) \otimes \hat{e}_b(u) \]
\[ = \frac{dt^{ab}}{du} \hat{e}_a(u) \otimes \hat{e}_b(u) + t^{ab} \frac{d\hat{e}_a(u)}{du} \otimes \hat{e}_b(u) + t^{ab} \frac{d\hat{e}_b(u)}{du} \otimes \hat{e}_a(u) \]

In terms of the coordinates \( x^a(u) \), we can write
\[ \frac{d\hat{e}_a(u)}{du} = \frac{d\hat{e}_a(u)}{dx^c} \frac{dx^c}{du}, \quad \frac{d\hat{e}_b(u)}{du} = \frac{d\hat{e}_b(u)}{dx^c} \frac{dx^b}{du} \]
and using the affine connection
\[ \partial_c \hat{e}_b = \frac{d\hat{e}_a(u)}{dx^c} \Gamma^f_{bc} \hat{e}_f \] (4.101)
we have
\[ \frac{d\tilde{e}_a(u)}{du} = \Gamma^f_{ac} \tilde{e}_f(u) \frac{dx^c}{du}, \quad \frac{d\tilde{e}_b(u)}{du} = \Gamma^f_{bc} \tilde{e}_f(u) \frac{dx^b}{du} \] (4.102)

so that
\[ \frac{dt(u)}{du} = \frac{dt^{ab}}{du} \tilde{e}_a(u) \otimes \tilde{e}_b(u) + \Gamma^f_{ac} \frac{dx^c}{du} \tilde{e}_f(u) \otimes \tilde{e}_b(u) + \Gamma^f_{bc} \frac{dx^b}{du} \tilde{e}_a(u) \otimes \tilde{e}_f(u). \] (4.103)

Replacing the dummy index \( a \) by \( f \) in the first term, we can write
\[ \frac{dt(u)}{du} = \frac{dt^{fb}}{du} \tilde{e}_f(u) \otimes \tilde{e}_b(u) + \Gamma^f_{ac} \frac{dx^c}{du} \tilde{e}_f(u) \otimes \tilde{e}_b(u) + \Gamma^f_{bc} \frac{dx^b}{du} \tilde{e}_a(u) \otimes \tilde{e}_f(u). \] (4.104)

Noting that by making the following dummy index change \( b \rightarrow d \) followed by \( a \rightarrow b \)
\[ t^{ab} \Gamma^f_{bc} \frac{dx^b}{du} \tilde{e}_a(u) \otimes \tilde{e}_f(u) = t^{ad} \Gamma^f_{dc} \frac{dx^d}{du} \tilde{e}_a(u) \otimes \tilde{e}_f(u) = t^{bd} \Gamma^f_{dc} \frac{dx^d}{du} \tilde{e}_b(u) \otimes \tilde{e}_f(u) \]
we find
\[ \frac{dt(u)}{du} = \left[ \frac{dt^{fb}}{du} + \frac{t^{ad} \Gamma^f_{dc} \frac{dx^d}{du}}{Du} + \frac{t^{bd} \Gamma^f_{dc} \frac{dx^d}{du}}{Du} \right] \tilde{e}_f(u) \otimes \tilde{e}_b(u). \] (4.106)

that we expressed as
\[ \frac{dt(u)}{du} = \frac{Dt^{fb}}{Du} \tilde{e}_f(u) \otimes \tilde{e}_b(u). \] (4.107)

where
\[ \frac{Dt^{fb}}{Du} = \frac{dt^{fb}}{du} + \frac{t^{ad} \Gamma^f_{dc} \frac{dx^d}{du}}{Du} + \frac{t^{bd} \Gamma^f_{dc} \frac{dx^d}{du}}{Du}, \] (4.108)

is called the intrinsic (absolute) derivative of the component \( t^{fb} \) along the curve defined by \( x^a(u) \). For the sake of convenience we make change of dummy indices \( (f \rightarrow a) \) in Eq. (4.107) and \( (a \rightarrow d) \) in Eq. (4.108) as
\[ \frac{dt(u)}{du} = \frac{Dt^{ab}}{Du} \tilde{e}_a(u) \otimes \tilde{e}_b(u). \] (4.109)

where
\[ \frac{Dt^{ab}}{Du} = \frac{dt^{ab}}{du} + \frac{t^{db} \Gamma^a_{dc} \frac{dx^c}{du}}{Du} + \frac{t^{bd} \Gamma^a_{dc} \frac{dx^c}{du}}{Du}. \] (4.110)

We can switch the dummy indices \( c \) and \( d \) in the third term as the affine is symmetric for Torsionless
It can be easily shown that

\[
\frac{dt}{du} (u) = \frac{Dt^{ab}}{Du} \hat{e}_a (u) \otimes \hat{e}_b (u) = \frac{Dt_{ab}}{Du} \hat{e}^a (u) \otimes \hat{e}^b (u) = \frac{Dt_b^a}{Du} \hat{e}^a (u) \otimes \hat{e}_b (u).
\]

(4.111)

Like vectors parallel-transported, we can say tensors are parallel-transported when

\[
\frac{Dt^{ab}}{Du} = 0.
\]

(4.112)

Suppose we pretend that the tensor depends on the entire manifold instead of submanifold defined by some curve \( C \), we can write

\[
\frac{dt^{ab}}{du} = \frac{\partial t^{ab}}{\partial x^c} \frac{dx^c}{du} = \partial_c t^{ab} \frac{dx^c}{du}
\]

so that the intrinsic derivative can be expressed as

\[
\frac{Dt^{ab}}{Du} = \partial_c t^{ab} \frac{dx^c}{du} + t^{db} \Gamma^a_{dc} \frac{dx^c}{du} + t^{bc} \Gamma^a_{cd} \frac{dx^c}{du} = (\partial_c t^{ab} + t^{db} \Gamma^a_{dc} + t^{bc} \Gamma^a_{cd}) \frac{dx^c}{du}
\]

Using the result in Eq. (4.93) we can write

\[
\frac{Dt^{ab}}{Du} = \nabla_c t^{ab} \frac{dx^c}{du}
\]

(4.113)
Chapter 5

Special relativity using tensors

In this chapter we will see how tensor calculus can be used in special relativity that introduced in chapter one.

5.1 The Minkowski spacetime in Cartesian coordinates

5.1.1 The metric tensor and the Affine connection

We recall the metric in the Minkowski spacetime is given by

\[ ds^2 = c^2 d\tau = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \]  

(5.1)

where \( x^0 = ct, x^1 = x, x^2 = y, x^3 = z \).

For the boost, for an inertial frame moving with a velocity, \( v \), along the positive \( x \)-direction,

\[ ds^2 = c^2 d\tau = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \]  

(5.2)

where the coordinates \( x^\alpha (\tau) \) parameterized by the proper time, \( \tau \). Then in terms of the components of the metric tensor, \( g_{ab} = \eta_{ab} \), the metric can be expressed as

\[ ds^2 = g_{ab} dx^a dx^b, \]

where

\[ G = [g_{ab}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \]  

(5.3)
Let’s determine the affine connections, obviously one can expect this to be zero as we are using Cartesian coordinates. We recall the relationship between the metric and the affine connection

$$
\Gamma_{bc}^a = \frac{g^{ad}}{2} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) ,
$$

which gives

$$
\Gamma_{bc}^a = 0
$$

This shows that the affine connections are zero everywhere in the Minkowski spacetime manifold (in Cartesian coordinates).

### 5.1.2 The Lorentz transformation

Lorentz transformation is Cartesian coordinates transformation defined by, for the boost,

$$
x^0 = \gamma x'^0 + \gamma \beta x'^1, x^1 = \gamma \beta x'^0 + \gamma x'^1, x^2 = x'^2, x^3 = x'^3,
$$

from the \(x^a \rightarrow x'^a\) or

$$
x'^0 = \gamma x^0 - \gamma \beta x^1, x'^1 = \gamma \beta x^0 - \gamma x^1, x'^2 = x^2, x'^3 = x^3.
$$

from the \(x'^a \rightarrow x^a\), where

$$
\beta = \frac{v}{c}, \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},
$$

The metric for this transformation, we have

$$
d s^2 = g_{ab} (x) \, dx^a dx^b = g'_{cd} (x') \, dx'^c dx'^d ,
$$

where

$$
g'_{cd} = \frac{\partial x'^c}{\partial x'^a} \frac{\partial x'^d}{\partial x'^b} g_{ab} , \quad g_{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} g_{cd} .
$$

Using Eqs. (5.6) and (5.7), one finds

$$
[A^c_a]_{ab} = \left[ \frac{\partial x'^c}{\partial x'^a} \right] = \begin{bmatrix}
\frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^1} & \frac{\partial x'^0}{\partial x^2} & \frac{\partial x'^0}{\partial x^3} \\
\frac{\partial x'^1}{\partial x^0} & \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^1}{\partial x^3} \\
\frac{\partial x'^2}{\partial x^0} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^2}{\partial x^3} \\
\frac{\partial x'^3}{\partial x^0} & \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3}
\end{bmatrix} = \begin{bmatrix}
\gamma & \gamma \beta & 0 & 0 \\
\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} ,
$$

and

$$
[A'^b_b] = \left[ \frac{\partial x^b}{\partial x'^a} \right] = \begin{bmatrix}
\frac{\partial x^0}{\partial x'^0} & \frac{\partial x^0}{\partial x'^1} & \frac{\partial x^0}{\partial x'^2} & \frac{\partial x^0}{\partial x'^3} \\
\frac{\partial x^1}{\partial x'^0} & \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\
\frac{\partial x^2}{\partial x'^0} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\
\frac{\partial x^3}{\partial x'^0} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3}
\end{bmatrix} = \begin{bmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} .
$$
5.1. THE MINKOWSKI SPACETIME IN CARTESIAN COORDINATES

There follows that
\[ x'^a = \Lambda^a_\nu x^\nu \] (5.13)
or
\[ x^a = \Lambda^\nu_a x'^\nu \] (5.14)

In terms of the rapidity parameter \( \psi \) defined by
\[ \psi = \tanh^{-1} (\beta), \beta = \frac{v}{c} = \begin{cases} 1 & v = c \\ 0 & v = 0 \end{cases} \] (5.15)
we can also write
\[
[\Lambda^c_a] = \left[ \frac{\partial x^c}{\partial x'^a} \right] = \begin{bmatrix}
cosh (\psi) & \sinh (\psi) & 0 & 0 \\
\sinh (\psi) & \cosh (\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (5.16)
and
\[
[\Lambda^b_d] = \left[ \frac{\partial x'^b}{\partial x^d} \right] = \begin{bmatrix}
cosh (\psi) & -\sinh (\psi) & 0 & 0 \\
-\sinh (\psi) & \cosh (\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (5.17)

Using Eqs. (5.16) and (5.17), we note that
\[
[\Lambda^c_a] [\Lambda^b_d] = \begin{bmatrix}
cosh (\psi) & \sinh (\psi) & 0 & 0 \\
\sinh (\psi) & \cosh (\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
cosh (\psi) & -\sinh (\psi) & 0 & 0 \\
-\sinh (\psi) & \cosh (\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = [\delta^c_d]
\] (5.18)

Homework: We recall that the metric tensor can be used to lower or raise indices.
Using this property of the metric tensor for Lorentz transformation in Cartesian coordinates for the boost find
\[
[\Lambda^b_d] = \begin{bmatrix}
cosh (\psi) & -\sinh (\psi) & 0 & 0 \\
-\sinh (\psi) & \cosh (\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (5.19)

from
\[
[\Lambda^c_a] = \begin{bmatrix}
cosh (\psi) & \sinh (\psi) & 0 & 0 \\
\sinh (\psi) & \cosh (\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (5.20)
The basis and dual basis vectors:

For the coordinate transformation, \( x^a \to x'^a \), we recall that the basis vectors transform according to

\[
\tilde{e}'^a = \frac{\partial x'^a}{\partial x^c} \tilde{e}^c, \quad \tilde{e}^a \Rightarrow \tilde{e}'^a = \Lambda^a_c \tilde{e}^c, \quad \tilde{e}'^a = \Lambda^a_c \tilde{e}^c.
\]  

(5.21)

so that using the results in Eqs. (5.11), one can easily write

\[
\tilde{e}'^a = \Lambda^a_c \tilde{e}^c, \quad \tilde{e}'^a = \Lambda^a_c \tilde{e}^c
\]  

(5.22)

Using the index lowering and raising tensor \((g^{ab})\), we can also show that

\[
\tilde{e}'^a = g^{ab} \tilde{e}_b.
\]  

(5.23)

We recall that the metric, \( g_{ab} \),

\[
g_{ab} = \tilde{e}_a \cdot \tilde{e}_b \Rightarrow [g_{ab}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\]  

and in terms of the basis vectors in \( x'^a \) coordinates, the metric similarly can be expressed as

\[
g'_{ab} = \tilde{e}'_a \cdot \tilde{e}'_b = \Lambda^a_c \tilde{e}^c \cdot \Lambda'^d_b \tilde{e}'^d = \Lambda^a_c \Lambda'^d_b (\tilde{e}^c \cdot \tilde{e}^d) = \Lambda^a_c \Lambda'^d_b g_{cd} = \Lambda^a_c g_{cd} \Lambda'^d_b
\]

and applying the result in Eq. (5.17), one can write

\[
[g'_{ab}] = \begin{bmatrix} \Lambda^a_c \end{bmatrix} [g_{cd}] \begin{bmatrix} \Lambda'^d_b \end{bmatrix}
\]

\[
= \begin{bmatrix} \cosh(\psi) & \sinh(\psi) & 0 & 0 \\ \sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cosh(\psi) & \sinh(\psi) & 0 & 0 \\ \sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \cosh(\psi) & \sinh(\psi) & 0 & 0 \\ \sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(\psi) & \sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & -\cosh(\psi) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\]

\[
= [g'_{ab}] = [\Lambda^a_c] [g_{cd}] [\Lambda'^d_b] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = [g_{ab}]
\]

There follows that

\[
g'_{ab} = g_{ab} \tag{5.24}
\]

Following a similar procedure one can easily show that

\[
g'^{ab} = g^{ab}, \quad g'^{a} = g^{a} = \delta^{a}_b
\]

(5.25)
5.1. THE MINKOWSKI SPACETIME IN CARTESIAN COORDINATES

5.1.3 Four vector and Lorentz transformation

The four vector: We recall that a vector (which is a geometrical entity that remain the same independent of coordinate transformation) on a manifold in terms of its contravariant components can be expressed as

\[ \vec{v} = u^b \vec{e}_b = u^a \vec{e}_a. \]  

(5.26)

The components are determined by using

\[ u^a = \vec{u} \cdot \vec{e}^a = u^b \vec{e}_b \cdot \vec{e}^a = u^b \vec{e}_b \cdot \Lambda^a_{\ c} \vec{e}^c = u^b \Lambda^a_{\ c} \delta^c_b = \Lambda^a_{\ b} u^b. \]  

(5.27)

Similarly

\[ u^b = \vec{u} \cdot \vec{e}^b = u^a \vec{e}_a \cdot \vec{e}^b = u^a \Lambda^c_{\ a} \vec{e}_c \cdot \vec{e}^b = u^a \Lambda^c_{\ a} \delta^c_b = \Lambda^b_{\ a} u^a. \]  

(5.28)

where

\[
[\Lambda^a_{\ b}] = \begin{bmatrix}
\cosh(\psi) & -\sinh(\psi) & 0 & 0 \\
-\sinh(\psi) & \cosh(\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(5.29)

and

\[
[\Lambda^a_{\ b}] = \begin{bmatrix}
\cosh(\psi) & \sinh(\psi) & 0 & 0 \\
\sinh(\psi) & \cosh(\psi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
\gamma & \gamma \beta & 0 & 0 \\
\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(5.30)

We can also express the components as

\[ u^a = u^b \vec{e}_b \cdot \Lambda^c_{\ a} \vec{e}^c = u^b \Lambda^c_{\ a} \vec{e}_b \cdot \vec{e}^c = u^b g_{bc} \Lambda^c_{\ a}. \]  

(5.31)

For the vector, \( \vec{u} \)

\[ \vec{u} \cdot \vec{u} = u^a \vec{e}_a \cdot u^b \vec{e}_b = u^a u^b \delta^a_b = g_{ab} u^a u^b \]  

(5.32)

for a four vector we say the vector is time-like, null vector, or space like, when

\[ g_{ab} u^a u^b = \begin{cases} 
> 0, & \text{time-like} \\
= 0, & \text{null} \\
< 0, & \text{Space-like}
\end{cases} \]  

(5.33)

The four velocity: In the Minkowski spacetime the trajectory of a particle form the worldline (a curve on the manifold). This curve can be defined in terms of some parameter. For a massive particle this parameter can be the proper time, \( \tau \), which is defined as

\[ (\frac{ds}{d\tau})^2 = c^2, \]  

(5.34)
where $ds$ is an infinitesimal displacement on the worldline. The worldline can then be defined in terms of the proper time using the coordinates $x^a (\tau)$. The four velocity, $u^a (\tau)$, of a particle then is the tangent vector on the worldline.

$$\vec{u} (\tau) = u^a \vec{e}_a = \frac{dx^a}{d\tau} \vec{e}_a$$

and the length of this tangent vector

$$u (\tau) \cdot u (\tau) = \left( \frac{ds}{d\tau} \right)^2 = c^2$$

or

$$[u^a] = \left[ \frac{dx^a}{d\tau} \right] = \left[ \frac{d(ct)}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right].$$

Recalling that

$$dt = \frac{d\tau}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow d\tau = \frac{dt}{\gamma_u}$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

we can write the four velocity

$$[u^a] = \left[ \frac{dx^a}{d\tau} \right] = \gamma_u \begin{bmatrix} c \frac{dx^1}{dt} ; \frac{dx^2}{dt} ; \frac{dx^3}{dt} \end{bmatrix} = \gamma_u [e, \vec{u}]$$

(5.39)

The 3-D velocity, $\vec{u}$, is the velocity of the particle by an observer in the rest inertial frame, $S$. In some other inertial from, $S'$, the four velocity is given by

$$\vec{u}' = u'^a \vec{e}'_a.$$  

(5.40)

The components can be determined using relation

$$u'^a = u \cdot \vec{e}'^a = u^b \vec{e}_b \cdot \vec{e}'^a = \Lambda'^a_b u^b.$$  

(5.41)

where we used the transformation

$$\vec{e}'_a = \Lambda'^a_c \vec{e}_c.$$  

(5.42)

Applying the relation

$$\Lambda'^a_c = \frac{\partial x'^a}{\partial x^c} = \begin{bmatrix} \gamma_v & -\gamma_v \beta & 0 & 0 \\ -\gamma_v \beta & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(5.43)
5.1. **THE MINKOWSKI SPACETIME IN CARTESIAN COORDINATES**

we have

\[
\begin{bmatrix}
  u^0 \\
  u^1 \\
  u^2 \\
  u^3 
\end{bmatrix} =
\begin{bmatrix}
  
\gamma_v & -\gamma_v \beta & 0 & 0 \\
  -\gamma_v \beta & \gamma_v & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
  u^0 \\
  u^1 \\
  u^2 \\
  u^3 
\end{bmatrix}
\]

(5.44)

so that

\[
\begin{bmatrix}
  u^0 \\
  u^1 \\
  u^2 \\
  u^3 
\end{bmatrix} =
\begin{bmatrix}
  \gamma_v (u^0 - \beta u^1) \\
  -\gamma_v (\beta u^0 - u^1) \\
  u^2 \\
  u^3 
\end{bmatrix}
\]

(5.45)

and substituting Eq. (5.39)

\[
\begin{bmatrix}
  u^0 \\
  u^1 \\
  u^2 \\
  u^3 
\end{bmatrix} =
\begin{bmatrix}
  \gamma_v c \\
  \gamma_v u_x \\
  \gamma_v u_y \\
  \gamma_v u_z 
\end{bmatrix}
\]

(5.46)

in to Eq. (5.45), one finds

\[
\begin{bmatrix}
  u^0 \\
  u^1 \\
  u^2 \\
  u^3 
\end{bmatrix} =
\begin{bmatrix}
  \gamma_v (u^0 - \beta u^1) \\
  -\gamma_v (\beta u^0 - u^1) \\
  \gamma_v \gamma_u (c - u_x \beta) \\
  \gamma_v \gamma_u (c^2 - u_x) 
\end{bmatrix}
\]

(5.47)

The proper time, \(\tau\), is measured by a clock at the instantaneous inertial frame (IIF) for the particle. The particle is moving with a speed \(u'\) for an observer on the \(S'\). Then one can write

\[
dt' = \frac{\frac{d\tau}{\sqrt{1 - \frac{u'^2}{c^2}}}}{\gamma_u} = \frac{d\tau}{\gamma_u}
\]

which means

\[
[u'] = \left[ \frac{dx'^0}{dt'} \right] = \gamma_u \left[ c, \frac{dx'^1}{dt'}, \frac{dx'^2}{dt'}, \frac{dx'^3}{dt'} \right] = \gamma_u [c, \bar{u}']
\]

(5.48)

Then Eq. (5.47) becomes

\[
\begin{bmatrix}
  \gamma_v c \\
  \gamma_v u_x \\
  \gamma_v u_y \\
  \gamma_v u_z 
\end{bmatrix} =
\begin{bmatrix}
  \gamma_v \gamma_u (c - u_x \beta) \\
  \gamma_v \gamma_u (u_x - c \beta) \\
  \gamma_u u_y \\
  \gamma_u u_z 
\end{bmatrix} =
\begin{bmatrix}
  \gamma_v \gamma_u \left( 1 - \frac{u_x \beta}{c} \right) \\
  \gamma_v \gamma_u \left( u_x - c \beta \right) \\
  \gamma_u u_y \\
  \gamma_u u_z 
\end{bmatrix}
\]

(5.49)

Using the relation for \(\gamma_u\) from the first row,

\[
\gamma_u' = \gamma_v \gamma_u \left( 1 - \frac{u_x \beta}{c} \right),
\]
one finds
\[
\begin{bmatrix}
  u'_x \\
  u'_y \\
  u'_z
\end{bmatrix} = \begin{bmatrix}
  \frac{u_x - v}{\gamma c} \\
  \frac{u_y}{\gamma} \\
  \frac{u_z}{\gamma c}
\end{bmatrix}.
\] (5.50)

Using
\[
\beta = \frac{v}{c}, \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\] (5.51)
we find what we derived in Chapter one for the velocity transformation.

\[
u_x' = \frac{u_x - v}{1 - \frac{v^2}{c^2} u_x},
\] (5.52)

\[
u_y' = \frac{u_y \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2} u_x},
\] (5.53)

\[
u_z' = \frac{u_z \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2} u_x}.
\] (5.54)

**Homework:** Following a similar procedure re-derive the four acceleration:

\[
a_x' = \frac{a_x}{\gamma^2 (1 - \frac{u_v}{c^2})^2}, \quad a_y' = \frac{1}{\gamma^2 (1 - \frac{u_v}{c^2})^2} a_y + \frac{u_y v}{\gamma^2 c^2 (1 - \frac{u_v}{c^2})^2} a_x,
\]

\[
a_z' = \frac{1}{\gamma^2 (1 - \frac{u_v}{c^2})^2} a_z + \frac{u_z v}{\gamma^2 c^2 (1 - \frac{u_v}{c^2})^2} a_x.
\] (5.55)

**Hint:** the four acceleration can be defined in terms of the four velocity as

\[
[a^a] = \frac{d u^a}{d \tau} = \gamma_u \frac{d u^a}{d t} = \gamma_u \frac{d}{d t} [\gamma u c, \gamma u u]
\] (5.56)

which gives

\[
[a^a] = \gamma_u \frac{d}{d \tau} [\gamma u c, \gamma u u] = \gamma_u \left[ \frac{d^2 \gamma_u}{d \tau^2} + \frac{d \gamma_u}{d \tau} \left( \gamma u u \right) \right] = \gamma_u \left[ \frac{d^2 \gamma_u}{d \tau^2} + \gamma u \frac{d u}{d \tau} \right]
\] (5.57)

### 5.2 The four-momentum of a particle

Suppose a particle (e.g. the Alien in Fig. 5.2) is described by the four-velocity

\[
\ddot{u} = u^a \dot{e}^a = \frac{d x^a}{d \tau} \dot{e}^a = \gamma_u \left[ \dot{c}, \frac{d x^1}{d \tau}, \frac{d x^2}{d \tau}, \frac{d x^3}{d \tau} \right] = \gamma_u [c, \ddot{u}]
\] (5.58)
as measured by an observer on \( S \) frame. Let the rest mass of the particle, that
I prefer to call it the "proper mass", be \( m_0 \). The proper mass is the mass measured by an observer moving with the same velocity of the particle (the IIF). The four momentum, \( \vec{P} \), on the \( S \) inertial frame is defined in terms of the proper mass (rest mass)

\[
\vec{P} = p^a \bar{e}_a = m_0 u^a \bar{e}_a = M_0 \left[ m_0 c, m_0 \frac{dx^1}{d\tau}, m_0 \frac{dx^2}{d\tau}, m_0 \frac{dx^3}{d\tau} \right] = \gamma_u \left[ m_0 c, m_0 \bar{u} \right]
\]

Note that here \( \bar{u} \) represent the three velocity. Suppose we define the rest mass energy which I also prefer to call it the "Proper energy" as

\[
E_0 = m_0 c^2;
\]

the four momentum can be expressed as

\[
[p^a] = \gamma_u \left[ \frac{E_0}{c}, m_0 \bar{u} \right]
\]

or

\[
\vec{P} = [p^a] = \left[ \frac{E}{c}, \vec{p} \right],
\]

where \( \vec{p} \) is the three-momentum and

\[
E = \frac{E_0}{\sqrt{1 - \bar{u}^2}} = \frac{m_0 c^2}{\sqrt{1 - \bar{u}^2}}
\]

is the relativistic energy of the particle as measured by an observer on the \( S \) frame and

\[
\vec{p} = \gamma_u m_0 \bar{u} = \frac{m_0 \bar{u}}{\sqrt{1 - \bar{u}^2}}.
\]
is the relativistic momentum of the particle (the three-momentum). In Eq. (5.59), we note that the magnitude of the four momentum
\[ \vec{P} \cdot \vec{P} = p^a p_a = m_0^2 u^a u_a = m_0^2 c^2 = \left( \frac{E_0}{c} \right)^2. \tag{5.65} \]
Using the metric tensor one can also write
\[ \vec{P} \cdot \vec{P} = p^a e^a \cdot e_b p_b = p^a \eta_{ab} = p^a \eta_{ab} p_b \]
\[ = \begin{bmatrix} p^0 & p^1 & p^2 & p^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} \]
\[ = \begin{bmatrix} p^0 & -p^1 & -p^2 & -p^3 \end{bmatrix} \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \]
\[ \Rightarrow \vec{P} \cdot \vec{P} = \left( \frac{E}{c} \right)^2 - \vec{p} \cdot \vec{p} = \left( \frac{E}{c} \right)^2 - p^2. \tag{5.66} \]
From these two expressions one easily find
\[ \left( \frac{E}{c} \right)^2 - p^2 = \left( \frac{E_0}{c} \right)^2 \Rightarrow p = \frac{1}{c} \sqrt{E^2 - E_0^2}. \tag{5.67} \]
This is the magnitude of the three-momentum in special theory of relativity that you may have seen in Intro physics or modern physics classes.

### 5.3 Four momentum of a photon and the Doppler effect

In the case of the particle the time measured by a clock aboard the particles frame (or by an observer at rest on HIF), the proper time \( \tau \), is used as our affine parameter. The reason for this is that
\[ d\tau = \frac{dt}{\gamma_u} = dt \sqrt{1 - \frac{u^2}{c^2}} \tag{5.68} \]
is never be zero since the particle never travel with a speed \( c \). In the case of a photon or other particles traveling the a speed of light (if it exists), we can not use the proper time as our affine parameter since for, \( u = c \),
\[ d\tau = \frac{dt}{\gamma_u} = dt \sqrt{1 - \frac{u^2}{c^2}} = 0. \tag{5.69} \]
5.3. **FOUR MOMENTUM OF A PHOTON AND THE DOPPLER EFFECT**

If $t$ is the tangent vector defining the worldline of a photon to the curve defining the trajectory of a photon in the Minkowski spacetime manifold

$$ds = t d\tau = 0,$$

which confirms the worldline of a photon is a null-curve. For a null curve we can not use such affine parameter since it gives a null vector and that does not define the geometry of the curve on the manifold. Therefore, we need to find a different affine parameter. For a photon traveling along the positive $x$-axis with speed, $c$, in the $S$ frame, we can define an affine parameter, $\varepsilon$, that would give us a none null tangent vector given by.

$$x^a = u^a \varepsilon$$

where for a photon traveling along the x-direction

$$x^0 = ct, x = x^1 = ct, y = x^2 = 0, z = x^3 = 0$$

$$\Rightarrow x^a = [ct, ct, 0, 0] = u^a \varepsilon = [1, 1, 0, 0] \varepsilon.$$  \hspace{1cm} (5.72)

Which means the tangent vector, $u^a$, that defines the curve for the photon in terms of this parameter is given by

$$\bar{u} = u^a \hat{e}_a = \frac{dx^a}{d\varepsilon} \hat{e}_a,$$

where

$$u^a = [1, 1, 0, 0],$$  \hspace{1cm} (5.74)

which is a none null vector. But the curve defined by

$$u.u = \left(\frac{ds}{d\varepsilon}\right)^2 = 0,$$

which is still a null curve. This tangent vector, which gives the four velocity of the photon is different from that of a massive particle, where we saw

$$u.u = \left(\frac{ds}{d\tau}\right)^2 = c^2.$$  \hspace{1cm} (5.76)

We also note that the equation of motion for a photon can also be expressed in terms of this parameter given by

$$\frac{du}{d\varepsilon} = \frac{d}{d\varepsilon} [1, 1, 0, 0] = 0.$$  \hspace{1cm} (5.77)

In the case of a massive particle to find the four momentum, we multiplied the four velocity by the rest mass ("proper mass"). We follow a similar approach here also. We multiply the four velocity by some parameter, $\alpha$, so that

$$\tilde{P} = \alpha \bar{u} \Rightarrow p^a = \alpha u^a.$$  \hspace{1cm} (5.78)
Like the massive particle, we represent the energy of the photon by \( E = \alpha u^0 = \alpha \) and the three momentum by \( \vec{p} = \alpha \vec{u} \) in the S frame, then the four momentum of a photon can be expressed as
\[
\vec{P} = p^a \vec{e}_a = \left( \frac{E}{c}, \vec{p} \right).
\] (5.79)

According to De Broglie, whether it is massive or none massive particle, all particles behave like wave as long as it has a momentum.

\[
p = \frac{\hbar}{\lambda}
\] (5.80)

The wavelength is often expressed in terms of the magnitude of the wavevector, \( \vec{k} \), which is defined as
\[
k = \frac{2\pi}{\lambda} \Rightarrow \vec{k} = \left( \frac{2\pi}{\lambda_x}, \frac{2\pi}{\lambda_y}, \frac{2\pi}{\lambda} \right).
\] (5.81)

Then the three momentum
\[
\vec{p} = \frac{\hbar \vec{k}}{2\pi} = \hbar \vec{k}.
\] (5.82)

For a photon the energy is given by
\[
E = \frac{hc}{\lambda} = \frac{h}{2\pi} \frac{2\pi}{\lambda} c = \hbar c
\]
and if we express the four momentum in terms of the four wavevector
\[
p^a = \frac{\hbar k^a}{2\pi} = \hbar k^a.
\] (5.83)

Then four wavevector \( \vec{K} \) can be expressed as
\[
\frac{\hbar k^a}{2\pi} = \left( \frac{h}{\lambda}, \frac{\hbar k}{2\pi} \right) \Rightarrow \vec{K} = k^a \vec{e}_a = \left( \frac{2\pi}{\lambda}, \vec{k} \right)
\] (5.84)
\[
\hbar k^a = \left( \frac{E}{c}, \vec{p} \right) = \left( \hbar k, \hbar \vec{k} \right) \Rightarrow \vec{K} = \left( k, \vec{k} \right).
\] (5.85)

*Note: Every quantity for the photon (wavelength, frequency, momentum) is measured on the S frame.*

Now let’s apply the four momentum to re-derive the Doppler shift we studied in chapter one. Let’s consider an observer, \( O \), on an inertial reference frames \( S \) defined by the coordinates \( x^a \). Suppose this observer received a photon of wave length, \( \lambda \), at angle, \( \theta \), as measured from the positive x axis. This photon is mounted on a spacecraft \( (S') \) moving with a velocity, \( v \), along the positive x direction. We define a set of none rotating coordinates, \( x'^a \), on the spacecraft. We want to find the wavelength, \( \lambda' \), and the angle of emission by the laser
pointer, $\theta'$, on the spacecraft as measured by an observer on the $S'$ frame (the Alien). To this end, we note that the four wavevector of the photon in the $S$ frame is given by

$$\vec{K} = k^u \vec{e}_u = \left( \frac{2\pi}{\lambda}, k \cos (\theta) , k \sin (\theta) , 0 \right) = \frac{2\pi}{\lambda} (1, \cos (\theta) , \sin (\theta) , 0) , \quad (5.86)$$

where we used

$$k = \frac{2\pi}{\lambda}. \quad (5.87)$$

In the $S'$ coordinate the four wavevector is given by

$$\vec{K}' = k'^u \vec{e}'_u = \left( \frac{2\pi}{\lambda'}, k' \cos (\theta') , k' \sin (\theta') , 0 \right) = \frac{2\pi}{\lambda'} (1, \cos (\theta') , \sin (\theta') , 0) . \quad (5.88)$$

We want to find the components, $k'^a$. These components can easily be determined from the fact that vectors or tensors on a manifold are geometrical properties and would remain the same. That means the four-wavevector must be the same in $S$ and $S'$ coordinate system. Therefore, we can write

$$k'^a = \vec{K}' \cdot \vec{e}'^a = k^b \vec{e}_b \cdot \vec{e}'^a = \Lambda^a_c \delta^c_b \cdot \vec{e}'^a = \Lambda^a_c \delta^c_b = \Lambda^a_k k^b . \quad (5.89)$$

Using

$$\Lambda^a_b = \begin{bmatrix} \gamma & -\gamma \beta & 0 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.90)$$
we have

$$\begin{bmatrix} k^0 \ u \end{bmatrix} = \frac{2\pi}{\lambda} \begin{bmatrix} \gamma & -\gamma \beta & 0 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \cos (\theta) \\ \sin (\theta) \\ 0 \end{bmatrix} = \frac{2\pi}{\lambda} \begin{bmatrix} \gamma - \gamma \beta \cos (\theta) \\ -\gamma \beta + \gamma \cos (\theta) \\ \sin (\theta) \\ 0 \end{bmatrix}$$

so that

$$\begin{bmatrix} k'^0 \ u \end{bmatrix} = \frac{2\pi}{\lambda} \begin{bmatrix} 1 \\ \cos (\theta') \\ \sin (\theta') \\ 0 \end{bmatrix} = \frac{2\pi}{\lambda} \begin{bmatrix} \gamma - \gamma \beta \cos (\theta) \\ -\gamma \beta + \gamma \cos (\theta) \\ \sin (\theta) \\ 0 \end{bmatrix}.$$  \hspace{1cm} (5.92)

From the first equation follows that

$$\frac{1}{\lambda'} = \frac{\gamma}{\lambda} \left( 1 - \beta \cos (\theta) \right) \Rightarrow \frac{\lambda}{\lambda'} = \gamma \left( 1 - \beta \cos (\theta) \right).$$  \hspace{1cm} (5.93)

From the second and third equations we find

$$\frac{2\pi}{\lambda} \cos (\theta') = \frac{2\pi\gamma}{\lambda} (\cos (\theta) - \beta) \Rightarrow \frac{\lambda}{\lambda'} \cos (\theta') = \gamma \left( \cos (\theta) - \beta \right)$$  \hspace{1cm} (5.94)

$$\frac{2\pi}{\lambda} \sin (\theta') = \frac{2\pi\gamma}{\lambda} \sin (\theta) \Rightarrow \frac{\lambda}{\lambda'} \sin (\theta') = \sin (\theta)$$  \hspace{1cm} (5.95)

Dividing Eq. (5.95) by Eq. (5.94), we find the angle of emission by the laser pointer

$$\frac{\sin (\theta')}{\cos (\theta')} = \frac{\sin (\theta)}{\gamma \left( \cos (\theta) - \beta \right)} \Rightarrow \tan (\theta') = \frac{\tan (\theta)}{\gamma \left[ 1 - \left( v/c \right) \sec (\theta) \right]}$$  \hspace{1cm} (5.96)

Eq. (5.96) is a version of the relativistic aberration formula. For $\theta = 0$, Eq. (5.93) reduces to

$$\frac{\lambda}{\lambda'} = \gamma \left( 1 - \beta \right) = \frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}},$$  \hspace{1cm} (5.97)

in terms of the frequencies ($\lambda = c/f$), we find

$$\frac{f'}{f} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}},$$  \hspace{1cm} (5.98)

On the other hand, for $\theta = \pi$,

$$\frac{\lambda}{\lambda'} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \Rightarrow \frac{f}{f'} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}},$$  \hspace{1cm} (5.99)

which is the same as what we derived in Chapter one.
Note: In our derivation of the Doppler effect even though the emission and observation of the photon occurs at two different events that took place at different time, we took the four momentum to be constant. This is because the equation of a photon in terms of the affine parameter, \( \varepsilon \), we used for the world-line is a constant

\[
\frac{d\bar{P}}{d\varepsilon} = 0 \quad (5.100)
\]

### 5.4 Relativistic mechanics for a massive particle

The equation of motion for a massive particle is given by "Newton’s second law" in terms of the four-momentum

\[
\vec{F} = \frac{d\bar{P}}{d\tau} = f^a \bar{e}_a = \bar{f}^a \bar{e}_a, \quad (5.101)
\]

which gives the four-force. We recall the intrinsic derivative of a vector in terms of it contravariant components on a curve, \( C \), on a manifold is given by

\[
\frac{Dv^a}{Du} = \frac{dv^a}{du} + \Gamma^a_{cb} v^c \frac{dx^b}{du}. \quad (5.102)
\]

For the Minkowski spacetime, we have shown that the affine connection is zero everywhere in Cartesian coordinates. Therefore, the four-momentum

\[
\vec{F} = \frac{d\bar{P}}{d\tau} = \frac{dp^a}{d\tau} \bar{e}_a \quad (5.103)
\]

and the components of the four-force can be expressed as

\[
f^a = \bar{F} \cdot \bar{e}^a = \frac{dp^b}{d\tau} \bar{e}_b \cdot \bar{e}^a = \frac{dp^b}{d\tau} \delta^a_b = \frac{dp^a}{d\tau}. \quad (5.104)
\]

We recall for a particle traveling with a speed, \( u \), the proper time is given by

\[
d\tau = \frac{dt}{\gamma_u}
\]

and we may write the four force as

\[
\left[ f^a \right] = \left[ \frac{dp^a}{d\tau} \right] = \gamma_u \left[ \frac{dp^a}{dt} \right] \gamma_u \frac{d}{dt} \left[ \frac{E}{c}, \bar{P} \right] = \gamma_u \left[ \frac{1}{c} \frac{dE}{dt}, \frac{dp}{dt} \right]. \quad (5.105)
\]

We recall the work energy theorem

\[
dE = dW = \bar{f} \cdot d\bar{D}, \quad (5.106)
\]

where \( d\bar{D} \) is the infinitesimal displacement of the massive particle in the \( S \) frame. In this frame if the particle has moved this displacement in a time interval, \( dt \), we can write

\[
\frac{dE}{dt} = \frac{dW}{dt} = \bar{f} \cdot \frac{d\bar{D}}{dt} = \bar{f} \cdot \bar{u}, \quad (5.107)
\]
where \( \vec{u} \) is the three-velocity and

\[
\vec{f} = \frac{d\vec{p}}{dt},
\]

(5.108)
is the three-force. Thus we can write the four-force as

\[
\vec{F} = \gamma_u \left[ \frac{\vec{f} \cdot \vec{u}}{c}, \vec{f} \right].
\]

(5.109)

We recall the energy and the three-momentum of the massive particle moving with a velocity, \( \vec{u} \),

\[
E = \frac{E_0}{\sqrt{1 - \frac{\vec{u}^2}{c^2}}} = \frac{m_0 c^2}{\sqrt{1 - \frac{\vec{u}^2}{c^2}}}, \quad \vec{p} = \gamma_u m_0 \vec{u} = \frac{m_0 \vec{u}}{\sqrt{1 - \frac{\vec{u}^2}{c^2}}}
\]

(5.110)

and the four-momentum

\[
\vec{P} = \left[ \frac{E}{c}, \vec{p} \right].
\]

(5.111)

**Pure force**: a force that does not alter the rest mass of a particle. Consider the inner product of the four-velocity and the four-force of a massive particle

\[
\vec{u} \cdot \vec{F} = u \cdot \frac{d\vec{P}}{d\tau}.
\]

(5.112)

If we write the four-momentum in terms of the mass, \( m \), and the four-velocity as

\[
\vec{P} = m \vec{u} = mu^a \epsilon_a,
\]

(5.113)

where \( m \) is the relativistic mass, then we have

\[
\vec{u} \cdot \vec{F} = \vec{u} \cdot \frac{d\vec{P}}{d\tau} = \vec{u} \cdot \frac{d}{d\tau} (m \vec{u}) = m \vec{u} \cdot \frac{d\vec{u}}{d\tau} + \vec{u} \cdot \vec{u} \frac{dm}{d\tau}.
\]

(5.114)

and noting that for the four-velocity

\[
\vec{u} \cdot \vec{u} = c^2 \Rightarrow 2\vec{u} \cdot \frac{d\vec{u}}{d\tau} = 0
\]

one can write

\[
\vec{u} \cdot \vec{F} = m \vec{u} \cdot \frac{d\vec{u}}{d\tau} + c^2 \frac{dm}{d\tau} = c^2 \frac{dm}{d\tau}.
\]

(5.115)

If the force is pure, the mass does not change and we have

\[
\frac{dm}{d\tau} = 0 \Rightarrow m = m_0
\]

and we find

\[
\vec{u} \cdot \vec{F} = 0.
\]

(5.116)
Noting that for pure force
\[
\frac{dE}{d\tau} = \frac{d}{d\tau}[mc^2] = 0
\] (5.117)
and
\[
\frac{d\vec{p}}{dt} = m_0 \frac{d\vec{u}}{dt}
\] (5.118)
the four acceleration \([a^a] = \left[ f^a \right] m_0 \) becomes
\[
\gamma_u \left[ \frac{dp^a}{dt} \right] = \gamma_u \frac{dp^a}{dt} \left[ \frac{E}{c}, \vec{p} \right] = \gamma_u \left[ 0, \frac{d\vec{p}}{dt} \right]
\] (5.119)
Using the three-momentum \(\vec{p} = m_0 \vec{u} \)
we find
\[
[a^a] = \gamma_u \left[ 0, \gamma_u \frac{d\vec{u}}{dt} \right].
\] (5.120)
This means the four acceleration becomes three-acceleration for a pure force. Generally, for none pure force the acceleration is given by
\[
[a^a] = \gamma_u \left[ \frac{c\gamma_u}{\gamma_u - 1} \frac{d\gamma_u}{dt} \gamma_u \vec{a} + u \frac{d\gamma_u}{dt} \right]
\] (5.121)
Free particles: For a free massive particles, where the force is zero, the equation of motion is given by
\[
\frac{d\vec{p}}{d\tau} = 0,
\] (5.122)
where the proper time, \(\tau\), is the affine parameter along the particles worldline. Similarly for photons, we have
\[
\frac{d\vec{P}}{d\epsilon} = 0
\] (5.123)
where \(\epsilon\) is the affine parameter along the photon worldline.

We learned that on a curve \(C\) on a manifold, when the intrinsic derivative of a vector vanishes
\[
\frac{Dv_a}{Du} = \frac{dv_a}{d\mu} - \Gamma^b_{ac}v_b \frac{dx^c}{d\mu} = 0,
\] (5.124)
where \(\mu\) is the parameter along the curve, we said the vector is parallel transported along the curve. In the case of a free particle or a photon in the Minkowski spacetime where the affine connection, \(\Gamma^b_{ac} = 0\), in Cartesian co-
ordinates, the four momentum becomes
\[
\frac{Dp_a}{D\tau} = \frac{dp_a}{d\tau} = 0, \quad \frac{Dp_a}{D\epsilon} = \frac{dp_a}{d\epsilon} = 0
\] (5.125)
along the respective worldlines. Thus, in special relativity the worldlines of free particles and photons are non null and null geodesics in Minkowski spacetime, respectively.
5.5 Relativistic collision and Compton scattering

The collision of an electron and a photon known as *compton scattering* can be described using relativistic four momentum. Consider a photon with four momentum, $\vec{P}$, collides with an electron that is at rest at the origin on an inertial $S$ frame that is at rest. Suppose the photon has a frequency, $\nu$, and the electron has a rest mass, $m_0$, in the $S$ frame. Before the photon collides with the electron, the four-momentum for the photon can be expressed as

$$\vec{P}_{ph} = p^a_{ph} \tilde{e}_a = \left( \frac{h \nu}{c}, \frac{h \nu}{c}, \frac{h k}{2\pi}, 0, 0 \right) = \left( \frac{h \nu}{c}, \frac{h \nu}{c}, 0, 0 \right)$$

$$\Rightarrow \vec{P}_{ph} = \frac{h \nu}{c} (1, 1, 0, 0). \quad (5.126)$$

where we used

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \nu}{c}. \quad (5.127)$$

For the electron, noting that the 3-velocity of the electron, $\vec{v} = 0$, the four-momentum before the collision can be expressed as

$$\vec{P}_{el} = p^a_{el} \tilde{e}_a = \left( m_0 c, 0, 0, 0 \right). \quad (5.128)$$

After the collision as shown in Fig. 5.1, the electron would transfer some of its momentum to the electron. As a result the photon frequency changes to $\nu'$ and the electron velocity to, $\vec{v} = \vec{u}$. Thus for the photon, the four momentum becomes

$$\vec{P}_{ph} = p^a_{ph} \tilde{e}_a = \left( \frac{h \nu'}{c}, \frac{h \nu' \cos(\theta)}{c}, \frac{h \nu' \sin(\theta)}{c}, 0 \right) = \frac{h \nu'}{c} (1, \cos(\theta), \sin(\theta), 0). \quad (5.129)$$

for the electron

$$\vec{P}_{el} = p^a_{el} \tilde{e}_a = \gamma_u (m_0 c, m_0 \vec{u}) = \gamma_u (m_0 c, m_0 u \cos(\phi), -m_0 u \sin(\phi), 0) \quad (5.130)$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}.$$  

Note that we need to find $u$. The total four momentum before and after the collision must be the same which leads to

$$\left( m_0 c, 0, 0, 0 \right) + \frac{h \nu}{c} (1, 1, 0, 0) = \gamma_u (m_0 c, m_0 u \cos(\phi), -m_0 u \sin(\phi), 0)$$

$$+ \frac{h \nu'}{c} (1, \cos(\theta), \sin(\theta), 0)$$

$$\Rightarrow \frac{h \nu}{c} \left( 1 + m_0 c^2 \frac{1}{h \nu'} \right), 1, 0, 0 \right)$$

$$= \frac{h \nu}{c} \left( 1 + \frac{\gamma_u m_0 c^2}{h \nu'}, \cos(\theta) + \frac{c}{h \nu} \gamma_u m_0 u \cos(\phi), \sin(\theta) - \frac{c}{h \nu} \gamma_u m_0 u \sin(\phi), 0 \right). \quad (5.131)$$
5.5. RELATIVISTIC COLLISION AND COMPTON SCATTERING

There follows that

\[ m_0c + \frac{h\nu}{c} = \gamma_u m_0c + \frac{h\tilde{\nu}}{c} \Rightarrow \gamma_u = 1 + \frac{h}{m_0c^2} (\nu - \tilde{\nu}), \quad (5.132) \]

\[ \frac{h\nu}{c} = \frac{h\tilde{\nu}}{c} \cos (\theta) + \gamma_u m_0u \cos (\phi) \Rightarrow \gamma_u m_0u \cos (\phi) = \frac{h\nu}{c} - \frac{h\tilde{\nu}}{c} \cos (\theta) \quad (5.133) \]

\[ 0 = \sin (\theta) - \frac{c}{h\tilde{\nu}} \gamma_u m_0u \sin (\phi) \Rightarrow \frac{c}{h\tilde{\nu}} \gamma_u m_0u \sin (\phi) = \sin (\theta) \quad (5.134) \]

Combining the last two equations, we have

\[
\cos (\phi) = \frac{\nu - \tilde{\nu} \cos (\theta)}{\tilde{\nu} \sin (\theta)} \sin (\phi) \Rightarrow \sin^2 (\phi) \left[ 1 + \left( \frac{\nu - \tilde{\nu} \cos (\theta)}{\tilde{\nu} \sin (\theta)} \right)^2 \right] = 1
\]

\[ \Rightarrow \sin^2 (\phi) = \frac{1}{1 + \left( \frac{\nu - \tilde{\nu} \cos (\theta)}{\tilde{\nu} \sin (\theta)} \right)^2} \quad (5.135) \]

Substituting this into the Eqs. (5.133) and (5.134) and also using Eq. (5.132), we can show that

\[ \tilde{\nu} = \nu \left[ 1 + \frac{h\nu}{m_0c^2} (1 - \cos (\theta)) \right]^{-1} \quad (5.136) \]

Homework: Show Eq. (5.136)
5.6 Accelerating observers and the tetrads

So far what we have considered are observers on an inertial frame (a frame moving with a constant velocity). When the observer is in a none inertial frame (accelerating frame), we have for the four-acceleration

\[ \ddot{a}(\tau) = \frac{d\dot{u}(\tau)}{d\tau}, \]

where \( u(\tau) \) is the four-velocity

\[ \dot{u}(\tau) = \frac{dx^a(\tau)}{d\tau} \epsilon_a, \]

which is tangent to the world line. Recalling that for the four velocity

\[ \ddot{u}(\tau) \cdot \dot{u}(\tau) = c^2 \]

we note that

\[ \ddot{a}(\tau) \cdot \dot{u}(\tau) = \frac{d\dot{u}(\tau)}{d\tau} \cdot \dot{u}(\tau) = \frac{1}{2} \frac{d}{d\tau} [\dot{u}(\tau) \cdot \dot{u}(\tau)] = \frac{1}{2} \frac{d}{d\tau} [c^2] = 0. \] (5.137)

Eq. (5.137) shows that the four acceleration is orthogonal to the four velocity. The parameter \( \tau \) (the proper time) we used to define the worldline in the Minkowski space time is measured by an observer that is at rest frame at all the time. An accelerating observer is not at rest at all time. However, we can find for such an observer what is known as instantaneous reference frame (IRF) \( S' \) on which the accelerating observer is at rest momentarily. We recall that the basis vectors on the worldline defined by

\[ ds = \dot{e}_a(\tau) dx^a = \dot{e}_a(\tau) \frac{dx^a(\tau)}{d\tau} d\tau = u^a(\tau) \dot{e}_a(\tau) d\tau \]

\[ \Rightarrow \frac{ds}{d\tau} = u^a(\tau) \dot{e}_a(\tau) = \ddot{u}, \] (5.138)

where \( \ddot{u} \) is the four velocity on the \( S \) frame. Similarly for another frame (accelerating), \( S' \), we can write

\[ ds' = \dot{e}_a'(\tau) dx'^a = \dot{e}_a'(\tau) \frac{dx'^a(\tau)}{d\tau} d\tau = u'^a(\tau) \dot{e}_a'(\tau) d\tau \Rightarrow \frac{ds'}{d\tau} = u'^a(\tau) \dot{e}_a'(\tau). \] (5.139)

If this reference frame is an IRF, where the observer is momentarily at rest, we must have

\[ u'^1(\tau) = u'^2(\tau) = u'^3(\tau) = 0 \Rightarrow \frac{ds'}{d\tau} = u'^0(\tau) \dot{e}_0(\tau). \] (5.140)

We know that vectors are geometrical enteties, whether is accelerating or none accelerating it must remain the same independent of the reference frame. Therefore

\[ \frac{ds}{d\tau} = \frac{ds'}{d\tau} \Rightarrow \ddot{u} = u^a(\tau) \dot{e}_a(\tau) = u'^0(\tau) \dot{e}_0(\tau) \] (5.141)
This means the timelike basis vector \( \tilde{e}_0 \) in the IRF frame is parallel to the four velocity of the accelerating observer \( \tilde{u}(\tau) \). The remaining spacelike basis vectors \( (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3) \) are orthogonal to \( \tilde{e}_0 \) and to one another and will depend on the relative velocity of \( S \) and \( S' \) and the relative orientation of their spacial axes. Therefore observation made by an observer at event, \( p \), on the worldline whether accelerating or none accelerating corresponds to the instantaneous rest frame (IRF) \( S' \) at \( p \). An observer carries along four orthogonal unit vectors, \( \tilde{e}_a(\tau) \), (or tetrad) which vary along his worldline but always satisfy
\[
\tilde{e}_a(\tau) \cdot \tilde{e}_b(\tau) = \eta_{ab}.
\] (5.142)

Normalizing the four-velocity of the observer with the speed of light
\[
\frac{d\tilde{s}}{d\tau} = \frac{u^a(\tau)}{c} \tilde{e}_a(\tau) = \tilde{u}.
\] (5.143)

the timelike unit vector which is parallel to the four-velocity of the accelerating observer can be expressed as
\[
\tilde{e}_0(\tau) = \tilde{u}.
\] (5.144)

At any event \( P \) along the observer’s worldline, the tetrad comprises the basis vectors of the Cartesian IRF at the event \( P \) and defines a time direction and three space directions to which the observer will refer all measurements. Thus, the results of any measurement made by the observer at the event \( P \) are given by projections of physical quantities (tensors) onto those tetrad vectors.

**Example 1** The worldline of an observer in an accelerating frame intersects with the worldline of some particle at the event \( P \). If \( \tilde{P} \) is the 4-momentum of the particle at this event, find the energy and 3-momentum of the particle as measured by the accelerating observer.

**Solution:** The 4-momentum measured by the accelerating observer is just the projection of the 4-momentum of the particle along the tetrad \( \tilde{e}_0' \). That means
\[
\frac{E'}{c} = \tilde{P} \cdot \tilde{e}_0' = \tilde{P} \cdot \tilde{u}
\] (5.145)

and the components of the 3-momentum (the spacial part)
\[
p_i' = \tilde{P} \cdot \tilde{e}_i
\] (5.146)
Chapter 6

Electromagnetism

In this chapter by applying what we have learned about tensor calculus, we will develop the entire theory of relativistic electromagnetism from one equation, the Lorentz force equation.

6.1 The Lorentz force

In introductory physics II you were introduced that a particle with charge, \( q \), moving with a velocity, \( \vec{v} \), in a region where there is both an electric field, \( \vec{E} \), and magnetic field, \( \vec{B} \), experiences an electrical force

\[
\vec{f}_E = q\vec{E}
\]  

and a magnetic force

\[
\vec{f}_B = q\vec{v} \times \vec{B}.
\]

The total force

\[
\vec{f} = q\vec{E} + q\vec{v} \times \vec{B}.
\]

is known as the Lorentz force. We recall that the four-force is given by

\[
[f^a] = \left[ \frac{dp^a}{d\tau} \right] = \gamma_v \left[ \frac{dp^a}{dt} \right] = \gamma_v \frac{d}{dt} \left( \frac{E}{c}, \vec{p} \right) = \gamma_v \left[ \frac{1}{c} \frac{dE}{dt} \frac{d\vec{p}}{dt} \right].
\]

We recall the work energy theorem

\[
dE = dW = \vec{f} \cdot d\vec{D},
\]

where \( d\vec{D} \) is the infinitessimal displacement of the particle. If the particle has moved this displacement in a time interval, \( dt \), we can write

\[
\frac{dE}{dt} = \frac{dW}{dt} = \vec{f} \cdot \frac{d\vec{D}}{dt} = \vec{f} \cdot \vec{v},
\]
where \( \vec{u} \) is the three-velocity and
\[
\vec{f} = \frac{d\vec{p}}{dt},
\]
is the three-force. Thus we can write the four-force as
\[
\vec{F} = \gamma_v \left[ \frac{\vec{v} \cdot \vec{f}}{c}, \vec{f} \right]
\]
(6.8)

For Lorentz force
\[
\vec{v} \cdot \vec{f} = q \left( \vec{v} \cdot \vec{E} \right) + q \vec{v} \cdot \left( \vec{v} \times \vec{B} \right) = q \left( \vec{v} \cdot \vec{E} \right)
\]
(6.9)

one can write
\[
\vec{F} = [F^a] = \gamma_v \left[ \frac{q \left( \vec{v} \cdot \vec{E} \right)}{c}, q\vec{E} + q\vec{v} \times \vec{B} \right]
\]
(6.10)

Using the four velocity
\[
\vec{u} = [u^b] = \gamma_v \left[ c, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right] = \gamma_v [c, \vec{v}]
\]
we find
\[
\vec{u} \cdot \vec{F} = u_a F^a = g_{ab} u^b = \gamma_v [c, -\vec{v}] \cdot \gamma_v \left[ \frac{q \left( \vec{v} \cdot \vec{E} \right)}{c}, q\vec{E} + q\vec{v} \times \vec{B} \right]
\]
\[
= \gamma_v^2 \left\{ q \left( \vec{v} \cdot \vec{E} \right) - \left[ q \left( \vec{v} \cdot \vec{E} \right) + q\vec{v} \cdot \left( \vec{v} \times \vec{B} \right) \right] \right\} = 0
\]
(6.11)

This shows that Lorentz force is a pure force. Consequently, we find
\[
\vec{u} \cdot \vec{F} = \vec{u} \cdot \frac{d}{dt} (m\vec{u}) = m\vec{u} \cdot \frac{d\vec{u}}{dt} + (\vec{u} \cdot \vec{u}) \frac{dm}{dt} = \frac{m}{2} \frac{d}{dt} (\vec{u} \cdot \vec{u}) + (\vec{u} \cdot \vec{u}) \frac{dm}{dt}
\]
\[
\Rightarrow \vec{u} \cdot \vec{F} = \frac{m}{2} \frac{dc^2}{dt} + c^2 \frac{dm}{dt} = c^2 \frac{dm}{dt} = 0 \Rightarrow m = m_0.
\]
(6.12)

that shows the mass of the particle does not not change.

From tensor calculus we know that you can get a vector (a 1st-rank tensor) from the inner product of a second-rank tensor and a first-rank tensor. For example, we have seen that the four-wave vector of a photon in Doppler effect in the \( S' \) frame is determined from four-wave vector in the \( S \) frame using
\[
k'^a = \bar{K}^a \cdot e'^a = k_b e'_b \cdot e'^a = \Lambda_c^a e'_b \cdot e' c = \Lambda_c^a \delta_b^c = \Lambda'^a_b k^b
\]
(6.13)

\[
\Rightarrow \begin{bmatrix} k'^0 \\ k'^1 \\ k'^2 \\ k'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma \beta & 0 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{bmatrix}.
\]
(6.14)
6.2. THE CHARGE AND THE CURRENT DENSITY

\[
\Lambda^a_b = \begin{bmatrix}
\gamma & -\gamma\beta & 0 & 0 \\
-\gamma\beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  
(6.15)

where \([\Lambda^a_b]\) is a second-rank tensor and \([k^b]\) is a first-rank tensor. Generally, Lorentz force shows the force depends on the velocity and the field. So let’s assume that both the electric and magnetic fields can be expressed in terms of some second-rank electromagnetic field tensor, \(\Omega\), and the velocity by a 4-velocity, \(\vec{u}\), in the four-dimensional manifold (Minkowski spacetime). Then we can express the Lorentz 4-force as

\[
\vec{F} = q (\vec{\Omega} \cdot \vec{u}).
\]  
(6.16)

If we express the EM field tensor, \(\vec{\Omega}\), in terms of its covariant components and the 4-velocity in terms of its contravariant components, we may write the components of the four force as

\[
F_a = q \Omega_{ab} u^b.
\]  
(6.17)

As we have shown earlier, Lorentz force is a pure force and we have

\[
\vec{F} \cdot \vec{u} = 0 \Rightarrow F_a u^a = q \Omega_{ab} u^b u^a = 0
\]  
(6.18)

or

\[
\vec{F} \cdot \vec{u} = 0 \Rightarrow F_b u^b = q \Omega_{ba} u^a u^b = 0
\]  
(6.19)

which leads to

\[
q (\Omega_{ab} + \Omega_{ba}) u^b u^a = 0.
\]  
(6.20)

There follows that

\[
\Omega_{ab} = -\Omega_{ba},
\]  
(6.21)

which means the EM tensor is \textit{antisymmetric} tensor. The mixed and covariant component of the EM tensor can be obtained from the metric tensor for Minkowski spacetime

\[
\Omega^b_a = g^{bc} \Omega_{ac}, \Omega^{ab} = g^{ad} \Omega^b_d = g^{ad} g^{bc} \Omega_{dc}
\]  
(6.22)

6.2 The charge and the current density

Let’s consider a cube with side length, \(l_0\), be the proper length of a cube with a uniform charge density. The proper number density for the charges in this cube, \(n_0\), can be written as

\[
n_0 = \frac{N}{l_0^3},
\]  
(6.23)
CHAPTER 6. ELECTROMAGNETISM

where \( N \) is the total number of charges in the cube. Note that the proper length and the proper number density are measured on a rest frame. Then for the proper charge density, \( \rho_0 \), and current density, \( \vec{j}_0 \), on can write

\[
\rho_0 = qn_0, \quad \vec{j}_0 = 0.
\]

where we assumed that each particle carries a charge, \( q \). Now let’s consider an observer on a different inertial frame, \( S' \), moving with a velocity, \( \vec{u} \), along the positive x-axis. For an observer on the \( S \) frame since the length of the cube along the x-direction is Lorentz contracted

\[
l = l_0 \sqrt{1 - \frac{u^2}{c^2}} = \frac{l_0}{\gamma_u}
\]

and the charges are moving with a velocity \( \vec{u} \), we have

\[
n = \gamma_u \frac{N}{l_0} = \gamma_u n_0, \quad \vec{j} = \rho \vec{u},
\]

so that the charge and current densities becomes

\[
\rho = \gamma_u \rho_0, \quad \vec{j} = \gamma_u \rho_0 \vec{u}.
\]

From Eqs. (6.25) and (6.26), we note that the transformation for the charge and current densities from the \( S \) to \( S' \) from is

\[
(\rho_0, \vec{j}_0) \rightarrow (\rho, \vec{j})
\]

This suggests that the 3-current can be replaced by a 4-current. If we define the zero-component of the 4-current in any inertial reference frame as

\[
\vec{j}_0 = c\rho
\]
we can write the 4-current as

\[ \mathbf{J} = (c\rho, \mathbf{j}) \]  \hspace{1cm} (6.29)

Recalling that tensors are geometrical properties to the manifold it would remain the same in any reference frame. Therefore, noting that in the \( S' \) frame

\[ \mathbf{J} = (c\rho_0, 0) \Rightarrow \mathbf{J} \cdot \mathbf{J} = (c\rho_0)^2 \]  \hspace{1cm} (6.30)

we must have in any other reference frame

\[ \mathbf{J} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{J} = (c\rho_0)^2. \]  \hspace{1cm} (6.31)

In terms of the 4-velocity, the contravariant components of the 4-current density can be expressed as

\[ [J^a] = \gamma_a c \rho_0 (c, \mathbf{u}) = (c\rho, \mathbf{j}). \]

Note that \( \rho \) is the charge density of a volume of charge moving with a velocity \( \mathbf{u} \) in the \( S \) frame.

6.3 The electromagnetic field equations

I am sure, you are familiar with Gauss’s law,

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]  \hspace{1cm} (6.32)

where \( \mathbf{E} \) is the electric field vector. But let’s pretend we do not know Gauss’s law. But we do know that EM fields are a result of charges and/or current. We have seen that both the charge and current densities can be described in terms of the 4-current density in the Minkowski spacetime manifold. In EM theory we have 2-nd rank tensor, \( \Omega \) which is the field and 1-st rank tensor which is the 4-current density. We do not have any other 1-st rank tensor that we can contracted with the EM field tensor to get the 1-st rank tensor (the 4-current density). But we do have the 4-gradient that we can use to contract the field tensor and get the 4-current density. That means

\[ \nabla \cdot \Omega = k \mathbf{J} \]  \hspace{1cm} (6.33)

where \( k \) is some constant that we do not know yet. If the field is expressed in terms of its contravariant components, we may write Eq. (6.33) as

\[ \partial_a \Omega^{ab} = k \mathbf{J}^b. \]  \hspace{1cm} (6.34)

*The law of conservation of charge (the continuity equation):* Let’s find \( \partial_b \) of Eq. (6.34)

\[ \partial_b \partial_a \Omega^{ab} = k \partial_b \mathbf{J}^b. \]  \hspace{1cm} (6.35)
We have shown that the EM field tensor is \textit{antisymmetric}, that means we can write
\[ \partial_b \partial_a \Omega^{ab} = - \partial_a \partial_b \Omega^{ab} \quad (6.36) \]

\textbf{Homework: Show that}
\[ \partial_b \partial_a \Omega^{ab} = - \partial_a \partial_b \Omega^{ab} \quad (6.37) \]

for \( a, b = 1, 2 \).

There follows that
\[ \partial_b J^b = 0 \Rightarrow \frac{\partial (cp)}{\partial (ct)} + \nabla \cdot \vec{J} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \quad (6.38) \]

Taking into account the antisymmetry of the EM field tensor
\[ \Omega^{ab} = -\Omega^{ba} \]
we can write
\[ \left[ \Omega^{ab} \right] = \begin{pmatrix} 0 & \Omega^{01} & \Omega^{02} & \Omega^{03} \\ -\Omega^{10} & 0 & \Omega^{12} & \Omega^{13} \\ -\Omega^{20} & -\Omega^{12} & 0 & \Omega^{23} \\ -\Omega^{30} & -\Omega^{13} & -\Omega^{23} & 0 \end{pmatrix} \quad (6.39) \]

Note that the diagonal elements are zero because \( \Omega^{aa} = -\Omega^{aa} \) only when \( \Omega^{aa} = 0 \). This means that there are 6 independent components that we need to determine. But so far we have only four filed equations given by
\[ \partial_a \Omega^{ab} = k_J^b \]
\[ \Rightarrow \partial_0 \Omega^{00} + \partial_1 \Omega^{10} + \partial_2 \Omega^{20} + \partial_3 \Omega^{30} = k_J^0 \]
\[ \partial_0 \Omega^{01} + \partial_1 \Omega^{11} + \partial_2 \Omega^{21} + \partial_3 \Omega^{31} = k_J^1 \]
\[ \partial_0 \Omega^{02} + \partial_1 \Omega^{12} + \partial_2 \Omega^{22} + \partial_3 \Omega^{32} = k_J^2 \]
\[ \partial_0 \Omega^{03} + \partial_1 \Omega^{13} + \partial_2 \Omega^{23} + \partial_3 \Omega^{33} = k_J^3 \quad (6.40) \]

Therefore we need to find two more equations in order to completely determine the EM field tensor. To this end, we express the covariant components of the EM tensor in terms of some 4-vector potential as
\[ \Omega_{bc} = \partial_b A_c - \partial_c A_b \quad (6.41) \]

Since the field equation involves the derivative of the EM field tensor, let’s find \( \partial_a \) of this equation
\[ \partial_a \Omega_{bc} = \partial_a \partial_b A_c - \partial_a \partial_c A_b \quad (6.42) \]

Applying the property of the metric tensor \( g_{ab} \) for Minkowski spacetime in Cartesian coordinates, we can write
\[ g^{ab} (\partial_a \Omega_{bc}) = g^{ab} (\partial_a \partial_b A_c - \partial_a \partial_c A_b) . \quad (6.43) \]
We recall that the metric tensor for Minkowski spacetime in Cartesian coordinates is a constant and therefore, we can write
\[
\partial_a g^{ab} \Omega_{bc} = g^{ab} (\partial_b \partial_c A_a - \partial_c \partial_a A_b) \Rightarrow \partial_a \Omega^a_{bc} = g^{ab} (\partial_b \partial_c A_a - \partial_c \partial_a A_b) \tag{6.44}
\]
Using the contravariant components, one can also write the
\[
\partial_a \Omega^{ab} = k J^b \Rightarrow \partial_a g_{ab} \Omega^{bc} = k g_{ab} J^b \Rightarrow \partial_a \Omega^a_{bc} = k J_a \Rightarrow \partial_a \Omega^a_{c} = k J_c. \tag{6.45}
\]
Combining Eqs. (6.44) and (6.45), we can write
\[
g^{ab} (\partial_a \partial_b A_c - \partial_b \partial_a A_c) = k J_c. \tag{6.46}
\]

**Homework:** Show that in terms of the field tensor, Eq. (6.46) this can be written as
\[
\partial_c \Omega^{ab} + \partial_a \Omega^{bc} + \partial_b \Omega^{ca} = 0 \tag{6.47}
\]

**Solution:**
\[
\partial_c \Omega^{ab} = \partial_c \partial_b A_a - \partial_c \partial_a A_b \tag{6.48}
\]

switching the indices a and b
\[
\partial_b \Omega^{ac} = \partial_b \partial_a A_c - \partial_b \partial_c A_a \Rightarrow \partial_b \Omega^{ca} = - (\partial_b \partial_a A_c - \partial_b \partial_c A_a) = -\partial_b \partial_a A_c + \partial_b \partial_c A_a \tag{6.49}
\]
and the indices a and c
\[
\partial_c \Omega^{ba} = \partial_c \partial_b A_a - \partial_c \partial_a A_b \Rightarrow \partial_c \Omega^{ab} = - (\partial_c \partial_b A_a - \partial_c \partial_a A_b) = -\partial_c \partial_b A_a + \partial_c \partial_a A_b \tag{6.50}
\]

**Note that in Eqs. (6.49) and (6.50) we used the antisymmetric property of the field tensor.** Now adding Eqs. (6.48)-(6.50), we find
\[
\partial_c \Omega^{ab} + \partial_a \Omega^{bc} + \partial_b \Omega^{ca} = -\partial_c \partial_b A_a + \partial_c \partial_a A_b + \partial_b \partial_a A_c - \partial_b \partial_a A_c + \partial_b \partial_c A_a + \partial_b \partial_c A_a \Rightarrow \partial_c \Omega^{ab} + \partial_a \Omega^{bc} + \partial_b \Omega^{ca} = 0
\]

Therefore the complete EM field equations are
\[
\partial_c \Omega^{ab} + \partial_a \Omega^{bc} + \partial_b \Omega^{ca} = 0
\]
\[
\partial_a \Omega^{ab} = k J^b \tag{6.51}
\]

where the constant, \(k = \mu_0\), in SI unit. Using the notations for antisymmetric permutation
\[
l_{[a_1 a_2 \ldots a_N]} = \frac{1}{N!} \text{(Alternating subtraction and addition over all permutations of the indices } a_1 a_2 \ldots a_N) \tag{6.52}
\]
one can write
\[
\partial_{[c} \Omega^{ab]} = 0,
\]
so the EM field equations
\[
\partial_{[c} \Omega^{ab]} = 0, \partial_a \Omega^{ab} = \mu_0 J^b
\]
very simple equation!
6.4 Electromagnetism in the Lorentz gauge

You will learn in E&M that you can choose a four vector potential, \([A^b]\), such that the divergence is zero
\[
\partial_b A^b = 0 \quad (6.53)
\]
This is called Lorentz gauge. In Lorentz gauge
\[
g^{ab} \partial_a \partial_b A_b = \partial_a \partial_b g^{ab} A_b = \partial_a \partial_b A^b = 0 \quad (6.54)
\]
Then the field equation
\[
g^{ab} (\partial_a \partial_b A_c - \partial_a \partial_c A_b) = k J_c. \quad (6.55)
\]
becomes
\[
g^{ab} \partial_a \partial_b A_c = \partial_a g^{ab} \partial_b A_c = \partial_a \partial^a A_c = k J_c \quad (6.56)
\]
For Minkowski spacetime in Cartesian coordinates, we have
\[
[\partial_a] = \begin{pmatrix}
\frac{1}{c} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{pmatrix},
\]
\[
\partial^a = [g^{ab} \partial_b] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{c} \frac{\partial}{\partial t} \\
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{c} \frac{\partial}{\partial t} \\
-\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y} \\
-\frac{\partial}{\partial z}
\end{bmatrix}
\]
so that one finds
\[
\Box = \partial^a \partial_a = \left( \frac{1}{c^2} \frac{\partial}{\partial t} \right) \left( \frac{1}{c^2} \frac{\partial}{\partial t} \right) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}
\]
which is known as the d’Alembertian operator. Using the d’Alembertian operator, one can put the field equation in the form
\[
\Box A_c = \mu_0 J_c \Rightarrow \Box g^{ac} A_c = \mu_0 g^{ac} J_c \Rightarrow \Box A^a = \mu_0 g^{ac} J_c = \mu_0 J^a \quad (6.57)
\]
We now invoke some of the symbols we got introduced in Theoretical Physics III (E & M II) to denote the electric potential, \(V\), and of course the 3-vector potential, \(\vec{A}\). If we express the four-vector potential in terms of these \(V\) and \(\vec{A}\) as
\[
[A^b] = \begin{pmatrix}
V \\
\vec{A}
\end{pmatrix} = \begin{pmatrix}
V \\
A_x, A_y, A_z
\end{pmatrix} \quad (6.58)
\]
In Lorentz gauge
\[
\partial_a A^a = 0 \Rightarrow \left( \frac{\partial}{c \partial t} \left( \frac{V}{c} \right), \nabla \cdot \vec{A} \right) = 0 \Rightarrow \frac{1}{c^2} \frac{\partial V}{\partial t} + \nabla \cdot \vec{A} = 0. \quad (6.59)
\]
The four field equations, we find

\[
\frac{1}{c^2} \frac{\partial^2 A^0}{\partial t^2} - \frac{\partial^2 A^0}{\partial x^2} - \frac{\partial^2 A^0}{\partial y^2} - \frac{\partial^2 A^0}{\partial z^2} = \mu_0 J^0, \quad (6.60)
\]

\[
\frac{1}{c^2} \frac{\partial^2 A^1}{\partial t^2} - \frac{\partial^2 A^1}{\partial x^2} - \frac{\partial^2 A^1}{\partial y^2} - \frac{\partial^2 A^1}{\partial z^2} = \mu_0 J^1, \quad (6.61)
\]

\[
\frac{1}{c^2} \frac{\partial^2 A^2}{\partial t^2} - \frac{\partial^2 A^2}{\partial x^2} - \frac{\partial^2 A^2}{\partial y^2} - \frac{\partial^2 A^2}{\partial z^2} = \mu_0 J^2, \quad (6.62)
\]

\[
\frac{1}{c^2} \frac{\partial^2 A^3}{\partial t^2} - \frac{\partial^2 A^3}{\partial x^2} - \frac{\partial^2 A^3}{\partial y^2} - \frac{\partial^2 A^3}{\partial z^2} = \mu_0 J^3. \quad (6.63)
\]

First let’s consider Eq. (6.60) Using the component \(J^0 = c\rho\), from the four-current density,

\[
[J^a] = \begin{pmatrix} \rho & \vec{j} \end{pmatrix}, \quad (6.64)
\]

Eq. (6.60) can be put in the form

\[
\frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = c\mu_0 J^0. \quad (6.65)
\]

Introducing a vector field, \(\vec{E}\), (electric field vector) defined by

\[
\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \Rightarrow \nabla \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} \right) \quad (6.66)
\]

one can express the Lorentz gauge, in terms of this vector field as

\[
\nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \Rightarrow \nabla \cdot \vec{E} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \nabla^2 V \quad (6.67)
\]

Upon substituting this into Eq. (6.65), we find

\[
\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \text{(Gauss’s law!)}
\]

Adding Eqs.(6.61)-(6.63) and noting that

\[
\vec{A} = (A^1, A^2, A^3), \quad \vec{j} = (J^1, J^2, J^3)
\]

one finds

\[
\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} \quad (6.68)
\]

Introducing a second vector field, \(\vec{B}\), (the magnetic field) defined by

\[
\vec{B} = \nabla \times \vec{A}. \quad (6.69)
\]

we have

\[
\nabla \times \vec{B} = \nabla \times \left( \nabla \times \vec{A} \right) = \nabla \left( \nabla \cdot \vec{A} \right) - \nabla^2 \vec{A}. \quad (6.70)
\]
so that using Lorentz gauge

\[ \nabla \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \Rightarrow \nabla \cdot \vec{E} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \nabla^2 V \quad (6.71) \]

one can write

\[ \nabla \times \vec{B} = \nabla \left( -\frac{1}{c^2} \frac{\partial V}{\partial t} \right) - \nabla^2 \vec{A} = -\frac{1}{c^2} \frac{\partial (\nabla V)}{\partial t} - \nabla^2 \vec{A} \]

\[ \Rightarrow \nabla^2 \vec{A} = -\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial (\nabla V)}{\partial t}. \quad (6.72) \]

Using this relation, Eq. (6.71) can be rewritten

\[ \nabla \times \vec{B} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} + \frac{1}{c^2} \frac{\partial (\nabla V)}{\partial t} = \mu_0 \vec{J} \Rightarrow \nabla \times \vec{B} + \frac{1}{c^2} \frac{\partial (\nabla V)}{\partial t} = \mu_0 \vec{J} \quad (6.73) \]

and recalling that

\[ \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \]

one finds

\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \Rightarrow \nabla \times \vec{B} = \varepsilon_0 \alpha_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \quad (\text{Ampere's law!}) \quad (6.74) \]

The other two Maxwell's equation easily obtained using properties of vector calculus

\[ \nabla \times \vec{E} = \nabla \times \left( -\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \left( -\nabla \times \nabla V - \frac{\partial}{\partial t} \left( \nabla \times \vec{A} \right) \right) \]

\[ = -\nabla \times (\nabla V) - \frac{\partial \vec{B}}{\partial t} \Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (\text{Faraday's law!}) (6.75) \]

where we used the relation

\[ \nabla \times (\nabla V) = 0. \quad (6.76) \]

For the magnetic field,

\[ \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) \quad (6.77) \]

since for any vector \( \vec{A} \)

\[ \nabla \cdot (\nabla \times \vec{A}) = 0 \quad (6.78) \]

we can easily see that

\[ \nabla \cdot \vec{B} = 0 \quad (\text{No name law!}) \quad (6.79) \]
We now proceed to determine the elements of the covariant components of the field tensor

\[
[\Omega_{ab}] = \begin{pmatrix}
0 & \Omega_{01} & \Omega_{02} & \Omega_{03} \\
-\Omega_{01} & 0 & \Omega_{12} & \Omega_{13} \\
-\Omega_{02} & -\Omega_{12} & 0 & \Omega_{23} \\
-\Omega_{03} & -\Omega_{13} & -\Omega_{23} & 0 \\
\end{pmatrix}, \tag{6.80}
\]

in terms of the electric and magnetic fields. To this end, using

\[
\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} 
\]

one can write

\[
E^1 = - \left( \partial_1 V + \frac{\partial A_1}{\partial t} \right), \quad E^2 = - \left( \partial_2 V + \frac{\partial A_2}{\partial t} \right), \quad E^3 = - \left( \partial_3 V + \frac{\partial A_3}{\partial t} \right), \quad \tag{6.82}
\]

and using

\[
c\partial_0 = \frac{\partial}{\partial t}, \quad cA^0 = V \tag{6.83}
\]

we may write

\[
E^1 = -c \left( \partial_1 A^0 + \partial_0 A^1 \right), \quad E^2 = -c \left( \partial_2 A^0 + \partial_0 A^2 \right), \quad E^3 = -c \left( \partial_3 A^0 + \partial_0 A^3 \right). \quad \tag{6.84}
\]

Using the metric tensor one can determine the covariant components for the four-vector potential

\[
A_a = g_{ab} A^b \Rightarrow \begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix} \begin{pmatrix}
A^0 \\
A^1 \\
A^2 \\
A^3 \\
\end{pmatrix} \Rightarrow \begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3 \\
\end{pmatrix} = \begin{pmatrix}
A^0 \\
-A^1 \\
-A^2 \\
-A^3 \\
\end{pmatrix}. \quad \tag{6.85}
\]

Applying this result in Eq. (6.84), we have

\[
E^1 = -c \left( \partial_1 A_0 - \partial_0 A_1 \right), \quad E^2 = -c \left( \partial_2 A_0 - \partial_0 A_2 \right), \quad E^3 = -c \left( \partial_3 A_0 - \partial_0 A_3 \right)
\]

\[
\Rightarrow E^i = -c \left( \partial_i A_0 - \partial_0 A_i \right) = -c \delta^{ij} \left( \partial_j A_0 - \partial_0 A_j \right). \quad \tag{6.86}
\]

We recall that the elements of the field tensor is related to the components of the four-vector potential by

\[
\Omega_{ab} = \partial_a A_b - \partial_b A_a. \tag{6.87}
\]

and using this equation, one can write

\[
E^i = -c \delta^{ij} \Omega_{j0}. \quad \tag{6.88}
\]
CHAPTER 6. ELECTROMAGNETISM

There follows that
\[
\begin{align*}
\Omega_{10} &= -\frac{E_1}{c}, \\
\Omega_{20} &= -\frac{E_2}{c}, \\
\Omega_{30} &= -\frac{E_3}{c}.
\end{align*}
\]  
(6.89)

From the magnetic field
\[
\vec{B} = \nabla \times \vec{A}
\]  
(6.90)
we have
\[
B^1 = \partial_2 A^3 - \partial_3 A^2, \\
B^2 = \partial_3 A^1 - \partial_1 A^3, \\
B^3 = \partial_1 A^2 - \partial_2 A^1.
\]  
(6.91)

Noting that for the controvariant components for spacial part of the four-vector can be determined from the corresponding covariant component by using the metric tensor
\[
A^a = g^{ab} A_b = -A_a
\]  
(6.92)
one can write
\[
-B^1 = \partial_2 A_3 - \partial_3 A_2, \\
-B^2 = \partial_3 A_1 - \partial_1 A_3, \\
-B^3 = \partial_1 A_2 - \partial_2 A_1
\]  
(6.93)
and comparing this with
\[
\Omega_{ab} = \partial_a A_b - \partial_b A_a.
\]  
(6.94)
we note that
\[
\Omega_{23} = -B^1, \Omega_{31} = -B^2, \Omega_{12} = -B^3.
\]  
(6.95)

We have determined three electric field components and three magnetic field components. Using the antisymmetric property of the field tensor one can write the EM field tensor in terms of its covariant components as
\[
[\Omega_{ab}] = \begin{pmatrix}
0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\
-\frac{E_1}{c} & 0 & -B^3 & B^2 \\
-\frac{E_2}{c} & B^3 & 0 & -B^1 \\
-\frac{E_3}{c} & -B^2 & -B^1 & 0
\end{pmatrix}.
\]  
(6.96)

In an another inertial reference frame \(S'\) the EM field tensor can easily be determined from the vector potential or the EM field tensor in the \(S\) frame using
\[
A'^a = \Lambda^a_b A^b, \Omega'^{ab} = \Lambda^a_c \Lambda^b_d \Omega_{cd}.
\]  
(6.97)

**Homework:**

6.5 **Electromagnetism in arbitrary coordinates**

The field equations: for Cartesian coordinates we recall the field equations are given by
\[
\partial_c \Omega_{ab} + \partial_a \Omega_{bc} + \partial_b \Omega_{ca} = 0 \text{ or } g^{ab} (\partial_a \partial_b A_c - \partial_a \partial_c A_b) = \mu_0 J_c
\]
\[
\partial_a \Omega^{ab} = \mu_0 J^b
\]  
(6.98)
In any other coordinates the metric tensor $\partial_a$ should also be replaced by the covariant derivative $\nabla_a$.

**Homework # 1:** Show that for the covariant derivatives of the mixed and covariant component of a 2-nd rank tensor $t$ are given by

\begin{align*}
\nabla_c t^a_b &= \partial_c t^a_b + \Gamma^a_{dc} t^d_b - \Gamma^d_{bc} t^a_d \\
\nabla_c t_{ab} &= \partial_c t_{ab} - \Gamma^d_{ac} t_{db} - \Gamma^d_{bc} t_{ad}
\end{align*}

(6.99) (6.100)

**Useful relation**

\[ \partial_c e^a = -\Gamma^a_{bc} e^b \]  

(6.101)

**Homework # 2:** Show that the covariant derivative of the metric tensor is zero

\[ \nabla g = 0 \]

Suppose we represent the metric tensor in terms of its controvariant components, $g^{ab}$, then you must show that the covariant derivative expressed as

\[ \nabla_c g^{ab} = \partial_c g^{ab} + \Gamma^a_{cd} g^{db} + \Gamma^b_{cd} g^{ad} = 0 \]

(6.102)

**Useful relations**, for example, the affine connection and the metric are related by

\[ \Gamma^f_{bc} = \frac{g^{fd}}{2} \left( \partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc} \right) \]

(6.103)

Then the field equations should be written as

\[ \nabla_c \Omega_{ab} + \nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} = 0 \] or \[ g^{ab} \left( \nabla_a \nabla_b A_c - \nabla_c \nabla_a A_b \right) = \mu_0 J_c \]

\[ \partial_a \Omega^{ab} = \mu_0 J^b \]

(6.104)
CHAPTER 6. ELECTROMAGNETISM

Gauge transformation: still we can chose a new vector potential

\[ A^{\text{new}}_a = A_a + \nabla_a \psi = A_a + \partial_a \psi \]  

(6.105)

such that, for example, in Lorentz gauge

\[ \nabla_a A^a = 0 \]  

(6.106)

and we still find

\[ \Box A_\lambda = a_0 J_\lambda \]  

(6.107)

but this time the d’Alembertian operator is given by

\[ \Box = g^{ab} \nabla_a \nabla_b \]

Charge conservation:

\[ \nabla_a J^a = 0 \]  

(6.108)

The EM field tensor and the 4-vector potential: When we transform from an inertial from \( S \) to \( S' \) for arbitrary coordinates should be determined from

\[ A'_a = \frac{\partial x'^a}{\partial x^b} A^b, \Omega^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} \Omega^{cd} \]  

(6.109)

6.6 Equation of motion for a charged particle

Consider a particle of rest mass, \( m_0 \), in a region where there is an EM field. We recall that EM force is a pure force and it does not alter the rest mass of the particle. Thus the equation of motion for the particle can be written as

\[ F^a = \frac{d p^a}{d \tau} = m_0 \frac{d u^a}{d \tau}. \]  

(6.110)

We recall the force in terms of the field tensor

\[ F_a = q \Omega_{ab} u^b \Rightarrow F^b = g^{ba} F_a \Rightarrow F^b = q g^{ba} \Omega_{ac} u^c = q \Omega^a_b u^c \]  

(6.111)

and the equation of motion becomes

\[ m_0 \frac{d u^a}{d \tau} = q \Omega^a_b u^b \]

This equation of motion is in Cartesian coordinates. We recall, generally for any coordinates, we have the intrinsic derivative for the controvariant component given by

\[ \frac{D u^a}{D \tau} = \frac{d u^a}{d \tau} + \Gamma^a_{cb} u^c \frac{d x^b}{d \tau} \]  

(6.112)

and the covariant component

\[ \frac{D u_a}{D \tau} = \frac{d u_a}{d \tau} - \Gamma^b_{ac} u_b \frac{d x^c}{d \tau}. \]  

(6.113)
For an arbitrary coordinates, one can then write the contravariant component of the force in terms of the corresponding components for the momentum

\[ F^a = \frac{Dp^a}{Dt} = m_0 \frac{Du^a}{Dt} = m_0 \left( \frac{du^a}{dt} + \Gamma^a_{cb} u^c \frac{dx^b}{dt} \right), \]  

(6.114)

where still \( EM \) force is pure. Then the equation of motion in terms of the contravariant components can be written as

\[ m_0 \left( \frac{du^a}{dt} + \Gamma^a_{cb} u^c \frac{dx^b}{dt} \right) = q\Omega^a_b u^b \]  

(6.115)

In terms of only the four-velocity

\[ \frac{dx^b}{dt} = u^b \]  

(6.116)

we may write

\[ m_0 \left( \frac{du^a}{dt} + \Gamma^a_{cb} u^b u^c \right) = q\Omega^a_b u^b. \]  

(6.117)

Or in terms of only the coordinates

\[ m_0 \left( \frac{d^2x^a}{dt^2} + \Gamma^a_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} \right) = q\Omega^a_b \frac{dx^b}{dt} \]  

(6.118)

which we can put in the form

\[ \frac{d^2x^a}{dt^2} + \Gamma^a_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = \frac{q}{m_0} \Omega^a_b \frac{dx^b}{dt} \]  

(6.119)

In the absence of EM field or for none charged particle (i.e. \( \Omega^a_b = 0 \) or \( q = 0 \)), we find

\[ \frac{d^2x^a}{dt^2} + \Gamma^a_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = 0 \]  

(6.120)

which means the motion of the particle is Geodesic!

**Homework # 3: Problem 6.3**

**Homework # 3: Problem 6.5**
Chapter 7

The equivalence principle and spacetime curvature

7.1 Newtonian gravity and the equivalence principle

Before we discuss Newtonian gravity, let’s talk about the force on a test charge, \( q \), due to an electric field. This force is given by

\[
\vec{F}(\vec{r}) = q \vec{E}(\vec{r}).
\]  

(7.1)

This electric field \( \vec{E} \) is usually determined from the electrostatic potential \( V(\vec{r}) \) as

\[
\vec{E}(\vec{r}) = -\nabla V(\vec{r}).
\]  

(7.2)

Suppose this electric field \( \vec{E} \) is due to some volume charge distribution with charge density, \( \rho \), we know from the divergence theorem that

\[
\nabla \cdot \vec{E}(\vec{r}) = \left( \frac{1}{\epsilon_0} \right) \rho(\vec{r}).
\]  

(7.3)

and in terms of the potential, this can be written as

\[
\nabla^2 V(\vec{r}) = -\left( \frac{1}{\epsilon_0} \right) \rho(\vec{r}).
\]  

(7.4)

Now let’s consider a test particle of gravitational mass \( m_G \) at some position, the gravitational force on this particle is given by

\[
\vec{F}(\vec{r}) = m_G \vec{g}(\vec{r}).
\]  

(7.5)

where \( \vec{g}(\vec{r}) \) is the gravitational field determined from the gravitational potential, \( \Phi \),

\[
\vec{g}(\vec{r}) = -\nabla \Phi(\vec{r}).
\]  

(7.6)
Like the electric field, the gravitational field is due to some mass distribution in space that can be described a mass density, $\rho (\vec{r})$. Then in view of Eq. (7.3), the gravitational field, $\vec{g}$, can be expressed as

$$\nabla \cdot \vec{g} (\vec{r}) = (4\pi G) \rho (\vec{r}), \quad (7.7)$$

or in terms of the gravitational potential,

$$\vec{g} (\vec{r}) = \nabla \Phi \Rightarrow \nabla^2 \Phi (\vec{r}) = (4\pi G) \rho (\vec{r}), \quad (7.8)$$

where $G$ is the universal gravitational constant. The result in Eq. (7.8) is not consistent with special theory of relativity since it does not display any time dependence. This means the gravitational field due to some mass distribution in space is instantaneously felt by another mass in the universe billion light years away. This requires a travel speed greater than the speed of light which is in contradiction with special theory of relativity. There is a second fundamental difference between electromagnetic and gravitational forces. The equation of motion for a particle with inertial mass, $m_I$, is given by the 3-force

$$m_I \frac{d^2 \vec{x}}{dt^2} = \vec{F} = m_G \vec{g} \Rightarrow \frac{d^2 \vec{x}}{dt^2} = \frac{m_G}{m_I} \vec{g} = -\frac{m_G}{m_I} \nabla \Phi. \quad (7.9)$$

However, it is a well-established experimental fact that $\frac{m_G}{m_I}$ is the same for all particles and by an appropriate choice of units we can set $\frac{m_G}{m_I} = 1$. Thus the equation of motion is given by

$$\frac{d^2 \vec{x}}{dt^2} = -\nabla \Phi, \quad (7.10)$$

which is independent of the nature of the particle. This result is different from the equation of motion for charged particle where it depends on the $\frac{q}{m} \vec{v}$ which varies from particle to particle.

The result in Eq. (7.10) shows there is an equivalence between the inertial mass, $m_I$, of a particle that determines the particle resistance to an applied force and gravitational mass, $m_G$, the quantity that determines the magnitude of the gravitational force.

The equivalence principle: In a freely falling (non-rotating) laboratory occupying a small region of spacetime, the laws of physics are those of special relativity.

### 7.2 Gravity as spacetime curvature and local Cartesian coordinates

Einstein proposal: Gravity should no longer be regarded as a force in the conventional sense but rather as a manifestation of the curvature of the spacetime, this curvature being induced by the presence of matter. This is the central idea underpinning the theory of general relativity. Then according to Einstein the
7.2. GRAVITY AS SPACETIME CURVATURE AND LOCAL CARTESIAN COORDINATES

Equation of motion of a particle

\[ F^a = \frac{d\tilde{P}^a}{d\tau} \]  

(7.11)

in a gravitational field becomes

\[ \frac{d\tilde{P}}{d\tau} = 0, \]

(7.12)

as the gravitational force is not part of the 4-force. Note that \( \tilde{F} \) is the 4-force on the particle excluding the gravitational force (which is zero), \( \tilde{P} \) is the 4-momentum of the particle, and \( \tau \) the proper time measured along the particle's worldline. We recall from the previous chapter, when the 4-force is zero

\[ \tilde{F} = \frac{d\tilde{P}}{d\tau} = \frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0, \]

(7.13)

which means the motion of a freely falling particle is geodesic in the curved spacetime!

The equivalence principle restricts the possible geometry of the curved space time to pseudo-Riemannian. Mathematically this means at any event point, \( p \), in the spacetime manifold we must be able to define a local coordinate system \( x^a = X^a \) such that in the neighborhood of point \( p \), the line element of spacetime takes the form

\[ ds^2 \simeq g_{ab}dX^adX^b, \]

(7.14)

where

\[ [g_{ab}] = \eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]
At the exact event point \( p \), we have
\[
ds^2 = g_{ab} dX^a dX^b. \tag{7.15}
\]
We recall that at the local coordinates the affine connection vanishes
\[
\Gamma^a_{\sigma \rho} (p) \simeq 0 \tag{7.16}
\]
and the geodesic
\[
\frac{d^2 X^a}{d\tau^2} + \Gamma^a_{bc} \frac{dX^b}{d\tau} \frac{dX^c}{d\tau} = 0
\]
becomes
\[
\frac{d^2 X^a}{d\tau^2} \simeq 0,
\]
where \( a = 0,1,2,3 \) and \( X^0 \) is replaced by \( ct \). This means \( X^a \) define a local Cartesian inertial frame in which the laws of special relativity hold locally. We can then construct such a system, spacetime is pseudo-Riemannian manifold where the metric is given by
\[
ds^2 = g_{ab} dx^a dx^b \tag{7.17}
\]
The curvature of spacetime means that it is not possible to find coordinates in which the metric, \( g_{ab} = \eta_{ab} \), at all points on the manifold. From the equivalence principle, however, we can always make a transformation at any event point, \( p \), in the curved spacetime, to a local inertial coordinate, \( X^a \), which in a limited region of spacetime about point \( p \), corresponds to a freely falling, non-rotating, Cartesian frame over a short time interval. This means from what we introduced in chapter 2, at the event point, \( p \)
\[
g_{ab} (p) = \eta_{ab} \text{ and } (\partial_{\sigma} g_{ab})_p = 0.
\]
This means
\[
\Gamma^a_{b\sigma} (p) = 0 \tag{7.18}
\]
and the coordinates basis vectors form an orthonormal set at point, \( p \)
\[
\hat{e}_a (p) \cdot \hat{e}_b (p) = \eta_{ab} \tag{7.19}
\]
There follows that,
\[
X^{*a} = A^a_b X^b \tag{7.20}
\]
where \( A^a_b \) defines the Lorentz transformation 2-rank tensor. Thus, local Cartesian freely falling (non-rotating) frames at an event point, \( p \) are related to one another by boosts, spatial rotations or combination of the two. For any of these coordinates
\[
\hat{e}_0 (p) \rightarrow \hat{u} (p) \text{-Time like, } \hat{e}_i (p) \text{ for } i = 1,2,3 \text{-Spacelike}
\]
For points near to the event at point \( p \), the metric in a local inertial coordinate system, \( X^a \)

\[
g_{ab}(X) = g_{ab}(p) + \frac{1}{2} \left( \partial_c \partial_d g_{ab} \right)_p X^c X^d + \ldots
\]  
(7.21)

or

\[
g_{ab}(X) = \eta_{ab}(p) + \frac{1}{2} \left( \partial_c \partial_d g_{ab} \right)_p X^c X^d + \ldots
\]  
(7.22)

where we used

\[ g_{ab}(p) = \eta_{ab} \text{ and } \left( \partial_c \partial_d g_{ab} \right)_p = 0. \]

### 7.3 Observers in a curved spacetime

We recall that an observer trace out some general (timelike) worldline \( x^a(\tau) \) through spacetime whether it is on accelerating or non-accelerating reference frame. His local laboratory, where he measured the physical observables, is the IRF described by the four orthonormal basis vectors (the tetrads) \( \hat{e}_a(\tau) \)

\[ \hat{e}_a(\tau) \cdot \hat{e}_b(\tau) = \eta_{ab}, \] 
(7.23)

where

\[ \hat{e}_0(\tau) \rightarrow \frac{\dot{u}(\tau)}{c} \text{-Time like, } \hat{e}_i(\tau) \text{ for } i = 1, 2, 3 \text{-Spacelike,} \] 
(7.24)

which are unrelated to \( \hat{e}_a(\tau) \) that we used to label the curved spacetime. If the observer has the four acceleration

\[ \ddot{a}(\tau) = \frac{d\ddot{u}}{d\tau}, \] 
(7.25)

but not rotating, the tetrad basis vectors are *Fermi-Walker-transported* along the observer’s worldline satisfying the equation

\[ \frac{d\hat{e}_a}{d\tau} = \frac{1}{c^2} \left[ (\ddot{u} \cdot \hat{e}_a) \frac{d\ddot{u}}{d\tau} - (\dot{a} \cdot \hat{e}_a) \ddot{u} \right]. \] 
(7.26)

For a non-rotating freely falling observer, we have

\[ \ddot{a} = \frac{d\ddot{u}}{d\tau} = 0 \] 
(7.27)

so that

\[ \frac{d\hat{e}_0}{d\tau} = 0, \frac{d\hat{e}_i}{d\tau} = 0 \] 
(7.28)

This means the tetrad basis vectors are parallel transported along the observers worldline. Hence in an arbitrary coordinate system \( x^a \) with dual basis vectors \( \tilde{e}^a \), the components of the tetrad vectors \( \hat{e}_b(\tau) \) along the \( \tilde{e}^a \), which we denote by \( \hat{e}_b^a(\tau) \) is given by

\[ \hat{e}_b^a(\tau) = \hat{e}_b(\tau) \cdot \tilde{e}^a \] 
(7.29)
and its evolution is determined by the equation

\[
\frac{D\hat{\epsilon}_b^a}{D\tau} = \frac{d\hat{\epsilon}_b^a}{d\tau} + \Gamma^a_{cd}(\hat{\epsilon}_c^b (\tau)) u^d = 0
\]  
(7.30)

This equation is useful for determining what a freely falling observer would measure at a given event in spacetime.

### 7.4 Weak gravitational fields and the Newtonian limit

We have seen that in general relativity the role of gravitational field is not to create a force in a conventional way but rather to curve the Minkowski spacetime which is flat otherwise. It is obvious that our description of gravity in terms of spacetime curvature reduces to special relativity in the local inertial frames. However, we have to show that the description of gravity in terms of spacetime curvature gives Newtonian gravity. In Newtonian gravity the spacetime is flat! The metric tensor in Cartesian coordinates is given by

\[
g_{ab} = \eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
\]  
(7.31)

This means if we want to see if the description of gravity in terms of spacetime reduces to Newtonian gravity, we must consider the case where the gravitational field is weak and the curvature is small. Then the metric can be approximated as

\[
g_{ab} \simeq \eta_{ab} + h_{ab},
\]  
(7.32)
where $|h_{ab}| << 1$. Let’s assume that in the coordinates system where Eq. (7.32) is valid, the metric is stationary. That means

$$\partial_0 g_{ab} = \frac{\partial g_{ab}}{\partial t} = 0$$  \hspace{1cm} (7.33)

In other words this mean that the basis vectors do not change with time. An example is a Cartesian coordinates on a none-rotating earth. We recall that the worldline of a freely falling object is geodesic

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0.$$  \hspace{1cm} (7.34)

In a weak gravitational field we can say the speed of the particle is much smaller than the speed of light. This requires each components of the three velocity of the particle must be smaller compared to the speed of light.

$$\frac{dx^i}{dt} << c \Rightarrow dx^i << d(ct) \Rightarrow \frac{dx^i}{d\tau} << \frac{d(ct)}{d\tau} \Rightarrow \frac{dx^i}{d\tau} << \frac{dx^\alpha}{d\tau},$$  \hspace{1cm} (7.35)

where $i = 1, 2, 3$. Applying Eq. (7.34), we find

$$\frac{d^2 x^0}{d\tau^2} + \Gamma^0_{00} \left( \frac{dx^0}{d\tau} \right)^2 = 0 \Rightarrow \frac{d^2 x^a}{d\tau^2} = -c^2 \Gamma^a_{00} \left( \frac{dt}{d\tau} \right)^2.$$  \hspace{1cm} (7.36)

Let’s determine the connection. We recall that the connection is related to the metric by

$$\Gamma^a_{bc} = \frac{g^{ad}}{2} \left( \partial_0 g_{cd} + \partial_c g_{db} - \partial_d g_{bc} \right)$$  \hspace{1cm} (7.37)

so that

$$\Gamma^a_{00} = \frac{g^{ad}}{2} \left( \partial_0 g_{0d} + \partial_d g_{00} - \partial_d g_{00} \right).$$  \hspace{1cm} (7.38)

Since the metric is assumed to be stationary

$$\partial_0 g_{0d} = \partial_0 g_{00} = 0$$

which leads to

$$\Gamma^a_{00} = \frac{g^{ad}}{2} \partial_0 g_{00} = -\left( \frac{g^{a0}}{2} \partial_0 g_{00} + \frac{g^{a1}}{2} \partial_1 g_{00} \right) = \frac{g^{a1}}{2} \partial_1 g_{00}.$$  \hspace{1cm} (7.39)

where $i = 1, 2, 3$. Using Eq. (7.32), we may write

$$\Gamma^a_{00} = -\left( \eta^{ai} + h^{ai} \right) \partial_i (\eta_{00} + h_{00}).$$  \hspace{1cm} (7.40)

Noting that the metric for Minkowski spacetime is a constant

$$\partial_i \eta_{00} = 0,$$
CHAPTER 7. THE EQUIVALENCE PRINCIPLE AND SPACETIME CURVATURE

we find
\[ \Gamma^0_{00} = -\frac{1}{2} (\eta^{\alpha\beta} \partial_\beta h_{00} + h^{\alpha\beta} \partial_\beta h_{00}) \simeq -\frac{1}{2} \eta^{\alpha\beta} \partial_\beta h_{00}. \]  
(7.41)

where we made the approximation
\[ h^{\alpha\beta} \partial_\beta h_{00} \simeq 0 \]
for small curvature, where \( |h_{ab}| << 1 \). Noting that
\[ \eta^{\alpha i} = -\delta^{\alpha i} \]
for \( i = 1, 2, 3 \), we find for the connection
\[ \Gamma^0_{00} = \frac{1}{2} \delta^{\alpha i} \partial_\beta h_{00}. \]  
(7.42)

and for \( a = 0, 1, 2, 3 \), we have
\[ \Gamma^0_{00} = \frac{1}{2} \delta^{0 i} \partial_\beta h_{00}, \Gamma^1_{00} = \frac{1}{2} \delta^{1 i} \partial_\beta h_{00}, \Gamma^2_{00} = \frac{1}{2} \delta^{2 i} \partial_\beta h_{00}, \Gamma^3_{00} = \frac{1}{2} \delta^{3 i} \partial_\beta h_{00}, \]  
(7.43)

which leads to
\[ \Gamma^0_{00} = 0, \Gamma^1_{00} = \frac{1}{2} \partial_1 h_{00}, \Gamma^2_{00} = \frac{1}{2} \partial_2 h_{00}, \Gamma^3_{00} = \frac{1}{2} \partial_3 h_{00}. \]  
(7.44)

From the result in Eq. (7.44) one can easily see that
\[ \Gamma^a_{00} = \frac{1}{2} \nabla h_{00}. \]  
(7.45)

Substituting this result into Eq. (7.36), we find
\[ \frac{d^2 \tilde{x}^a}{d\tau^2} = -c^2 \Gamma^a_{00} \left( \frac{dt}{d\tau} \right)^2 \Rightarrow \frac{d^2 \tilde{x}}{d\tau^2} = -\frac{c^2}{2} \nabla h_{00} \left( \frac{dt}{d\tau} \right)^2 \]
which can be rewritten as
\[ \frac{d^2 \tilde{x}}{d\tau^2} = -\frac{c^2}{2} \nabla h_{00} = \nabla \left( -\frac{c^2}{2} h_{00} \right). \]  
(7.46)

We recall in Newtonian gravity
\[ \frac{d^2 \tilde{x}}{dt^2} = -\frac{m_G}{m_I} \nabla \Phi = \nabla \left( -\frac{m_G}{m_I} \Phi \right) \]  
(7.47)

Now comparing Eqs. (7.46) and (7.47), we can easily see that
\[ \frac{c^2}{2} h_{00} = \frac{m_G}{m_I} \Phi \]  
(7.48)

which gives
\[ h_{00} = \frac{2}{c^2} \frac{m_G}{m_I} \Phi \]
7.4. WEAK GRAVITATIONAL FIELDS AND THE NEWTONIAN LIMIT

and for \( m G / m I = 1 \),

\[
h_{00} = \frac{2\Phi}{c^2} \quad (7.49)
\]

Therefore, the metric in the weak field limit can be approximated as

\[
g_{00} = \eta_{00} + h_{00} = 1 + \frac{2\Phi}{c^2}. \quad (7.50)
\]

We recall the solution to Poisson’s equation, from EM

\[
\nabla^2 V (\vec{r}) = -\frac{\rho (\vec{r})}{\epsilon_0}, \quad (7.51)
\]

is given by

\[
V (\vec{r}) = \frac{1}{4\pi \epsilon_0} \int \rho (r') dr' = \frac{1}{4\pi \epsilon_0} \frac{Q}{r}. \quad (7.52)
\]

Then for

\[
\nabla^2 \Phi = (4\pi G) \rho. \quad (7.53)
\]

one can write

\[
\Phi (\vec{r}) = -\frac{GM}{r} \quad (7.54)
\]

so that

\[
\frac{2\Phi}{c^2} = -\left( \frac{GM}{c^2} \right) \frac{1}{r}. \quad (7.55)
\]

Let’s look at the values of this on a surface of the earth, the sun, and a white dwarf star.

<table>
<thead>
<tr>
<th>Object</th>
<th>radius, ( r = R ) in m</th>
<th>Mass, ( M ) in Kg</th>
<th>( \frac{2\Phi}{c^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>( 6.37 \times 10^6 )</td>
<td>( 5.97 \times 10^{24} )</td>
<td>( -10^{-9} )</td>
</tr>
<tr>
<td>Sun</td>
<td>( 6.96 \times 10^8 )</td>
<td>( 1.99 \times 10^{30} )</td>
<td>( -10^{-6} )</td>
</tr>
<tr>
<td>white dwarf star</td>
<td></td>
<td></td>
<td>( -10^{-4} )</td>
</tr>
</tbody>
</table>

where we used for the universal gravitational constant, \( G \)

\[
G = 6.67408 \times 10^{-11} \frac{m^3}{\text{kg} \cdot \text{s}^2} \quad (7.56)
\]

**Homework: find the value for \( \frac{2\Phi}{c^2} \) the white dwarf star.**

Consider a freely falling massive particles in a weak gravitational field produced by another particle. In each of these particles there are two clocks at rest,

\[
\frac{dx^i}{dt} = 0 \quad (7.57)
\]

Suppose an observer on the freely falling particle frame measures a proper time interval, \( dr \), between two clicks (two events), is given by

\[
(ds')^2 = c^2 (d\tau)^2. \quad (7.58)
\]

This interval must be the same as the interval measured on the reference frame
of the second massive particle that creates the gravitational field,

\[(ds)^2 = g_{ab}dx^a dx^b.\]  

(7.59)

In the weak field approximation, we found that

\[(ds)^2 = g_{00} (dx^0)^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 (dt)^2.\]  

(7.60)

as \(dx^i \simeq 0\) for \(i = 1, 2, 3\). Using

\[\frac{2\Phi}{c^2} = -\frac{GM}{c^2} \frac{1}{R},\]  

(7.61)

where \(R\) is the radius of the massive object responsible for the gravitational "disturbance", we have

\[(ds)^2 = \left[1 - \frac{GM}{c^2} \frac{1}{R}\right] c^2 (dt)^2.\]  

(7.62)

Eqs. (7.58) and (7.62) must be the same since the interval is independent of the coordinates system. Therefore

\[\left[1 - \frac{GM}{c^2} \frac{1}{R}\right] c^2 (dt)^2 = c^2 (d\tau)^2 \Rightarrow d\tau = \sqrt{1 - \left(\frac{GM}{c^2}\right) \frac{1}{R}} dt\]

This shows that \(d\tau < dt\). If we introduce the speed as

\[v^2 = \frac{GM}{R}\]  

(7.63)

we may write

\[d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt\]  

(7.64)
7.5. ELECTROMAGNETISM IN CURVED SPACETIME

Time dilation in a weak gravitational field! For example for a particle accelerated by earth gravitational field, using the result we estimated

\[ \frac{2\Phi}{c^2} = - \frac{GM}{c^2} \frac{1}{R} = 10^{-9} \Rightarrow v^2 = \frac{GM}{R} = 10^{-9} c^2 \]  \hspace{1cm} (7.65)

we find

\[ d\tau \simeq dt. \]  \hspace{1cm} (7.66)

7.5 Electromagnetism in curved spacetime

We recall that EM in uncurved spacetime (Minkowski spacetime) the governing equations for the fields are

\[ \partial_a \Omega_{ab} + \partial_b \Omega_{ca} = 0, \partial_a \Omega^{ab} = k J^b. \]  \hspace{1cm} (7.67)

The difference is that since now spacetime is curved and the basis vectors are no longer constant, instead of partial derivatives, \( \partial_b \), is replaced by the covariant derivative, \( \nabla_b \) such that

\[ \nabla_b J^a = \partial_b J^a + \Gamma^a_{cb} J^c, \]
\[ \nabla_c \Omega_b^a = \partial_c \Omega_b^a + \Gamma^a_{dc} \Omega^d_b - \Gamma^d_b \Omega^a_{dc}, \] \hspace{1cm} (7.68)
\[ \nabla_c \Omega_{ab} = \partial_c \Omega_{ab} - \Gamma^d_{ac} \Omega_{bd} - \Gamma^d_{bc} \Omega_{ad}, \] \hspace{1cm} (7.69)

that changes the field equations to

\[ \nabla_c \Omega_{ab} + \nabla_a \Omega_{bc} + \nabla_b \Omega_{ca} = 0, \nabla_a \Omega^{ab} = k J^b. \] \hspace{1cm} (7.70)

The equation of motion for a charged particle will also be the same

\[ \frac{Du^a}{D\tau} = \frac{q}{m_0} \Omega_b^a u^b \] \hspace{1cm} (7.71)

where \( F^a_b \) is the electromagnetic field tensor and

\[ \frac{Du^a}{D\tau} = \frac{du^a}{d\tau} + \Gamma^a_{cb} u^c \frac{dx^b}{d\tau}. \] \hspace{1cm} (7.72)

But the particle is moving under the influence of gravitational and EM fields. The effect of the gravitational field is just to change the curvature of the spacetime. This can also be put in terms of the coordinates variables, \( x^a \), as

\[ \frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = \frac{q}{m_0} \Omega_b^a \frac{dx^b}{d\tau}. \] \hspace{1cm} (7.73)
7.6 The curvature tensor

If the geometry of a manifold or a region of manifold is defined by *Riemannian geometry*, this geometry is determined by the equation for the metric

\[(ds)^2 = g_{ab} dx^a dx^b.\]  
(7.74)

If this geometry is not curved (if it is has a plane geometry), the metric takes the form

\[ds^2 = \gamma_1 (dx^1)^2 + \gamma_2 (dx^2)^2 + \ldots \gamma_n (dx^n)^2,\]  
(7.75)

where \(\gamma_n = \pm 1\). Next we want to generally quantify the curvature of a *Riemannian geometry* defined by

\[(ds)^2 = g_{ab} dx^a dx^b.\]  
(7.76)

It is not always easy to find out a coordinate transformation exists that allows us to write the metric in the form of Eq. (7.75) so that one can find out whether the manifold is curved or not. Therefore, we need to seek a way we can measure the curvature of a manifold or a region of manifold. The curvature tensor (or Riemann tensor) is a tensor that we use to measure the curvature of a manifold. We next derive this tensor. We recall for a vector, \(\vec{v}\), expressed in terms of its covariant components

\[\vec{v} = v_a \hat{e}^a\]  
(7.77)

the covariant derivative is given by

\[\partial_b \vec{v} = \partial_b (v_a \hat{e}^a) = \partial_b (v_a) \hat{e}^a + v_a \partial_b (\hat{e}^a) = (\partial_b v_a - \Gamma^c_{ab} v_c) \hat{e}^a = (\nabla_b v_a) \hat{e}^a.\]  
(7.78)

where

\[\nabla_b v_a = \partial_b v_a - \Gamma^c_{ab} v_c.\]  
(7.79)

Then taking the covariant derivative one more time for this vector, one can also write

\[\nabla_c \nabla_b v_a = \nabla_c (\partial_b v_a - \Gamma^c_{ab} v_c) = \nabla_c (\partial_b v_a) - \nabla_c (\Gamma^e_{ab} v_e) = \nabla_c (\partial_b v_a) - \Gamma^e_{ab} \nabla_c v_e - \nabla_c (\Gamma^e_{ab}) v_e.\]  
(7.80)

Noting that

\[\nabla_b v_a = t_{ba}\]

is a second rank tensor and for a second rank tensor the covariant derivative is given by

\[\nabla_c t_{ba} = \partial_c t_{ba} - \Gamma^d_{bc} t_{da} - \Gamma^d_{ac} t_{bd}\]

one can write

\[\nabla_c (\nabla_b v_a) = \partial_c (\nabla_b v_a) - \Gamma^d_{bc} \nabla_d v_a - \Gamma^d_{ac} \nabla_b v_d.\]  
(7.82)

In applying the relation in Eq. (7.79), we have

\[\nabla_b v_a = \partial_b v_a - \Gamma^c_{ab} v_c, \nabla_d v_a = \partial_d v_a - \Gamma^c_{ad} v_c, \nabla_b v_d = \partial_b v_d - \Gamma^c_{db} v_c\]  
(7.83)
so that using these expressions one finds

\[ \nabla_c (\nabla_b v_a) = \partial_c (\partial_b v_a - \Gamma^e_{ab} v_e) - \Gamma^d_{bc} (\partial_d v_a - \Gamma^e_{ad} v_e) - \Gamma^d_{ac} (\partial_d v_b - \Gamma^e_{db} v_e) \]

\[ = \partial_c \partial_b v_a - \partial_c (\Gamma^e_{ab} v_e) - \Gamma^d_{bc} (\partial_d v_a - \Gamma^e_{ad} v_e) - \Gamma^d_{ac} (\partial_d v_b - \Gamma^e_{db} v_e) \]  (7.84)

Which can be evaluated to give

\[ \nabla_c \nabla_b v_a = \partial_c \partial_b v_a - (\partial_c \Gamma^e_{ab}) v_e - \Gamma^d_{bc} (\partial_d v_a - \Gamma^e_{ad} v_e) - \Gamma^d_{ac} (\partial_d v_b - \Gamma^e_{db} v_e) \]  (7.85)

\textbf{Homework: Derive Eq. (7.85)}

Swapping the indices \( b \) and \( c \), one can also express

\[ \nabla_b \nabla_c v_a = \partial_b \partial_c v_a - (\partial_b \Gamma^e_{ac}) v_e - \Gamma^d_{cb} (\partial_d v_a - \Gamma^e_{ad} v_e) - \Gamma^d_{bc} (\partial_c v_b - \Gamma^e_{cd} v_e) \]  (7.86)

Subtracting Eq. (7.86) from (7.85), we find

\[ \nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = (\partial_b \Gamma^e_{ac} - \partial_c \Gamma^e_{ab}) v_e + \Gamma^e_{ac} \partial_b v_e - \Gamma^e_{ab} \partial_c v_e \\
+ \Gamma^d_{bc} (\partial_d v_a - \Gamma^e_{ad} v_e) - \Gamma^d_{ac} (\partial_d v_b - \Gamma^e_{db} v_e) \\
+ \Gamma^d_{cd} \partial_d v_a - \Gamma^d_{ac} \partial_d v_b + \Gamma^d_{bc} \partial_c v_e \\
\]

\[ = (\partial_b \Gamma^e_{ac} - \partial_c \Gamma^e_{ab}) v_e + \Gamma^d_{ac} \partial_b v_e - \Gamma^d_{ab} \partial_c v_e \\
+ \Gamma^d_{cd} \partial_d v_a - \Gamma^d_{ac} \partial_d v_b + \Gamma^d_{bc} \partial_c v_e \]  (7.87)

Swapping the indices \( b \) and \( c \), one can write

\[ \Gamma^e_{ac} \partial_b v_e = \Gamma^e_{ab} \partial_c v_e, \quad \Gamma^e_{ab} \partial_c v_e = \Gamma^e_{ac} \partial_b v_e \]  (7.88)

so that we find

\[ \nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = (\partial_b \Gamma^e_{ac} - \partial_c \Gamma^e_{ab}) v_e + \Gamma^e_{ac} \partial_d v_e + \Gamma^d_{cd} \partial_d v_a - \Gamma^d_{ac} \partial_d v_b - \Gamma^d_{bc} \partial_c v_e \]

where

\[ R^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac} \Gamma^d_{eb} - \Gamma^e_{ab} \Gamma^d_{ec} \]

is the \textit{curvature tensor}. In a flat region of a manifold we can choose coordinates \( x^a \) such that

\[ ds^2 = \epsilon_1 (dx^1)^2 + \epsilon_2 (dx^2)^2 + \ldots + \epsilon_n (dx^N)^2, \]  (7.89)

where \( \epsilon_n = \pm 1 \). We recall the relation between the affine connection and the metric,

\[ \Gamma^f_{bc} = \frac{g^{fd}}{2} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \]  (7.90)

Noting that for the flat region of the manifold

\[ g_{cd} = \delta_{cd} \epsilon_c \Rightarrow \Gamma^f_{bc} = 0 \]  (7.91)

the connections in the curvature tensor vanishes, which leads to

\[ R^d_{abc} = 0. \]  (7.92)
Conversely, if Eq. (7.92) is true, then we can find coordinates $x^a$ such that the interval can be expressed as Eq. (7.89).

The symmetry: In order to find out the symmetry of the curvature tensor, we lower the index. Using the metric tensor

$$R_{abcd} = g_{ac} R^c_{bdc} = g_{ac} \left( \partial_d \Gamma_{bc}^e - \partial_c \Gamma_{bd}^e + \Gamma_{bd}^f \Gamma_{fc}^e - \Gamma_{bd}^f \Gamma_{fc}^e \right).$$

(7.93)

Homework: Applying

$$\Gamma_{bc}^a = \frac{g^{ad}}{2} \left( \partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

(7.94)

show that

$$R_{abcd} = \frac{1}{2} \left( \partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ac} + \partial_c \partial_a g_{bd} - \partial_c \partial_b g_{ad} \right)$$

$$- g^{ef} \left( \Gamma_{ead} \Gamma_{fbc} - \Gamma_{eaf} \Gamma_{cbd} \right).$$

(7.95)

We recall that tensor equation is invariant under coordinate transformation. Then if we determine the symmetries of the curvature tensor in one coordinate system then it must be the same in another. So let’s consider a Geodesic coordinate at some arbitrary point on the manifold so that the connection be zero, $\Gamma_{bd}^e = 0$, but not necessarily of its derivative. Then the curvature tensor in Eq. (7.95) becomes

$$R_{abcd} = \frac{1}{2} \left( \partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ac} + \partial_c \partial_a g_{bd} - \partial_c \partial_b g_{ad} \right).$$

(7.96)

There follows that

$$R_{bacd} = \frac{1}{2} \left( \partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ac} + \partial_c \partial_a g_{bd} - \partial_c \partial_b g_{ad} \right)$$

$$= \frac{1}{2} \left( \partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ac} + \partial_c \partial_a g_{bd} - \partial_c \partial_b g_{ad} \right)$$

$$\Rightarrow R_{bacd} = - R_{abcd}$$

(7.97)

Similarly, one can show that

$$R_{abcd} = - R_{abdc}$$

(7.98)

which leads to

$$R_{abcd} = R_{cdef}$$

(7.99)

Homework: Show Eqs. (7.98) and (7.99)

The cyclic identity: the curvature tensor satisfies the cyclic identity for the last three indices

$$R_{abcd} + R_{adbc} + R_{acdb} = 0,$$

(7.100)

or using the symmetries in Eqs. (7.98) and (7.99) along with the notation we introduced to, we can write

$$R_{a[bcdf]} = 0,$$

(7.101)
7.6. THE CURVATURE TENSOR

Homework: Show that

\[ R_a{}^{bcd} = R_{abcd} + R_{adbc} + R_{acdb} = 0, \]  
(7.102)

Taking into account the symmetry and cyclic properties, for \( N \) dimensional manifold, the components of the curvature tensor are \( N^2(N^2 - 1)/12 \) instead of \( N^4 \). For example for

\[ N = 2 \rightarrow 1 \text{ component} \]
\[ N = 3 \rightarrow 6 \text{ components} \]
\[ N = 4 \rightarrow 20 \text{ components} \]  
(7.103)

The differential identity: the curvature tensor also satisfies the differential identity at a point, \( p \), where the metric is a constant and diagonal and the affine connections vanish

\[ g_{ae} = \delta_{ae} \epsilon_a, \Gamma^f_{bc} = 0 \]  
(7.104)

we have

\[
R_{abcd} = g_{ae} R^e_{bcd} = g_{ae} \left( \partial_c \Gamma^e_{bd} - \partial_d \Gamma^e_{bc} + \Gamma^f_{bd} \Gamma^e_{fc} - \Gamma^f_{bc} \Gamma^e_{fd} \right)
\]
\[
= \partial_c (g_{ae} \Gamma^e_{bd}) - \partial_d (g_{ae} \Gamma^e_{bc}) + \Gamma^f_{bd} g_{ae} \Gamma^e_{fc} - \Gamma^f_{bc} g_{ae} \Gamma^e_{fd}
\]
\[
\Rightarrow R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc}
\]  
(7.105)

so that

\[
(\nabla_e R_{abcd})_p = (\partial_e R_{abcd})_p = (\partial_c \partial_e \Gamma_{abd} - \partial_e \partial_d \Gamma_{abc})_p
\]  
(7.106)

\[
\nabla_e R_{abcd} + \nabla_e R_{abde} + \nabla_d R_{abec} = 0
\]  
(7.107)

This is known as the Bianchi identity and using the antisymmetry relation

\[
\nabla_e R_{abcd} = 0
\]  
(7.108)

Raising the first index in the curvature tensor \( R_{abcd} \)

\[
R^e_{bcd} = g^{ea} R_{abcd}
\]  
(7.109)

and then contracting on the first two indices, we have

\[
R_{cd} = R^b_{bcd} = g^{ba} R_{abcd}
\]  
(7.110)

Similarly for \( R_{bacd} \), we have

\[
R^a_{acd} = g^{ab} R_{bacd} \Rightarrow R_{cd} = R^a_{acd} = g^{ab} R_{bacd}
\]  
(7.111)

so that applying the antisymmetric properties of the curvature tensor

\[
R_{bacd} = -R_{abcd}
\]

we have

\[
R_{cd} = R^a_{acd} = -g^{ab} R_{abcd}
\]  
(7.112)
Combining Eqs. (7.110) and (7.112), we can easily see that
\[ R_{cd} = R^a_{abcd} = 0 \]  
(7.113)

This shows that raising the first index and then contracting the first and second indices of the curvature tensor gives a zero tensor. However, raising the first index and then contracting the first and the last indices gives a non-zero tensor which is known as the Ricci tensor.

\[ R^e_{bcd} = g^{ea} R_{abcd} \]  
(7.114)

or
\[ R^e_{bcd} = g^{ea} R_{abcd} \]  
(7.115)

The Ricci tensor is symmetric. We can prove this using the cyclic identity in Eq. (7.100). Raising the index \( a \)

\[ g^{fa} R_{abcd} + g^{fa} R_{acdb} + g^{fa} R_{adbc} = 0, \]  
(7.116)

and contracting with the index \( d \)

\[ g^{da} R_{abcd} + g^{da} R_{acdb} + g^{da} R_{adbc} = 0, \]  
(7.117)

which we may write as

\[ R^d_{bcd} + R^d_{cde} + R^d_{dhe} = 0 \]  
(7.118)

Applying the relation in Eq. (7.113), we have

\[ R_{bc} = R^d_{dhe} = 0 \]  
(7.119)

so that

\[ R^d_{bcd} + R^d_{cde} = 0. \]  
(7.120)

Noting that

\[ g^{fa} R_{abcd} = -g^{fa} R_{abdc} \Rightarrow g^{da} R_{abcd} = -g^{da} R_{abdc} \Rightarrow R^d_{bcd} = -R^d_{bdc} \]

we can write Eq. (7.120) as

\[ R^d_{bcd} = R^d_{cde} \Rightarrow R_{bc} = R_{cb} \]  
(7.121)

which shows that the Ricci tensor is symmetric. By raising the first indices on both side of Eq. (7.121), we can write

\[ R^d_{cde} = R^d_{ced} \]  
(7.122)

A further contraction of the curvature tensor leads to the curvature scalar. From Eq.(7.115), we have

\[ R_{ab} = R^d_{abcd} = g^{de} R_{eabcd} \]  
(7.123)

upon contracting this, the curvature scalar is given by

\[ R = g^{ab} R_{ab} = R^a_a \]  
(7.124)
7.7 The Einstein Tensor

We recall the Bianchi identity

$$
\nabla_e R_{abcd} + \nabla_e R_{abdc} + \nabla_d R_{abec} = 0
$$

(7.125)

raising $a$ and contracting it with $d$ ($d = a$) leads to

$$
\nabla_e R_{bca} + \nabla_e R_{bac} + \nabla_a R_{bec} = \nabla_e R_{bc} + \nabla_c R_{be} + \nabla_a R_{bec} = 0.
$$

(7.126)

From the antisymmetric property of the curvature tensor

$$
R_{abcd} = -R_{abdc}
$$

(7.127)

we may write

$$
R_{abde} = -R_{abed} \quad R_{abde} = -R_{bedc} \quad R_{abed} = -R_{bea} = -R_{be}
$$

(7.128)

so that Eq. (7.126) becomes

$$\nabla_e R_{bc} - \nabla_c R_{be} + \nabla_a R_{bec} = 0,
$$

(7.129)

Now raising $b$ we have

$$\nabla_e R^b_c - \nabla_c R^b_e + \nabla_a R^{ab}_{ce} = 0
$$

(7.130)

so that contracting with $c$ ($c = b$), we find

$$\nabla_b R^b_c - \nabla_c R^b_e + \nabla_a R^{ab}_{ce} = 0.
$$

(7.131)

In terms of the curvature scalar

$$
R = R^b_b
$$

we can write

$$\nabla_b R^b_c - \nabla_c R + \nabla_a R^{ab}_{ce} = 0.
$$

(7.132)

Once more invoking the antisymmetric properties of the curvature tensor

$$
R_{baced} = -R_{abced}, R_{abedc} = -R_{abdec}, R_{abcd} = R_{cdab}
$$

(7.133)

we have

$$
R^{ba}_{cd} = -R^{ab}_{cd}, R^{ab}_{cd} = -R^{ab}_{dc}, R^{ab}_{cd} = R^{cd}_{ab}
$$

(7.134)

$$\nabla_a R^{ba}_{cd} = -\nabla_a R^{ab}_{cd}, \quad \nabla_a R^{ab}_{cd} = -\nabla_a R^{ab}_{cd}
$$

(7.135)

contracting $b$ and $d$

$$\nabla_a R^{ba}_{cb} = -\nabla_a R^{ab}_{cb} \Rightarrow \nabla_a R^{ab}_{cb} = \nabla_a R^{ab}_{cb} = \nabla_a R^a_c
$$
where we contracted $d$. The indices are dummy indices and we can write
\[ \nabla_a R^{ab}_{bc} = \nabla_a R^{a}_{c} = \nabla_b R^{b}_{c} \]  
(7.136)
Substituting this into Eq. (7.132) gives
\[ \nabla_b R^{b}_{c} - \nabla_c R + \nabla_b R^{b}_{c} = 0 \Rightarrow 2 \nabla_b R^{b}_{c} - \nabla_c R = 0 \]  
(7.137)
Noting that
\[ \nabla_c = \delta^c_b \nabla_b \]  
(7.138)
we may write Eq. (7.137) as
\[ 2 \nabla_b R^{b}_{c} - \delta^b_c \nabla_b R = \nabla_b \left( 2 R^{b}_{c} - \delta^b_c R \right) = 0 \]  
(7.139)
Raising the index $c$ by multiplying with the metric tensor, we have
\[ \nabla_b \left[ 2 g^{cd} R^{b}_{d} - g^{cd} \delta^b_c R \right] = 0 \]  
(7.140)
which leads to
\[ \nabla_b G^{ab} = 0 \]  
(7.141)
where
\[ G^{ab} = R^{bc} - \frac{1}{2} g^{cb} R \]  
(7.142)
is known as the Einstein tensor. We have proved earlier that the Ricci tensor is symmetric and we already know the metric tensor is symmetric. Therefore, the Einstein tensor is a symmetric tensor.

### 7.8 Curvature and parallel transport

We recall that suppose we have a curve $C$ parametrized with the variable, $u$, on a given manifold, a vector field
\[ \vec{v} = v_a(u) \hat{e}^a(u) \]  
(7.143)
parallel transported along this curve is when the intrinsic derivative of this vector is zero
\[ \frac{Dv_a}{Du} = \frac{dv_a}{du} - \Gamma^b_{ac} v^c \frac{dx^b}{du} = 0. \]  
(7.144)
or in terms of its controvariant components
\[ \frac{Dv^a}{Du} = \frac{dv^a}{du} + \Gamma^a_{cb} v^c \frac{dx^b}{du} = 0. \]  
(7.145)
Thus when a vector is parallel transported, we must have
\[ \frac{dv_a}{du} = \Gamma^b_{ac} v^c \frac{dx^b}{du} \]  
(7.146)
7.8. CURVATURE AND PARALLEL TRANSPORT

or

\[ \frac{dv^a}{du} = -\Gamma^a_{bc} \frac{dx^c}{du} \]  \hspace{1cm} (7.147)

Let’s consider a parallel transported vector along a closed curve \( C \) on a manifold. Before we derive the condition for a parallel transport of a vector field on a curved manifold, it is helpful if we refresh our memory about Stoke’s theorem from Theoretical Physics I.

**Stokes’ Theorem**: Stokes theorem states that

\[ \oint_{\text{Curve bounding } A} \vec{D} \cdot d\vec{\sigma} = \int_{\text{Surface } A} \nabla \times \vec{D} \cdot \hat{n} dA. \]

where the surface is any surface that has \( C \), as its boundary. It could be an a cyclic shown in the figure or a hemisphere If the vector is constant (both its magnitude and direction), we have

\[ \nabla \times \vec{D} = 0 \]

which leads to

\[ \oint_{\text{Curve bounding } A} \vec{D} \cdot d\vec{\sigma} = 0 \Rightarrow \oint_{\text{Curve bounding } A} d\vec{\sigma} = 0 \]

Now let’s say that this curve \( C \) is on some manifold instead of the Euclidean space in which we know the Stoke’s theorem is valid. Also let’s consider a vector field

\[ \vec{v} = v_a(u) \hat{e}^a(u) \] \hspace{1cm} (7.148)

that is being parallel transported in this closed curve. When this vector is being parallel transported in curve \( C \), it is transported by dividing the area bounded
CHAPTER 7. THE EQUIVALENCE PRINCIPLE AND SPACETIME CURVATURE

by the curve $C$ into small areas each bounded by a small curve $c_N$. Suppose we assume there is a change $\Delta v^a$ in the components of the vector during the transportation of the vector along this curve. This change can be expressed as

$$\Delta v^a = \sum_N (\Delta v^a)_N,$$

where $(\Delta v^a)_N$ is the change resulting from the transportation of the vector along the tiny curve $c_N$.

If the vector is parallel transported along the tiny curve $c_N$, we must have

$$\frac{dv^a}{du} = -\Gamma^a_{bc} v^b dx^c du$$  \hspace{1cm} (7.149)

at all point on the curve. Suppose the transportation took from some point $p$ on $c_N$ to another point which we may describe by the coordinate $x^e(u)$, then we may write

$$\int_{u_p}^u \frac{dv^a}{du} du = -\int_{u_p}^u \Gamma^a_{bc} v^b dx^c du$$  \hspace{1cm} (7.150)

which gives

$$v^a(u) = v^a(u_p) - \int_{u_p}^u \Gamma^a_{bc} v^b dx^c du.$$  \hspace{1cm} (7.151)

Noting that we can make $c_N$ as small as we want, a series expansion about point $p$ allows us to make the approximations

$$v^b(u) \simeq v^b(u_p) - \Gamma^b_{cf}(u_p) v^c(u_p) (x^f(u) - x^f(u_p)).$$  \hspace{1cm} (7.152)

$$\Gamma^a_{bc}(u) \simeq \Gamma^a_{bc}(u_p) + (\partial_d \Gamma^a_{bc})_{u_p} (x^d(u) - x^d(u_p)).$$  \hspace{1cm} (7.153)

**Homework: derive Eq. (7.152)**

Substituting Eqs. (7.152) and (7.153) into Eq. (7.151), we have

$$v^a(u) = v^a(u_p) - \int_{u_p}^u \left[ \Gamma^a_{bc}(u_p) + (\partial_d \Gamma^a_{bc})_{u_p} (x^d(u) - x^d(u_p)) \right] \frac{dx^c}{du} du (7.154)$$

$$\times \left[ v^b(u_p) - \Gamma^b_{cf}(u_p) v^c(u_p) (x^f(u) - x^f(u_p)) \right] dx^c$$

so that keeping terms up to the first order in $(x^c(u) - x^c(u_p))$, we can write

$$v^a(u) = v^a(u_p)$$

$$- \int_{u_p}^u \Gamma^a_{bc}(u_p) v^b(u_p) dx^c + \int_{u_p}^u \Gamma^a_{bc}(u_p) \Gamma^b_{cf}(u_p) v^c(u_p) (x^f(u) - x^f(u_p)) dx^c$$

$$- \int_{u_p}^u (\partial_d \Gamma^a_{bc})_{u_p} (x^d(u) - x^d(u_p)) v^b(u_p) dx^c$$  \hspace{1cm} (7.156)
\[ v^a(u) = v^a(u_p) \]

\[ -\Gamma^b_{bc}(u_p) v^b(u_p) \int_p^u dx^c + \Gamma^b_{bc}(u_p) \Gamma^c_{ef}(u_p) v^e(u_p) \int_p^u \left( x^f(u) - x^f(u_p) \right) dx^c \]

\[ - (\partial_d \Gamma^a_{bc})_{u_p} v^b(u_p) \int_p^u \left( x^d(u) - x^d(u_p) \right) dx^c \]

(7.157)

Now if we close the tiny loop, we have

\[ \int_p^u dx^c = \oint dx^c = 0. \]

\[ \oint x^f(u_p) dx^c = x^f(u_p) \oint dx^c = 0, \quad \oint x^d(u_p) dx^c = x^d(u_p) \oint dx^c = 0 \]

so that for a closed curve Eq. (7.157) reduces to

\[ \Delta v^a = v^a(u) - v^a(u_p) = \Gamma^b_{bc}(u_p) \Gamma^c_{ef}(u_p) v^e(u_p) \oint x^f dx^c \]

(7.158)

\[ - (\partial_d \Gamma^a_{bc})_{u_p} v^b(u_p) \oint x^d dx^c. \]

Changing the dummy indices \( f \) to \( d \), and swapping the places of \( e \) and \( b \), we can write

\[ \Gamma^a_{bc}(u_p) \Gamma^b_{ef}(u_p) \oint x^f dx^c = \Gamma^a_{cc}(u_p) \Gamma^b_{bd}(u_p) v^b(u_p) \oint x^d dx^c \]

so that Eq. (7.158) can be expressed as

\[ \Delta v^a = - \left( \partial_d \Gamma^a_{bc} \right)_{u_p} v^b(u_p) \oint x^d dx^c. \]

(7.159)

Swapping the indices \( d \) and \( c \), we can also write Eq. (7.159) as

\[ \Delta v^a = - \left( \partial_d \Gamma^a_{bd} \right)_{u_p} v^b(u_p) \oint x^c dx^d. \]

(7.160)

Noting that for the closed loop

\[ \oint x^d dx^d = \oint x^c dx^c + \oint x^d dx^c = 0 \Rightarrow \oint x^c dx^c = - \oint x^c dx^d \]

(7.161)
we may rewrite Eq. (7.159)

$$\Delta v^a = \left[ (\partial_d \Gamma_{bc}^a)_{up} - \Gamma_{ec}^a (u_p) \Gamma_{bd}^e (u_p) \right] v^b (u_p) \int_{c_n} x^e dx^d. \quad (7.162)$$

Upon adding Eqs. (7.160) and (7.162),

$$2\Delta v^a = \left[ (\partial_d \Gamma_{bc}^a)_{up} - \Gamma_{ec}^a (u_p) \Gamma_{bd}^e (u_p) - (\partial_c \Gamma_{bd}^a)_{up} + \Gamma_{ed}^a (u_p) \Gamma_{bc}^e (u_p) \right] v^b (u_p) \int_{c_n} x^e dx^d. \quad (7.163)$$

and simplification of this leads to

$$\Delta v^a = \frac{1}{2} \left[ (\partial_c \Gamma_{bd}^a)_{up} - (\partial_d \Gamma_{bc}^a)_{up} + \Gamma_{ec}^a (u_p) \Gamma_{bd}^e (u_p) - \Gamma_{ed}^a (u_p) \Gamma_{bc}^e (u_p) \right] v^b (u_p) \int_{c_n} x^e dx^d. \quad (7.164)$$

Applying the curvature tensor

$$R_d^{\ a \ bc} = \partial_b \Gamma_d^{\ a \ ec} - \partial_c \Gamma_d^{\ a \ eb} + \Gamma_e^{\ \ a \ ec} \Gamma_d^{\ eb} - \Gamma_e^{\ \ a \ eb} \Gamma_d^{\ ec}; \quad (7.165)$$

we find

$$\Delta v^a = -\frac{1}{2} (R_{bcd}^a)_{p} v^b (u_p) \int_{c_n} x^e dx^d. \quad (7.166)$$

where

$$(R_{bcd}^a)_{p} = (\partial_c \Gamma_{bd}^a)_{u_p} - (\partial_d \Gamma_{bc}^a)_{u_p} + \Gamma_{ec}^a (u_p) \Gamma_{bd}^e (u_p) - \Gamma_{ed}^a (u_p) \Gamma_{bc}^e (u_p).$$

For a parallel transport of a vector in the tiny loop $c_N$ the components must remain unchanged. This requires that for a vector field to be parallel transported on $c_N$ we must have

$$\Delta v^a = -\frac{1}{2} (R_{bcd}^a)_{p} v^b (u_p) \int_{c_n} x^e dx^d = 0 \quad (7.167)$$

This is possible if and only if the curvature tensor vanishes at point $p$

$$(R_{bcd}^a)_{p} = 0. \quad (7.168)$$

Then for a parallel transport of the vector field on a closed curve $C$ on the manifold, the components must not change over the entire area bounded by the curve $C$ so that

$$\Delta v^a = \sum_N (\Delta v^a)_N = \sum_N \left( -\frac{1}{2} (R_{bcd}^a)_{p} v^b (u_p) \int_{c_n} x^e dx^d \right)_N = 0.$$
This can be satisfied if and only if the curvature tensor vanishes

\[(R^a_{\ bcd})_p = 0.\]  \hspace{1cm} (7.169)

over the area \(A\) bounded by the curve \(C\).

**Homework:** Show that a vector can be parallel transported on a cylindrical surface but not on a spherical surface. (Refer to Appendix 7A in the text book)
Chapter 8

The gravitational field equations

In our discussion of electromagnetism we saw that the fields are created by some changes in some charge distribution (i.e. the four-current density) happens somewhere close by or light-years away from the place of the fields are detected. We determined the field equations that governs how the fields are related to this four-current density. In this chapter we will see how the gravitational fields are related to matter (mass). When we determined the field equations in electromagnetism we first found the four-current (1-rank tensor) the zero component of which is just the charge density. So we follow a similar approach here.

8.1 The energy-momentum tensor

Unlike electromagnetism where the current responsible for the fields is represented by a 1-st rank tensor (a vector), the mass responsible for the creation of the gravitational field is a 2-nd rank tensor known as the energy-momentum tensor (or the stress-energy tensor). Next we will see how this tensor is obtained. To this end, let’s consider time-dependent distribution of non-interacting particles, each with rest mass $m_0$ in a cube of length $l_0$ commonly known as dust. At each event $p$ in space-time we can characterize the mass distribution of this particle using the density $\rho$ and the 3-velocity $\vec{u}$ as measured in some inertial reference frame. This means basically we have a fluid of density $\rho$ flowing with a velocity $\vec{u}$. Suppose the inertial reference frame is the instantaneous reference frame (IRF) $S$ at $p$ in which $\vec{u} = 0$. In the IRF the number density of the fluid is $n_0$ and the mass of each particle that make up the fluid is $m_0$. Then the proper mass density $\rho_0$ can be expressed as

$$\rho_0 = m_0 n_0 = \frac{m_0 N}{l_0^3},$$

(8.1)
where $N$ is the number of particles in the cube of length $l_0$. In some other frame $S'$, moving with speed $v$ relative to $S$ we recall that

\[
l = l_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{l_0}{\gamma_v}, \tag{8.2}
\]

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma_v m_0, \tag{8.3}
\]

so that the mass density, $\rho$, in $S'$ would be

\[
\rho = mn = \gamma_v^2 \frac{m_0 N}{l_0^3} = \gamma_v^2 \rho_0. \tag{8.4}
\]

So what would this suggest about how we represent the mass responsible for the gravitational field? Well to answer this let’s go back to our discussion about electromagnetism. In electromagnetism, we saw that the charge and current densities transformed as

\[
\rho' = \gamma_u \rho_0, \quad J' = \gamma_u \rho_0 \vec{v}. \tag{8.5}
\]

which is first-order in $\gamma_u$. This has lead us to representing the charge and current densities with 4-current density (a vector-1st rank tensor)

\[
[J^a] = \gamma_u \rho_0 (c, \vec{v}) = \gamma_u \rho_0 u^a. \tag{8.6}
\]

in terms of the 4-velocity $u^a$. In the case of mass density as we can see from Eq. (8.4), the mass density is second order in $\gamma_u$. This suggests that we can make a similar argument and express the mass density as tensor product of the four velocity $u^a (x)$ so that we can at least sure that the 00 component will lead to the right mass density transformation. Therefore, one can then define a 2-nd rank tensor (the energy-momentum tensor) $T(x)$ for the mass responsible for the existence of the gravitational field as

\[
T (x) = \bar{u} (x) \otimes \bar{u} (x) \tag{8.7}
\]

where we denoted the proper density of the fluid $\rho_0$ by $\rho$ and $\bar{u} (x)$ is its 4-velocity. Then the covariant components of the energy-momentum tensor are given by

\[
T^{ab} = \rho u^a u^b. \tag{8.8}
\]

Using

\[
u^a = \gamma_u (c, \vec{u}) = \gamma_u (c, u^i) \tag{8.9}
\]

it is not hard to make the interpretation given in the table below.
8.2 A PERFECT FLUID

For a real fluid the contravariant components of the energy-momentum tensor represents the quantities described in the table below:

<table>
<thead>
<tr>
<th>Components</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^{00}$</td>
<td>$\rho u^0 u^0 = \frac{\gamma^2 c^2 \rho_0}{c}$</td>
</tr>
<tr>
<td>$T^{0i}$</td>
<td>$\rho u^0 u^i = \frac{\gamma^2 u^i \rho_0}{c}$</td>
</tr>
<tr>
<td>$T^{ij}$</td>
<td>$\rho u^i u^j = \frac{\gamma^2 u^i \rho_0 u^j}{c}$</td>
</tr>
</tbody>
</table>

For a perfect fluid we have

<table>
<thead>
<tr>
<th>Components</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^{00}$</td>
<td>no forces between particles</td>
</tr>
<tr>
<td>$T^{0i}$</td>
<td>no heat conduction</td>
</tr>
<tr>
<td>$T^{ij}$</td>
<td>no heat conduction then no momentum carried by the energy</td>
</tr>
<tr>
<td>$T^{ij}$</td>
<td>no viscous stresses in the fluid.</td>
</tr>
</tbody>
</table>

so that the none zero elements of the energy-momentum tensor would be

$$T = \begin{bmatrix} c^2 \rho & 0 & 0 & 0 \\ 0 & \rho u^1 u^1 & 0 & 0 \\ 0 & 0 & \rho u^2 u^2 & 0 \\ 0 & 0 & 0 & \rho u^3 u^3 \end{bmatrix}$$

(8.10)

We have stated that the random thermal motions of the particles will give rise
CHAPTER 8. THE GRAVITATIONAL FIELD EQUATIONS

to momentum flow and $T^i{}_i$ is the isotropic pressure in the i-direction. This means

$$T^i{}_i = \rho u^i u^i = \frac{\text{Change in momentum}}{\text{change in time} \times \text{area}} = \text{Pressure}$$  \hspace{1cm} (8.11)

and we can express Eq. (8.10) as

$$[T^{ab}] = \begin{bmatrix} c^2 \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \right)$$  \hspace{1cm} (8.12)

HW  Homework: show that

$$T^{ab} = \left( \rho + \frac{p}{c^2} \right) u^a u^b - p g^{ab}$$  \hspace{1cm} (8.13)

Sol:

$$[T^{ab}] = \begin{bmatrix} c^2 \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$  \hspace{1cm} (8.14)

$$\Rightarrow [T^{ab}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$  \hspace{1cm} (8.15)

$$\Rightarrow [T^{ab}] = \begin{bmatrix} (\rho - \frac{p}{c^2}) \ c.c. & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - p \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$  \hspace{1cm} (8.16)

There follows that

$$T^{ab} = \left( \rho + \frac{p}{c^2} \right) u^a u^b - p g^{ab}.$$  \hspace{1cm} (8.18)
8.3. CONSERVATION OF ENERGY AND MOMENTUM

Eq. (8.13) is obtained for a local Cartesian coordinates. The expression of the energy-momentum tensor for a perfect fluid in an arbitrary coordinate system is given by simply replacing $\eta^{ab}$ by $g^{ab}$.

$$T^{ab} = \left( \rho + \frac{p}{c^2} \right) u^a u^b - pg^{ab}$$  \hspace{1cm} (8.19)

Note that the energy-momentum tensor for a perfect fluid is symmetric. In the limit $p \to 0$, the tensor becomes

$$T = \rho u^{ab}$$  \hspace{1cm} (8.20)

and a perfect fluid becomes dust.

8.3 Conservation of energy and momentum

By analogy with conservation of the 4-current

$$\partial_a J^a = 0$$  \hspace{1cm} (8.21)

we can write conservation of energy and momentum

$$\partial_a T^{ab} = 0$$  \hspace{1cm} (8.22)

Instead of deriving this relation, we will prove that this is the right equation for conservation of energy and momentum by reproducing the conservation laws for a Newtonian perfect fluid. Substituting Eq. (8.13) into (8.22), we have

$$\partial_a T^{ab} = \partial_a \left[ \left( \rho + \frac{p}{c^2} \right) u^a u^b - p g^{ab} \right] = 0$$  \hspace{1cm} (8.23)

Generally, the density, the pressure, and the velocity are all depend on the coordinate $x^a$, thus

$$\partial_a \left( \rho + \frac{p}{c^2} \right) u^a u^b + \left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) u^b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) - \partial_a (\rho) \eta^{ab} = 0$$  \hspace{1cm} (8.24)

We recall that for the 4-velocity

$$u^b u_b = c^2$$  \hspace{1cm} (8.25)

so that

$$\left( \partial_a u^b \right) u_b + u^b (\partial_a u_b) = \partial_a \left( u^b u_b \right) = \partial_a (c^2) = 0$$

$$\Rightarrow 2 \left( \partial_a u^b \right) u_b = 0 \Rightarrow (\partial_a u^b) u_b = 0.$$  \hspace{1cm} (8.26)

Contracting Eq. (8.24) by $u_b$, we have

$$\left[ \partial_a \left( \rho + \frac{p}{c^2} \right) \right] u^a u^b u_b + \left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) u^b u_b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) u_b$$

$$- \partial_a (p) \eta^{ab} u_b = 0$$  \hspace{1cm} (8.27)
so that for the first and second terms applying Eq. (8.25) and for the third term using Eq. (8.26), one finds
\[
\begin{align*}
\left[ \partial_a \left( \rho + \frac{p}{c^2} \right) \right] u^a u^b & = \left[ \partial_a \left( \rho + \frac{p}{c^2} \right) \right] u^a c^2 \\
\left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) u^b u_b & = \left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) c^2 \\
\left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) u_b & = 0
\end{align*}
\] (8.28)

which leads to
\[
\begin{align*}
\left[ \partial_a \left( \rho + \frac{p}{c^2} \right) \right] u^a c^2 + \left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) c^2 - \partial_a (p) \eta^{ab} u_b &= 0 \\
\Rightarrow \left[ \partial_a \left( c^2 \rho + p \right) \right] u^a + (c^2 \rho + p) \partial_a (u^a) - \partial_a (p) \eta^{ab} u_b &= 0 \\
\Rightarrow \partial_a \left[ (c^2 \rho + p) u^a \right] - \partial_a (p) \eta^{ab} u_b &= 0
\end{align*}
\] (8.29)

Noting that
\[
\eta^{ab} u_b = u^a
\] (8.30)

we find
\[
\begin{align*}
\partial_a \left[ (c^2 \rho + p) u^a \right] - \partial_a (p) u^a &= \partial_a \left[ (c^2 \rho + p) u^a \right] - \partial_a (p u^a) + p \partial_a (u^a) = 0 \\
\Rightarrow \partial_a \left[ c^2 \rho u^a + p u^a - pu^a \right] + p \partial_a (u^a) &= 0 \\
\Rightarrow \partial_a (\rho u^a) + \frac{p}{c^2} \partial_a u^a &= 0
\end{align*}
\] (8.31)

Eq. (8.29) is the relativistic equation of continuity. Now we rearrange Eq. (8.24) as
\[
\begin{align*}
\partial_a \left( \rho + \frac{p}{c^2} \right) u^a u^b + \left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) u^b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) - \partial_a (p) \eta^{ab} &= 0 \\
\left\{ \partial_a \left( \rho + \frac{p}{c^2} \right) u^a + \left( \rho + \frac{p}{c^2} \right) (\partial_a u^a) \right\} u^b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) - \partial_a (p) \eta^{ab} &= 0 \\
\left\{ (\partial_a \rho) u^a + \rho (\partial_a u^a) + \frac{p}{c^2} \partial_a u^a + \frac{1}{c^2} (\partial_a \rho) u^a \right\} u^b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) - \partial_a (p) \eta^{ab} &= 0 \\
\left\{ \partial_a (\rho u^a) + \frac{p}{c^2} \partial_a u^a + \frac{1}{c^2} (\partial_a \rho) u^a \right\} u^b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) - \partial_a (p) \eta^{ab}
\end{align*}
\] (8.32)

so that upon using the relativistic equation of continuity in Eq. (8.29), we find
\[
\frac{1}{c^2} (\partial_a \rho) u^a u^b + \left( \rho + \frac{p}{c^2} \right) u^a (\partial_a u^b) - \partial_a (p) \eta^{ab} = 0
\] (8.33)

which we may write as
\[
\left( \rho + \frac{p}{c^2} \right) (\partial_a u^b) u^a = \left( \eta^{ab} - \frac{u^a u^b}{c^2} \right) \partial_a p
\] (8.34)

Eq. (8.34) is the relativistic equation of motion for a perfect fluid.
8.4 Classical limit

In the classical (Newtonian) limit the fluid is a slowly moving (i.e. \( u/c \ll 1 \Rightarrow \gamma_u \simeq 1 \)). In this limit the four velocity

\[
    u^a = \gamma_u \left( c, \bar{u} \right) = \gamma_u \left( c, u^i \right)
\]

would be approximated to be

\[
    u^a \simeq \left( c, \bar{u} \right).
\]

Moreover, in the classical limit

\[
    \frac{p^i}{c^2} = \frac{F^i}{A_c^2} = \frac{m_0 \frac{d u^i}{dt}}{A_c d t} = \frac{m_0}{A_c} \left( \frac{u^i}{c} \right) \simeq 0
\]

\[
    \Rightarrow p \simeq 0
\]

Thus the relativistic continuity equation Eq. (8.3) would reduce to

\[
    \partial_a \left( \rho u^a \right) = 0 \Rightarrow \partial_b \left( \rho u^0 \right) + \partial_i \left( \rho u^i \right) = 0
\]

\[
    \Rightarrow \frac{\partial}{\partial t} \left( \rho c \right) + \left( \frac{\partial (\rho u^x)}{\partial x} + \frac{\partial (\rho u^y)}{\partial y} + \frac{\partial (\rho u^z)}{\partial z} \right) = 0
\]

which gives

\[
    \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0
\]

which is the classical equation of continuity for a fluid. In low pressure limit the relativistic equation of motion in Eq. (8.34) reduces to

\[
    \rho \left( \partial_a u^b \right) u^a = \left( \delta^{ab} - \frac{u^a u^b}{c^2} \right) \partial_a p.
\]

Noting that for \( a = 0 \) this equation gives

\[
    \rho \left( \partial_0 u^b \right) u^0 = \left( \delta^{0b} - \frac{u^0 u^b}{c^2} \right) \partial_0 p \Rightarrow \rho \left( \partial_i u^i \right) u^0 = \left( \delta^{0i} - \frac{u^i}{c} \right) \partial_0 p.
\]

where we used

\[
    \partial_0 u^0 = \frac{\partial c}{\partial t} = 0, \quad \eta^{00} - \frac{u^0}{c} = 1 - \frac{c}{c} = 0.
\]

Noting that in the classical limit, \( u^i/c \ll 1 \),

\[
    \eta^{0i} = -1 \Rightarrow \eta^{0i} - \frac{u^i}{c} = - \left( 1 + \frac{u^i}{c} \right) \simeq -1,
\]

Eq. (8.41) reduces to

\[
    c \rho \left( \partial_i u^i \right) = -\partial_0 p = \frac{\partial p}{\partial t}.
\]
In the classical limit we have a slowly moving fluid where
\[
\frac{dp}{cdt} = \frac{m_0}{c} \frac{d}{dt} \left( \frac{u}{c} \right) \approx 0
\] (8.43)
and also
\[
\partial_t u^i = \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} \approx 0
\] (8.44)
Therefore both sides of the equation
\[
c \rho (\partial_t u^i) = \frac{\partial p}{\partial t} \approx 0 \text{ for } i = 1, 2, 3.
\] (8.45)
are zero. Therefore, in the classical limit
\[
\rho (\partial_0 u^i) u^0 = 0
\]
Furthermore, noting that in the classical limit one can write
\[
\rho (\partial_0 u^b) u^i = \left( \eta^{ib} - \frac{u^i u^b}{c^2} \right) \partial_b \rho \approx \eta^{ib} \partial_b \rho = -\delta^{ib} \partial_b \rho,
\] (8.46)
where we used the classical limit
\[
\frac{u^i u^b}{c^2} \approx 0,
\] (8.47)
and also
\[
\eta^{ib} = -\delta^{ib} \text{ for } i, b = 1, 2, 3.
\] (8.48)
which we may rewrite as
\[
\rho (\partial_\alpha u^i) u^\alpha = -\delta^{ij} \partial_j \rho.
\] (8.49)
There follows that
\[
\rho (\partial_0 u^i) u^0 + \rho (\partial_1 u^i) u^1 + \rho (\partial_2 u^i) u^2 + \rho (\partial_3 u^i) u^3 = -\delta^{ij} \partial_j \rho
\]
which leads to
\[
\rho (\partial_0 u^1) u^0 + \rho (\partial_1 u^1) u^1 + \rho (\partial_2 u^1) u^2 + \rho (\partial_3 u^1) u^3 = -\partial_1 p,
\] (8.50)
\[
\rho (\partial_0 u^2) u^0 + \rho (\partial_1 u^2) u^1 + \rho (\partial_2 u^2) u^2 + \rho (\partial_3 u^2) u^3 = -\partial_2 p,
\] (8.51)
\[
\rho (\partial_0 u^3) u^0 + \rho (\partial_1 u^3) u^1 + \rho (\partial_2 u^3) u^2 + \rho (\partial_3 u^3) u^3 = -\partial_3 p.
\] (8.52)
Upon adding Eqs. (8.50) - (8.52), we have
\[
\rho \left[ \partial_0 u^1 \tilde{x} + \partial_0 u^2 \tilde{y} + \partial_0 u^3 \tilde{z} \right] u^0 + \rho \left[ \partial_1 u^1 \tilde{x} + \partial_1 u^2 \tilde{y} + \partial_1 u^3 \tilde{z} \right] u^1
\]
\[
+ \rho \left[ \partial_2 u^1 \tilde{x} + \partial_2 u^2 \tilde{y} + \partial_2 u^3 \tilde{z} \right] u^2 + \rho \left[ \partial_3 u^1 \tilde{x} + \partial_3 u^2 \tilde{y} + \partial_3 u^3 \tilde{z} \right] u^3
\]
\[
= -\partial_1 p \tilde{x} - \partial_2 p \tilde{y} - \partial_3 p \tilde{z},
\]
\[
\Rightarrow \rho u^0 (\partial_0 \tilde{u}) + \rho u^1 (\partial_1 \tilde{u}) + \rho u^2 (\partial_2 \tilde{u}) + \rho u^3 (\partial_3 \tilde{u}) = -\nabla p,
\]
\[
\Rightarrow \rho u^0 (\partial_0 \tilde{u}) + (\rho u^1 \partial_1 + \rho u^2 \partial_2 + \rho u^3 \partial_3) (\tilde{u}) = -\nabla p,
\]
\[
\Rightarrow \rho \frac{\partial \tilde{u}}{c \partial t} + \rho (\tilde{u} \cdot \nabla) \tilde{u} = -\nabla p,
\] (8.53)
so that
\[ \rho \left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{u} = -\nabla \rho, \] (8.54)

Eq. (8.54) is Euler's classical equation of motion for a perfect fluid. Note that the conservation of energy and momentum
\[ \partial_a T^{ab} = 0 \] (8.55)
for an arbitrary coordinates is expressed as
\[ \nabla_a T^{ab} = 0 \]

Homework: derive the expression for
\[ \nabla_a T^{ab} = 0. \]

8.5 The Einstein equations

In order to deduce the gravitational field equations proposed by Einstein, it is important to recall some of the results we obtained in the sections

(a) Newtonian gravitational field equation:
\[ \nabla^2 \Phi = (4\pi G) \rho. \] (8.56)

(b) Weak gravitational field-for weak field space-time is "slightly" curved, the metric can be approximated as
\[ g_{ab} \simeq \eta_{ab} + h_{ab} \] (8.57)
and static
\[ \partial_0 g_{ab} = \frac{\partial g_{ab}}{\partial 0} = 0 \] (8.58)
and field and the metric are related by
\[ g_{00} = \left( 1 + 2 \frac{\Phi}{c^2} \right) \] (8.59)

(c) Perfect fluid in the IRF:
\[ T_{00} = \rho c^2 \]

Consider the weak gravitational and low velocity limit. In this limit we can combine the results (a) to (c) and the field equation that relates the field to the source (mass) as
\[ \nabla^2 \left( \frac{\Phi c^2}{c^4} \right) = \left( 4\pi G \right) \rho c^2 = \left( 4\pi G \right) T_{00} \] (8.60)

\[ \Rightarrow \nabla^2 \left( \frac{2\Phi}{c^2} \right) = \left( \frac{8\pi G}{c^4} \right) T_{00} \Rightarrow \nabla^2 \left( 1 + \frac{2\Phi}{c^2} \right) = \left( \frac{8\pi G}{c^4} \right) T_{00} \] (8.61)
which can be rewritten as
\[ \nabla^2 g_{00} = \left( \frac{8\pi G}{c^4} \right) T_{00} \]  
(8.62)

Note that we used
\[ \nabla^2 1 = 0. \]  
(8.63)

The result for this limit suggest that the gravitational field must be some 2-nd rank tensor \( K_{ab} \) that is linearly proportional to the energy-momentum tensor \( T_{ab} \),
\[ K_{ab} = \kappa T_{ab} \]  
(8.64)

where
\[ \kappa = \frac{8\pi G}{c^4}. \]  
(8.65)

We also recall that curvature of the Minkowski space time that is determined by curvature tensor \( R_{abcd} \). How strongly or weakly the space-time is curved determined by how the gravitational field be strong or weak. Therefore, the field tensor, \( K_{ab} \), must be constructed from the curvature tensor. Furthermore, this field tensor must satisfy the following two conditions:

(a) **The Newtonian limit:** According to Eq. (8.62)
\[ K_{ab} \text{ (in the Newtonian limit)} = \nabla^2 g_{00} \]  
(8.66)

This suggests that \( K_{ab} \) should contain terms no higher than linear in the second-order derivative of the metric tensor.

(b) **Symmetry:** Since \( K_{ab} \) is proportional to the energy momentum tensor \( T_{ab} \) which is symmetric, the field tensor must also be symmetric.

We recall that the curvature tensor
\[ R_{abcd} = \frac{1}{2} \left( \partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ac} + \partial_c \partial_b g_{ad} - \partial_c \partial_a g_{bd} \right) - g^{ef} (\Gamma_{eac} \Gamma_{fbd} - \Gamma_{ead} \Gamma_{fbc}). \]  
(8.67)

is linear in the second derivative of the metric. We also know that the Ricci tensor, \( R_{ab} \), is obtained from the curvature tensor by raising the first index and followed by contraction with the fourth index
\[ R_{bced} = g^{ea} R_{abcd} \Rightarrow R_{bc} = R_{bced} = g^{da} R_{abcd}. \]  
(8.68)

We have also seen that a further contraction of the curvature tensor leads to the **curvature scalar**
\[ R = g^{ab} R_{ab} = R_a^a. \]  
(8.69)

Therefore the field tensor, \( K_{ab} \), that satisfy the above two stated conditions can be expressed in terms of the Ricci tensor, the curvature scalar which are constructed from the curvature tensor and the metric tensor as
\[ K_{ab} = \alpha R_{ab} + \beta R g_{ab} + \lambda g_{ab}, \]  
(8.70)
where $\alpha, \beta,$ and $\lambda$ are constants to be determined. The requirement stated in (a) leads to $\lambda = 0$ but this will be relaxed later only when we introduce the cosmology constant. For now we use

$$K_{ab} = \alpha R_{ab} + \beta R g_{ab} = \kappa T_{ab} \quad (8.71)$$

and find the constants $\alpha$ and $\beta$. To this end, we recall the energy-momentum conservation

$$\nabla_a T_{ab} = 0$$

so that

$$\nabla_a (\alpha R_{ab} + \beta R g_{ab}) = 0. \quad (8.72)$$

In the previous chapter, we have shown that for the Einstein tensor, $G^{ab}$,

$$\nabla_a G^{ab} = \nabla_a \left( R^{ab} - \frac{1}{2} R g^{ab} \right) = 0. \quad (8.73)$$

Multiplying this equation by $\alpha$, we have

$$\nabla_a \left( \alpha R^{ab} - \frac{\alpha}{2} R g^{ab} \right) = 0 \quad (8.74)$$

so that upon subtracting Eq. (8.74) from Eq. (8.72), we find

$$\left( \frac{1}{2} \alpha + \beta \right) \nabla_a (R g^{ab}) = \left( \frac{1}{2} \alpha + \beta \right) (R \nabla_a g^{ab} + g^{ab} \nabla_a R) = 0. \quad (8.75)$$

We have shown that (in one of the homework problem) that the covariant derivative of the metric is zero

$$\nabla_a g^{ab} = 0.$$ 

Employing this result in Eq. (8.75), we find

$$\left( \frac{1}{2} \alpha + \beta \right) g^{ab} \nabla_a R = 0. \quad (8.76)$$

In Eq. (8.76) we note that

(a) $g^{ab}$ can not be zero, generally

(b) $\nabla_a R$ is zero only when the spacetime is is not curved (no gravitational field)

Therefore, the only way that Eq. (8.76) can be true is

$$\frac{1}{2} \alpha + \beta = 0 \Rightarrow \beta = -\frac{\alpha}{2}$$

and the field equation, Eq. (8.77), becomes

$$\beta \left( R_{ab} - \frac{1}{2} R g_{ab} \right) = \kappa T_{ab} \quad (8.77)$$
Homework: show that consistency with Newtonian gravity requires that $\beta = -1$.

Einstein’s gravitational field equations, which forms the mathematical basis of the theory of general relativity is given by

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = -\kappa T_{ab}, \quad (8.78)$$

where

$$\kappa = \frac{8\pi G}{c^4}.$$

$G_{ab}$ is the Einstein tensor we saw in the previous chapter. By multiplying Eq. (8.78) by $g^{ca}$, we have

$$g^{ca} R_{ab} - \frac{1}{2} g^{ca} g_{ab} R = -\kappa g^{ca} T_{ab} \Rightarrow R^c_b - \frac{1}{2} \delta^c_b R = -\kappa T^c_b \quad (8.79)$$

and contracting Eq. (8.79) by setting $c = b$, we have

$$R^b_b - \frac{1}{2} \delta^b_b R = -\kappa T^b_b \quad (8.80)$$

so that noting that

$$R^b_b = R, T^b_b = T, \delta^b_b = 4 \quad (8.81)$$

we find

$$R - 2R = -\kappa T \Rightarrow R = \kappa T \quad (8.82)$$

Applying Eq. (8.82) into Eq. (8.78), we find

$$R_{ab} - \frac{1}{2} g_{ab} \kappa T = -\kappa T_{ab}, \quad (8.83)$$

which can be put in the form

$$R_{ab} = -\kappa \left( T_{ab} - \frac{T}{2} g_{ab} \right) \quad (8.84)$$

Eq. (8.84) is an alternative form of the Einstein gravitational field equation. In general $T_{ab}$ contains all forms of energy and momentum that belongs to not only matter but also to electromagnetic radiation.

### 8.5.1 The Einstein field equations in vacuum

A region of space in which all components of the energy-momentum tensor are zero ($T_{ab} = 0$) is called empty (vacuum). This region of space is devoid of matter and charge. Therefore, for an empty region of space, we have

$$T_{ab} = 0 \Rightarrow T = T^{a}_{a} = g^{ap} T_{pa} = 0$$

and the Einstein field equation becomes

$$R_{ab} = 0. \quad (8.85)$$
8.5. THE EINSTEIN EQUATIONS

The following tables show the field and the curvature equations resulting from $N = 2, 3$ and 4 spacetime dimension.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$R_{ab}$</th>
<th>$R_{abcd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$R_{00}, R_{01}, R_{11}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$R_{00}, R_{01}, R_{02}, R_{10}, R_{12}, R_{20}, R_{21}, R_{22}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$R_{00}, R_{01}, R_{02}, R_{03}, R_{10}, R_{11}, R_{12}, R_{13}, R_{20}, R_{21}, R_{22}, R_{23}, R_{30}, R_{31}, R_{32}, R_{33}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N$</th>
<th># of $R_{ab}$</th>
<th>Independent $R_{\mu\nu}$</th>
<th># of Indp. $R_{ab}$</th>
<th># of $R_{abcd}$</th>
<th># of Indp. $R_{abcd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>$R_{00}, R_{01}, R_{11}$</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>$R_{00}, R_{01}, R_{02}, R_{11}, R_{12}, R_{22}$</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>$R_{00}, R_{01}, R_{02}, R_{03}, R_{11}, R_{12}, R_{13}, R_{20}, R_{21}, R_{22}, R_{23}, R_{30}, R_{31}, R_{32}, R_{33}$</td>
<td>10</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

**Homework:** find $R_{abcd}$ and the total # of $R_{abcd}$

We recall that the Ricci tensor, $R_{ab}$, is obtained from the curvature tensor by raising the first index and then contracting the first and the last indices

$$R_{bcd} = g^{da} R_{abcd}. \quad (8.86)$$

As we can see from these tables for $N = 2$ or 3, in an empty part of space the number of independent field equations (which are zero) is greater or equal to the corresponding independent number of the curvature equations (i.e $3 > 1$ for 2-D and $6 = 6$ for 3-D) guarantee that the full curvature tensor must vanish for 2-D and 3-D spacetime. This means spacetime in 2-D and 3-D for an empty part of space is flat. On the other hand for 4-D spacetime since the number of independent field equations (10) is greater than the number of independent curvature equations (20), spacetime is still curved for an empty part of space even if it is devoid of matter and charge. Therefore, we can make the conclusion that gravitational field exists only in dimension equal or greater than 4-D.

**Question:** so can we create a 2-D or 3D spacetime (i.e 1-D and 2-D space) in a laboratory where the gravitational field is zero and objects become weightless?

### 8.5.2 The Einstein field equations in the weak-field limit

We recall that for a weak gravitational field spacetime is "slightly" curved, the metric is given by

$$g_{ab} \simeq \eta_{ab} + h_{ab}. \quad (8.87)$$

where $|h_{ab}| << 1$ and it is static,

$$\partial_0 g_{ab} = \frac{\partial g_{ab}}{\partial t} = 0. \quad (8.88)$$

In this limit we have shown that field and the metric are related by

$$g_{00} = \left(1 + \frac{2\Phi}{c^2}\right). \quad (8.89)$$
CHAPTER 8. THE GRAVITATIONAL FIELD EQUATIONS

Taking these into consideration, we can write the Einstein field equation as

\[ R_{00} = -\kappa \left( T_{00} - \frac{T}{2} g_{00} \right). \]  

(8.90)

Recall that the curvature tensor is given by

\[ R^d_{\ abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac} \Gamma^d_{eb} - \Gamma^e_{ac} \Gamma^d_{eb}, \]  

(8.91)

and Ricci tensor, \( R_{ab} \), is obtained from the curvature tensor by raising the first index (Eq. (8.91)) and then contracting the first and the last indices, we have

\[ R_{ab} = R^c_{abc} = \partial_b \Gamma^c_{ac} - \partial_c \Gamma^c_{ab} + \Gamma^e_{ac} \Gamma^c_{eb} - \Gamma^e_{ac} \Gamma^c_{eb}, \]  

(8.92)

so that

\[ R_{00} = \partial_0 \Gamma^c_{0c} - \partial_c \Gamma^c_{00} + \Gamma^e_{0c} \Gamma^c_{e0} - \Gamma^e_{0c} \Gamma^c_{e0}. \]  

(8.93)

Note that the connections are related to the metric by

\[ \Gamma^a_{bc} = \frac{g^{ac}}{2} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}), \]  

(8.94)

For the first term in Eq. (8.93) we find

\[ \partial_0 \Gamma^c_{0c} = \frac{g^{cd}}{2} (\partial_0 g_{cd} + \partial_d g_{0c}) = \frac{g^{cd}}{2} (\partial_c g_{0d} - \partial_d g_{0c}) \approx 0, \]

where we have used the metric is static and the space is slightly curved in the weak-field limit. For the same reason, for the last two terms of Eq. (8.93), we can make the approximation

\[ \Gamma^c_{0c} \Gamma^c_{e0} = \Gamma^c_{00} \Gamma^c_{e0} \approx 0. \]  

(8.95)

Thus Eq. (8.93) becomes

\[ R_{00} = -\partial_c \Gamma^c_{00} \]  

(8.96)

Since

\[ \partial_0 \Gamma^a_{00} = 0 \]  

(8.97)

for a static metric, we have

\[ R_{00} = -\partial_j \Gamma^j_{00} \]  

(8.98)

where \( j = 1, 2, 3 \). Using the result we obtained in the previous chapter, in the Newtonian limit,

\[ \Gamma^j_{00} = \frac{1}{2} \delta^{ij} \partial_i h_{00}, \]  

(8.99)

we may write

\[ R_{00} = -\partial_j \left[ \frac{1}{2} \delta^{ij} \partial_i h_{00} \right] = -\frac{1}{2} \delta^{ij} \partial_j \partial_i h_{00} \]  

(8.100)
Applying Eq. (8.100), we find for the field equation Eq. (8.90) becomes
\[ \frac{1}{2} \delta^{ij} \partial_j \partial_i h_{00} = \kappa \left( T_{00} - \frac{T}{2} \right) g_{00} \] \hspace{1cm} (8.101)

In the weak field limit, we can make the approximation
\[ g_{00} \simeq 1 \]
so that
\[ \frac{1}{2} \delta^{ij} \partial_j \partial_i h_{00} = \kappa \left( T_{00} - \frac{T}{2} \right) \] \hspace{1cm} (8.102)

We will consider the gravitational field is due to some perfect fluid for which energy-momentum tensor is given by
\[ T_{ab} = \left( \rho + \frac{p}{c^2} \right) u_a u_b - p \delta_{ab} = \left[ \left( \rho + \frac{p}{c^2} \right) \frac{u_a u_b}{c^2} - \frac{p}{c^2} \eta_{ab} \right] c^2 \] \hspace{1cm} (8.103)

For most classical matter distribution
\[ \frac{p}{c^2} << \rho \]
so that
\[ T_{ab} = \rho u_a u_b. \] \hspace{1cm} (8.104)

There follows that
\[ T_{00} = \rho u_0 u_0 = \rho c^2 \] \hspace{1cm} (8.105)
and also
\[ T = T^b_b = g^{ba} T_{ab} = \rho u_b u_b = \rho c^2 \] \hspace{1cm} (8.106)

Then the field equation becomes
\[ \delta^{ij} \partial_j \partial_i h_{00} \simeq \kappa \rho c^2 \Rightarrow (\partial_0^2 + \partial_1^2 + \partial_2^2) h_{00} \simeq \kappa \rho c^2 \] \hspace{1cm} (8.107)
\[ \Rightarrow \nabla^2 h_{00} \simeq \kappa \rho c^2 \] \hspace{1cm} (8.108)

Recalling that for slightly curved spacetime
\[ g_{00} = \left( 1 + \frac{2\Phi}{c^2} \right) = \eta_{00} + h_{00} = 1 + h_{00} \] \hspace{1cm} (8.109)
\[ \Rightarrow h_{00} = \frac{2\Phi}{c^2} \] \hspace{1cm} (8.110)
and
\[ \kappa = \frac{8\pi G}{c^4}. \]

we finally find that
\[ \nabla^2 \left( \frac{2\Phi}{c^2} \right) \simeq \frac{8\pi G}{c^4} \rho c^2 \Rightarrow \nabla^2 \Phi \simeq 4\pi G \rho \] \hspace{1cm} (8.111)
CHAPTER 8. THE GRAVITATIONAL FIELD EQUATIONS

If the field is a result of some perfect fluid in a volume of sphere somewhere in the universe, we may write

\[ \int_V [\nabla \cdot (\nabla \Phi)] \, d^3x \simeq 4\pi G \int_V \rho \, d^3x = 4\pi GM \]  
(8.112)

Using the divergence theorem

\[ \int_V [\nabla \cdot (\nabla \Phi)] \, d^3x = \oint_S (\nabla \Phi) \cdot d\vec{a} = \oint_S \vec{g} \cdot d\vec{a} \]  
(8.113)

where \( g \) is the gravitational field vector like the electric field vector. For a very spherical surface of radius \( r \) enclosing the perfect fluid, we may write

\[ \oint_S \vec{g} \cdot d\vec{a} = g 4\pi r^2 \]  
(8.114)

so that

\[ g 4\pi r^2 = 4\pi GM \Rightarrow g = \frac{GM}{r^2} \]

Then the gravitational field vector can be expressed as

\[ \vec{g} = \frac{GM}{r^2} \hat{r} \]

Newton’s law of gravitational force on an object of mass, \( m \) at a distance \( r \) from the center of the perfect fluid can then be written as

\[ \vec{F}_G = m\vec{g} = \frac{GMm}{r^2} \hat{r} \]

Suppose the perfect fluid is our planet earth, \( M_E \) and the object is you or me on earth (\( r = R_E \)), you will find

\[ \vec{g} = \frac{GM}{r^2} \hat{r} = 9.8 \frac{m}{s^2} \]

and the force

\[ \vec{F}_G = \text{your (or my) mass} \times 9.8 \frac{m}{s^2} \hat{r}. \]