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# Piecewise linear prewavelets over type-2 triangulations

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In this article, we study the construction of piecewise linear prewavelets over type-2 triangulations. Different from a so-called semi-prewavelet approach, we investigate the orthogonal conditions directly and obtain parameterized prewavelets with a smaller support. The conditions for parameterized prewavelet basis on the parameters are also given.

*Keywords:* Linear splines; Prewavelets; Smaller support; Triangulation

*AMS 2000 Classifications:* 41A1541A63; 65D07; 68U05

## 1. Introduction

While the construction of univariate wavelets is well understood (see [1] and [2] for example), however, most of real world applications are multivariate or multiparameter in nature. The construction of multivariate wavelets are much more challenging. In fact, even the case of continuous piecewise linear wavelets construction is unexpectedly complicated, see [7] and references therein. Here, for higher degree splines, we only mention that a hierarchical basis for $C^1$ cubic bivariate splines over quadrangulations is used for surface compression in [9] very recently.

Because of the simplicity in computing with the linear splines, the piecewise linear element becomes one of the most important and useful elements in solving boundary value problems. In the literature on the finite element solutions of differential and integral equations, bases of piecewise linear prewavelets with small support have been constructed in [3–6,11–15]. In [8], a characterization of minimum support piecewise linear prewavelets with 10 non-zero coefficients in the mask is given on

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a bounded domain with a type-1 triangulation. A construction of wavelets over arbitrary triangulations is presented in [15] and the prewavelets have 23 non-zero coefficients in the mask. The construction is also applicable in higher dimensions. In [3], a construction of piecewise linear prewavelets over a general triangulation of a bounded domain was studied under an unusual requirement that the degree of vertices of the triangulation must be at most 21. Later, the same authors presented a so-called semi-prewavelet skill in [4] and constructed piecewise linear prewavelets particularly on a bounded type-1 triangulation with 13 non-zero coefficients in the mask. The restriction on the degree of vertices over an arbitrary triangulation is removed by applying the semi-wavelet approach and using a property of the positive definite matrices in [5]. Piecewise linear prewavelets over a type-2 triangulation are constructed in [6] with 13 non-zero coefficients in the mask associated with the interior vertices over a bounded domain by using semi-prewavelet approach. In [10], all the possible semi-prewavelets over uniform refinements of a regular triangulation are constructed, and a corresponding theorem is given to ensure the linear independence of a set of different pre-wavelets obtained by summing pairs of these semi-prewavelets.

In this article, we construct piecewise linear prewavelets over a bounded domain with a type-2 triangulation by investigating the orthogonal conditions directly and obtain parameterized prewavelets and it turns out that the prewavelets constructed in [6] become a special set of wavelets obtained by assigning a specific value for the parameter of the parameterized prewavelets. In particular, we obtain a smaller support for the second kind of interior prewavelets with only 11 non-zero coefficients in the mask. We also provide conditions on the parameters to ensure that these prewavelets become a basis. Since there are so many different kinds of cases for basis consideration for a type-2 triangulation, it becomes very complicated to obtain a similar characteristic result of [8] as on a type-1 triangulation. The article is organized as follows. Preliminaries are introduced in section 2. In section 3, we construct the smaller support prewavelets associated with an interior vertex and a boundary vertex, respectively. In section 4, parameterized prewavelets are constructed and conditions on the parameters for a basis are provided.

## 2. Preliminaries

Let $\Omega = [0, m] \times [0, n]$ be a rectangle. For $x = i,\ i = 0, 1, \ldots, m$ and $y = j,\ j = 0, \ldots, n$, $\Omega$ is divided into $mn$ small rectangles $\Omega_{ij} = [i, i+1] \times [j, j+1],\ i = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, n-1$, by mesh lines $x = i$ and $y = j$. The triangulation generated by drawing all northeast and northwest diagonals in all small rectangles is called a type-2 triangulation and is denoted by $\Delta^0 = \Delta^{(2)}_{mn}$, see figure 1. We will assume, for the sake of simplicity, that $m \geq 2$ and $n \geq 2$, though wavelet constructions can be made in a similar way when either $m = 1$ or $n = 1$ (or both). We let $V^0$ and $E^0$ denote the vertices and edges, respectively, in $\Delta^0$, so that

$$V^0 = \{(i,j); 0 \leq i \leq m, 0 \leq j \leq n\} \cup \left\{\left(i + \frac{1}{2}, j + \frac{1}{2}\right); 0 \leq i \leq m-1, 0 \leq j \leq n-1\right\}.$$

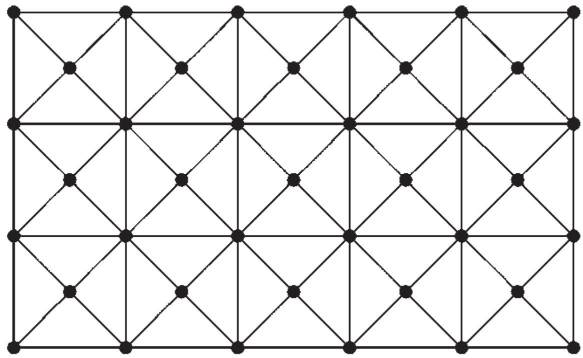

Figure 1. A type-2 triangulation.

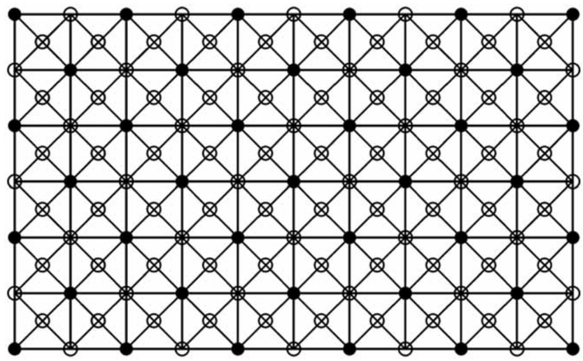

Figure 2. A refinement of $\Delta^0$.

Let $S^0 = S_1^0(\Delta^0)$ be the linear space of continuous linear spline functions over $\Delta^0$. Then a basis for $S^0$ is given by the piecewise linear nodal functions $\phi_v^0$ in $S^0$, for $v \in V^0$, satisfying $\phi_v^0(w) = \delta_{vw}$, where $\delta_{vw}$ is the Kronecker delta function. The support of $\phi_{i+(1/2),j+(1/2)}$ is the square $S_{ij}$, while the support of $\phi_{ij}^0$ is the diamond enclosed by the polygon with vertices $(i-1,j)$, $(i,j-1)$, $(i+1,j)$, $(i,j+1)$, with a suitable truncation if the point $(i,j)$ lies on the boundary of the domain $D = [0,m] \times [0,n]$.

Next we consider the refined triangulation $\Delta^1$ of $\Delta^0$, again a type-2 triangulation, formed by adding the mesh lines in the four directions halfway between each pair of existing parallel lines, shown as in figure 2. We define $V^1$, $E^1$, the linear space $S^1$, and the basis $\phi_u^1$, for $u \in V^1$ accordingly. Then $S^0$ is a subspace of $S^1$ with a refinement equation relating the coarse nodal function $\phi_v^0$ to the fine ones $\phi_v^1$. In order to formulate this equation we define

$$V_v^0 = \{w \in V^0;\ w \text{ and } v \text{ are neighbors in } V^0\},$$

and

$$V_v^1 = \left\{u = \frac{(w+v)}{2} \in V^1;\ w \in V_v^0\right\}.$$

Then $V_v^0$ is the set of neighbors of $v$ and $V_v^1$ is the set of midpoints between $v$ and its coarse neighbors. For example when $v$ is an interior vertex, there are two cases:

$$V^1_{i+(1/2),j+(1/2)} = \left\{(i+x, j+y);\ x, y = \frac{1}{4}, \frac{3}{4}\right\},$$

and

$$V^1_{i,j} = \{(i+x, j+y);\quad x=0,\ y=\pm 1/2,\ x=\pm 1/2,\ y=0,\quad \text{and } x=\pm 1/4,\ y=\pm 1/4\}.$$

It is easily to see the following refinement equation

$$\phi_v^0 = \phi_v^1 + \frac{1}{2}\sum_{u\in V_v^1}\phi_u^1(x).$$

If $W^0$ denotes the orthogonal complement space of $S^0$ in $S^1$, then $S^1 = S^0 \oplus W^0$, here the spaces $S^0$ and $S^1$ are equipped with the inner product

$$\langle f, g\rangle = \int_\Omega f(x)g(x)\mathrm{d}x, \quad f, g \in L^2(\Omega).$$

Similarly, if we define the wavelet space $W^j$ to be the orthogonal complement at every refinement level $j$, that is

$$S^{j+1} = S^j \oplus W^j,$$

then we obtain the decomposition

$$S^k = S^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{k-1},$$

for any $k \geq 1$.

The space $W^j$ can be used to represent the parts of functions in $S^{j+1}$ that cannot be represented in the space $S^j$. We can call $W^j$ the correcting space. Using $j$-step corrections, we have

$$\begin{aligned} S^j &= S^{j-1} \oplus W^{j-1} = S^{j-2} \oplus W^{j-2} \oplus W^{j-1} = \cdots \\ &= S^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{j-1}. \end{aligned}$$

Suppose $\Psi = \{\psi_{j,\ell}\}_{\ell\in L}$ forms a basis of $W^j$. If $\Psi$ is an orthonormal basis of $W^j$, then the elements $\psi_{j,\ell}$ of $\Psi$ are called wavelets, otherwise, they are called prewavelets.

We would like to obtain a basis of functions with small support for the purpose of conveniently representing the decomposition of a given function $f^{j+1} \in S^{j+1}$ into its two unique components $f^j \in S^j$ and $g^j \in W^j$ : $f^{j+1} = f^j \oplus g^j$. Note that the basis elements of any $W^k$ can simply be obtained from the basis of $W^0$ using dilations, we can restrict our study only to $W^0$.

To construct a small support basis, we follow the idea presented in [3]. For a 'new' vertex $u \in V^1 \setminus V^0$, we try to construct a non-trivial (prewavelet) $\psi_u \in W^0$ associated with the vertex $u$, whose support is around $u$, i.e., it has the form

$$\psi_u(x) = \sum_{w \in V(u)} q_{w,u} \phi^1_w(x),$$

where $V(u) \subset V^1$ is a small set of vertices of $\Delta^1$ which are near to $u$.

In [6], the authors introduced a so-called semi-(prewavelets) approach to seek prewavelets for the space $W^0$ in terms of sum pairs of elements $\sigma_{v_1,u}$ and $\sigma_{v_2,u}$ (called semi-wavelets) of $S^1$ which have small support and are close to being in the wavelet space $W^0$, in the sense that they are orthogonal to all but two of the nodal functions in the coarse space, where $u$ is the midpoint of the edge $[v_1, v_2]$. Depending on the locations of $v_1$, either $v_1 = (i + 1/2, j + 1/2)$ or $v_1 = (i, j)$, there are three interior semi-wavelets that generate two interior prewavelets $\psi_u$ in the sense that $v_1$ and $v_2$ are both interior vertices of $\Delta^0$ up to rotation and symmetries. Similarly, there are three edge prewavelets $\psi_u$ for which one of $v_1$ and $v_2$ is an interior vertex while the other one lies on the boundary but not the corner. The remaining two prewavelets are corner prewavelets. Requiring the prewavelets being the sum pairs of semi-wavelets, it has been shown in [6] that those seven kinds of prewavelets are uniquely determined. For the second kind of interior prewavelets, there are 13 non-zero coefficients in the masks. We take the same structures of $V(u)$ described in [6] and try to construct prewavelets with fewer coefficients in the masks by investigating the orthogonal conditions directly. It turns out that we obtain parameterized prewavelets. For the second kind of interior prewavelets, we have only 11 non-zero coefficients in the masks.

## 3. Construction of smaller support prewavelets

In the following, we study the second kind of interior prewavelet with a smaller support based on the same structure of the second interior prewavelet constructed in [6]. The support vertices are labeled $(1, 2, 3, \ldots, 12)$ and some vertices in $V^0$ are also labeled $(P_1, P_2, \ldots, P_8)$ as shown in figure 3.

Let us assume that the prewavelet at $u$ in $W^0$ has the following expression:

$$\psi^0_u = b_1\phi^1_1 + a\phi^1_u + b_4\phi^1_4 + b_5\phi^1_5 + b_6\phi^1_6 + b_7\phi^1_7 + b_8\phi^1_8 + b_9\phi^1_9 + b_{10}\phi^1_{10} + b_{11}\phi^1_{11} + b_{12}\phi^1_{12},$$

where $\phi^1_i$, $i = u, 1, \ldots, 12$ are nodal basis functions in $S^1$. By the orthogonal conditions, $\langle \psi^0_u, \phi^0_{P_i} \rangle = 0$, $i = 1, \ldots, 8$ and $\langle \psi^0_u, \phi^0_1 \rangle = 0$, $\langle \psi^0_u, \phi^0_{10} \rangle = 0$, we will obtain the following linear equations:

$$M_1 x_1 = 0, \tag{3.1}$$

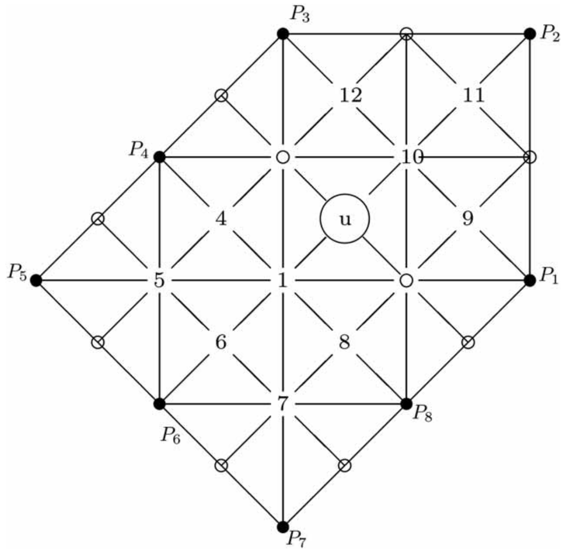


Figure 3. The second smaller support interior prewavelet.

where

$$M_1 = \begin{bmatrix} 24 & 8 & 8 & 12 & 8 & 12 & 8 & 1 & 3 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 8 & 3 & 1 & 0 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 20 & 6 & 6 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 8 \\ 1 & 0 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 4 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 8 & 1 \end{bmatrix},$$

and $x_1 = [b_1, a, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}]^T$.

We solve the matrix equation (3.1) and obtain the following solutions

$$b_1 = -15t_1,\ a = \frac{253}{6}t_1,\ b_4 = \frac{11}{6}t_1,\ b_5 = t_1,\ b_6 = \frac{7}{6}t_1,\ b_7 = t_1$$
$$b_8 = \frac{11}{6}t_1,\ b_9 = t_1,\ b_{10} = -14t_1,\ b_{11} = 5t_1,\ b_{12} = t_1$$

where $t_1$ is a non-zero arbitrary real number.

Notice that the second smaller support interior prewavelet only needs 11 points of support, but second interior prewavelet in [6] needs 13 points of support.

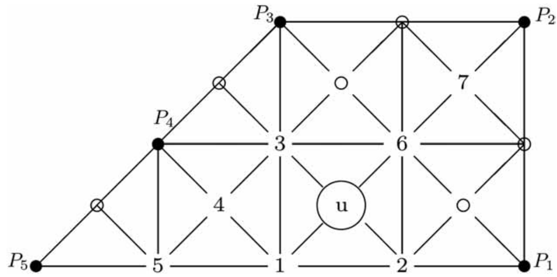


Figure 4. The second smaller support boundary prewavelet.

Next, we work on the second boundary prewavelet. Support vertices are labeled in the following figure 4 and $P_i$ $i = 1, \ldots, 6$ are labeled for the coarse vertices. This wavelet is called the second smaller support boundary prewavelet.

Let $\psi_u^0$ be the prewavelet function in $W^0$ at $u$ which has the following expression:

$$\psi_u^0 = a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1,$$

where $\phi_i^1,\ i = u, 1, \ldots, 7$ are nodal basis functions in $S^1$. By the orthogonal conditions, the following inner products must be zeros,

$$\langle \psi_u^0, \phi_1^0 \rangle = 0,\ \left\langle \psi_u^0, \phi_{p_i}^0 \right\rangle = 0, \quad i = 1, \ldots, 5,\ \langle \psi_u^0, \phi_6^0 \rangle = 0.$$

By the orthogonal conditions and directly computation, we obtain the following linear equation:

$$M_2 x_2 = 0, \tag{3.2}$$

where

$$M_2 = \begin{bmatrix} 8 & 12 & 6 & 12 & 8 & 6 & 3 & 0 \\ 1 & \frac{1}{2} & 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 8 \\ 1 & 1 & 0 & 12 & 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 6 & 0 & 0 \\ 6 & 1 & 4 & 4 & 0 & 0 & 20 & 6 \end{bmatrix},$$

and $x_2 = [a, b_1, b_2, b_3, b_4, b_5, b_6, b_7]^T$. Solving the equation (3.2), we obtain the following solutions:

$$x_2 = \frac{1}{2}\left[\frac{204}{5}t_2, -\frac{144}{5}t_2, \frac{6}{5}t_2, \frac{8}{5}t_2, \frac{12}{5}t_2, 2t_2, -\frac{64}{5}t_2, \frac{24}{5}t_2\right]^T,$$

where $t_2$ is a non-zero arbitrary real number.

Note that the second smaller support boundary prewavelet only needs 8 points of support but second boundary prewavelet in [6] needs 9 points of support. The prewavelets we obtained here are not unique. They are determined by the choices of values of the parameter $t_2$.

Due to symmetry of type-2 triangulations, the prewavelets with the same structures as these two prewavelets can be obtained by rotations. In the next section, we will construct edge prewavelets and also provide conditions for these prewavelets becoming a basis.

## 4. Parameterized prewavelet basis

In the following, we will construct the parameterized wavelet basis over type-2 triangulations. The two smaller support wavelets are discussed in section 3. (See the figures 3 and 4.) Since there are parameters in these two wavelets, we call these prewavelets parameterized prewavelet-1 and parameterized prewavelet-2, respectively as shown in figures 5 and 6.

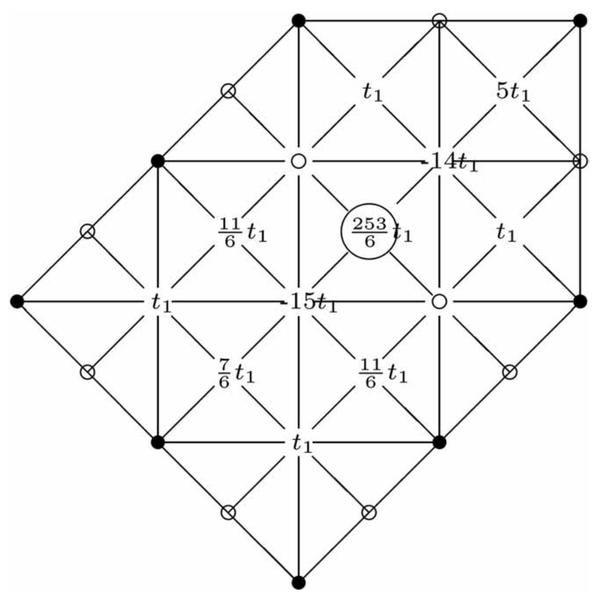


Figure 5. Parameterized prewavelet-1.

As we mentioned in section 2, there are seven kinds of prewavelets. We will construct the other five parameterized prewavelets directly and provide a theorem to ensure that these prewavelets form a basis.

Label the vertices as shown in figure 7. Let $\psi_u$ be the prewavelet in $W^0$ at vertex $u$ with the following expression:

$$\begin{aligned}\psi_u = a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 + b_9\phi_9^1 \\ + b_{10}\phi_{10}^1 + b_{11}\phi_{11}^1 + b_{12}\phi_{12}^1 + b_{13}\phi_{13}^1 + b_{14}\phi_{14}^1 + b_{15}\phi_{15}^1 + b_{16}\phi_{16}^1.\end{aligned}$$

Here $a$ and $b_i$ $i = 1, \ldots, 16$ will be determined by using the orthogonality conditions. By using the orthogonal conditions $\langle \psi_u, \phi_{P_i}^0 \rangle = 0$, $i = 1, \ldots, 12$ and $\langle \psi_u, \phi_1^0 \rangle = 0$, $\langle \psi_u, \phi_{13}^0 \rangle = 0$, we obtain the following equation:

$$M_3 x_3 = 0, \tag{4.1}$$

where

$$M_3 = \begin{bmatrix} 4 & 1 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 6 \\ 0 & 1 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 6 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 12 & 1 \\ 12 & 24 & 8 & 12 & 8 & 12 & 8 & 12 & 8 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 12 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 12 & 8 & 12 & 24 & 8 & 12 & 8 \end{bmatrix}$$

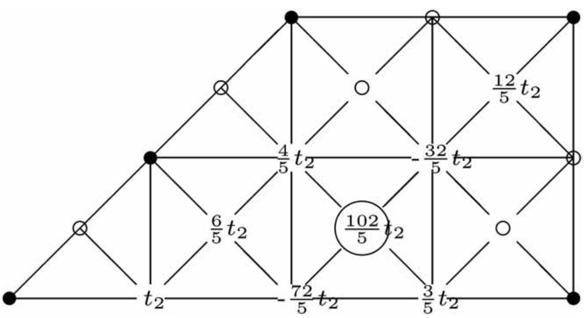


Figure 6. Parameterized prewavelet-2.

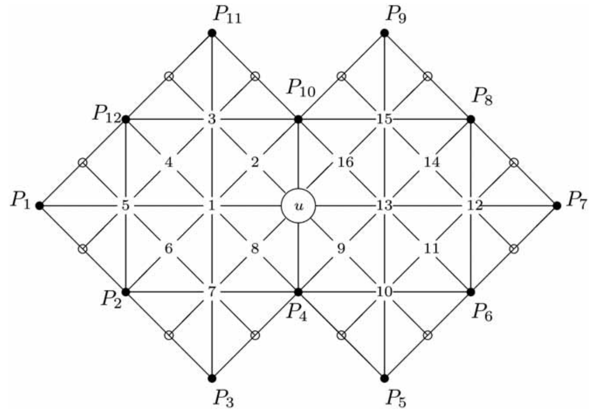


Figure 7. Parameterized prewavelet-3.

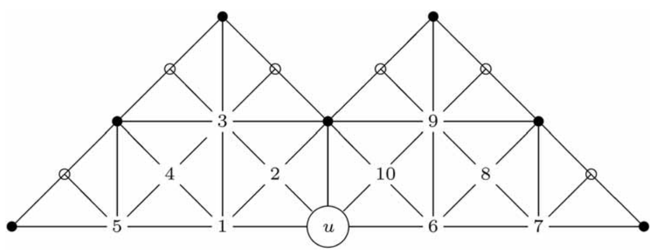


Figure 8. Parameterized prewavelet-4.

and $x_3 = [a, b_1, \ldots, b_{16}]^T$.

We solve the equation (4.1) by letting $b_{14} = 0$, $b_{11} = 0$, $b_6 = 0$, and $b_4 = 0$ and obtain the following solutions

$$x_3 = [38t_3, -12t_3, -12t_3, 2t_3, 0, t_3, 0, 2t_3, -12t_3, -12t_3, 2t_3, 0, t_3, -12t_3, 0, 2t_3, -12t_3]^T,$$

where $t_3$ is an arbitrary non-zero real number. This prewavelet is called the parameterized prewavelet-3.

In a similar way, we label the support vertices and compute the coefficients over the figure 8 for the parameterized prewavelet-4.

First, we assume that the prewavelet at vertex $u$ has the following expression:

$$\psi_u = a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 + b_9\phi_9^1 + b_{10}\phi_{10}^1.$$

Applying the orthogonality conditions, we can obtain the system of homogenous linear equations:

$$M_4 x_4 = 0,$$

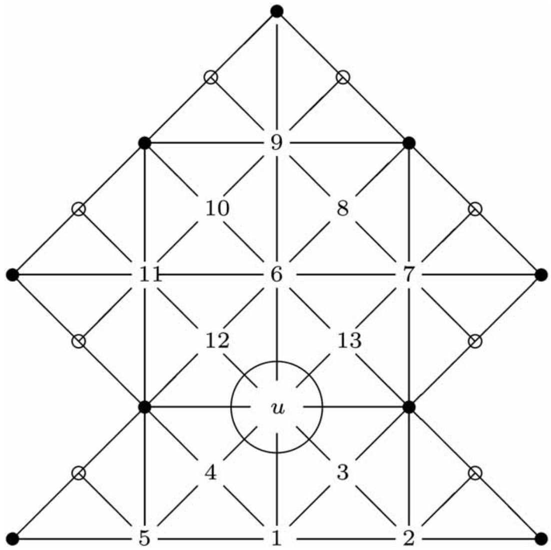


Figure 9. Parameterized prewavelet-5.

where

$$M_4 = \begin{bmatrix} 6 & 12 & 8 & 12 & 8 & 6 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 6 & \frac{1}{2} & 1 & 0 & 0 & 0 & 12 & 6 & 8 & 12 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 12 & 1 \\ 4 & 1 & 6 & 4 & 0 & 0 & 1 & 0 & 0 & 4 & 6 \\ 0 & 1 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $x_4 = [a, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}]^T$.

Solving the linear equation $M_4 x_4 = 0$ by choosing $b_4 = 0$ and $b_8 = 0$, we obtain the following.

$$x_4 = [38t_4, -12t_4, -12t_4, 2t_4, 0, t_4, -12t_4, t_4, 0, 2t_4, -12t_4]^T,$$

where $t_4$ is an arbitrary non-zero real number.

Now, we consider the boundary prewavelet at $u$ as shown in the figure 9 and it has the expression:

$$\psi_u = a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 + b_9\phi_9^1$$
$$+ b_{10}\phi_{10}^1 + b_{11}\phi_{11}^1 + b_{12}\phi_{12}^1 + b_{13}\phi_{13}^1.$$

By the orthogonality conditions, we obtain the following linear equation:

$$M_5 x_5 = 0$$

with the coefficient matrix

$$M_5 = \begin{bmatrix} 12 & 1 & 0 & 1 & 1 & 0 & 24 & 12 & 8 & 12 & 8 & 12 & 8 & 8 \\ 12 & 12 & 6 & 8 & 8 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \frac{1}{2} & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 4 & 6 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 12 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 12 & 1 & 0 \\ 4 & 1 & 0 & 0 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 4 & 6 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the vector

$$x_5 = [a, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}]^T.$$

we obtain the following for $b_{10} = 0$ and $b_8 = 0$:

$$x_5 = [39t_5, -24t_5, 4t_5, -12t_5, -12t_5, 4t_5, \; -12t_5, 2t_5, 0, t_5, 0, 2t_5, -12t_5, -12t_5]^T$$

with $t_5 \neq 0$, an arbitrary non-zero real number. We call this prewavelet the parameterized prewavelet-5.

To construct parameterized prewavelet-6, we label the vertices as shown in the figure 10.

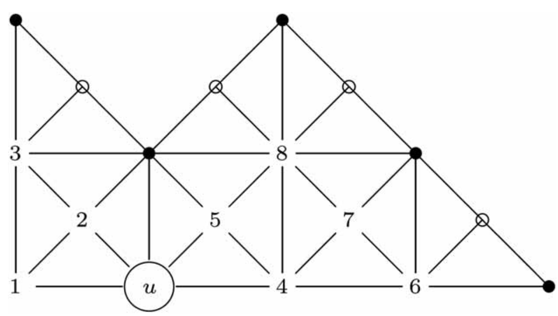


Figure 10. Parameterized wavelet-6.

Let $\psi_u \in W^0$ be the prewavelet on the vertex $u$ and suppose it has the following expression:

$$\psi_u = a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1.$$

By the orthogonality conditions, we obtain the linear equation for $b_7 = 0$:

$$M_6 x_6 = 0,$$

where

$$M_6 = \begin{bmatrix} 6 & 6 & 8 & 6 & \frac{1}{2} & 1 & 0 & 0 \\ 6 & \frac{1}{2} & 1 & 0 & 12 & 8 & 6 & 12 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 12 \\ 4 & 1 & 6 & 4 & 1 & 6 & 0 & 4 \\ 0 & \frac{1}{2} & 1 & 6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $x_6 = [a, b_1, b_2, b_3, b_4, b_5, b_6, b_8]^T$.

Solving the linear equation $M_6 x_6 = 0$, we obtain

$$x_6 = [39t_6, -24t_6, -12t_6, 4t_6, -12t_6, -12t_6, t_6, 2t_6]^T,$$

where $t_6$ is an arbitrary non-zero real number.

Finally, let us consider the seventh kind of prewavelets, the parameterized prewavelets with its support and vertices as shown in the figure 11.

Let $\psi_u$ be the wavelet on vertex $u$ in the parameterized prewavelet-7 and suppose it has the expression:

$$\psi_u = a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7.$$

By the orthogonality conditions and letting $b_2 = 0$ and $b_3 = 0$, we obtain linear equation

$$M_7 x_7 = 0,$$

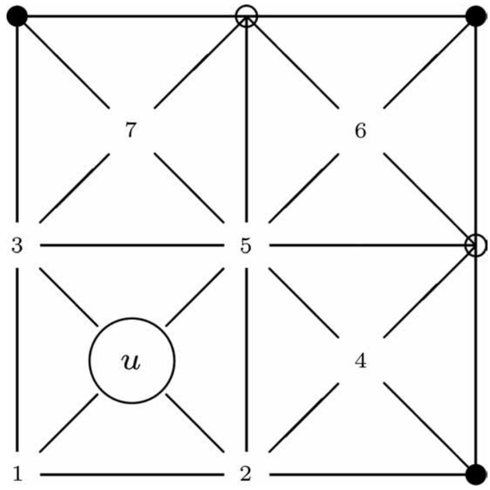


Figure 11. Parameterized prewavelet-7.

where

$$M_7 = \begin{bmatrix} 6 & 1 & 6 & 20 & 6 & 6 \\ 8 & 6 & 1 & 3 & 0 & 1 \\ 1 & \frac{1}{2} & 8 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 8 & 1 \\ 1 & \frac{1}{2} & 0 & 3 & 1 & 8 \end{bmatrix}$$

and

$$x_7 = [a, b_1, b_4, b_5, b_6, b_7]^T.$$

Solving the linear equation $M_7 x_7 = 0$, we obtain

$$x_7 = [20t_7, -24t_7, t_7, -6t_7, 2t_7, t_7]^T,$$

where $t_7$ is an arbitrary non-zero real number.

Due to the symmetry of the type-2 triangulations, any prewavelet functions on the new vertex in $u \in V^1 \setminus V^0$ can be obtained by the rotations through the above seven parameterized prewavelets. So we can obtain all prewavelet functions in $W^0$. In particular, for $t_3 = 2$, $t_4 = 2$, $t_5 = 2$, $t_6 = 4$, and $t_7 = 4$, the above five prewavelets can be transformed into the first interior prewavelet, the first boundary prewavelet, the third boundary prewavelet, the first corner prewavelet, and the second corner prewavelet in [6]. The other two parameterized prewavelets have smaller support than the ones in [6] and they are not unique but depending on the parameters $t_i$, $i = 1, 2$.

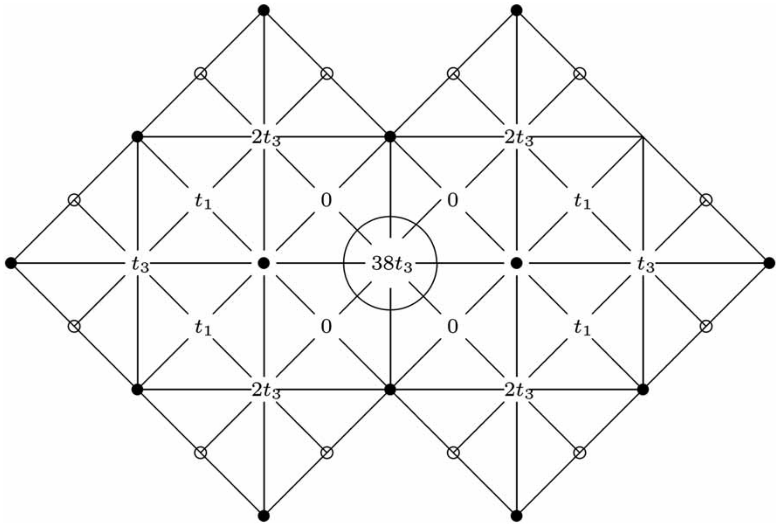


Figure 12. Dominant elements in the parameterized wavelets matrix $Q$.

In the following, we will give sufficient conditions of these parameters $t_i$, $i = 1, \ldots, 7$ to ensure that these parameterized prewavelets can form a basis of $W^0$.

THEOREM 1 *For the seven kinds of parameterized prewavelets-i, $i = 1, \ldots, 7$ constructed above, if $t_i$, $i = 1, \ldots, 7$ in the parameterized prewavelets satisfy the following conditions,*

$$\frac{144}{149}|t_3| < |t_1| < \min\{7|t_3|, 5|t_7| - 6|t_6|\}$$
$$\frac{5}{96}\left(\frac{41}{6}|t_1| + 12|t_4| + 12|t_5|\right) < |t_2| < \min\left(\frac{5}{8}(18|t_4| - 4|t_5|\right),$$
$$\frac{5}{8}\left(39|t_5| - 4|t_4| - 5|t_3| - 2|t_1|, \frac{1}{2}(35|t_6| - 4|t_5| - 5|t_4|)\right)$$

*where $t_i \neq 0$, then these parameterized prewavelets become a basis of $W^0$.*

*Proof* Let $Q = (\phi_u^1(v))_{v \in V^1 \setminus V^0}$ be the matrix evaluated at $u$ by every parameterized prewavelet. Figure 12 shows the non-zero values of the row in matrix $Q$ corresponding to the prewavelet-3. We can see that the row vector correspondingly in the matrix $Q$ is

$$[0, 0, \ldots, 0, t_3, t_1, 2t_3, 0, t_1, 2t_3, 0, 38t_3, 0, 2t_3, t_1, t_3, t_1, 2t_3, 0, 0 \ldots, 0]$$

and the entry $38t_3$ is the diagonal element of the row in the matrix $Q$.

In a similar way, we can draw figures and write down all rows associated with the rest of prewavelets in the matrix $Q$. We can verify that if the $t_i, i = 1, \ldots, 7$ satisfy the above conditions, then matrix $Q$ is a diagonal dominant matrix, and therefore, $Q$ is non-singular. Hence, these parameterized prewavelets can form a basis for the wavelet space $W^0$ .

As an example, $t_1 = 1, t_2 = 2, t_3 = 1, t_4 = 1, t_5 = 1, t_6 = 1, t_7 = 2$ gives one solution satisfying the conditions in the theorem. ■

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