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# Construction of wavelets and prewavelets over triangulations

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**Abstract**

Constructions of wavelets and prewavelets over triangulations with an emphasis of the continuous piecewise polynomial setting are discussed. Some recent results on piecewise linear prewavelets and orthogonal wavelets are presented.


## 1. Introduction

In recent years, multiresolution analysis has been intensively studied and has found applications in number of areas including in signal processing and image compression, computer graphics, and numerical solutions of differential and integral equations. Basically speaking, a multiresolution is a decomposition of a function space into mutually orthogonal subspaces, each of which is endowed with a basis. The basis functions of each subspace are called wavelets if they are mutually orthogonal and prewavelets otherwise. The subspaces are called wavelet spaces and prewavelet spaces accordingly.

While the construction of univariate wavelets is well understood, however, most of real world applications are multivariate or multiparameter in nature. The construction of multivariate wavelets are much more challenging. In fact, even the case of continuous piecewise linear wavelets construction is unexpectedly complicated, see [5–11,17,20] and the references therein. Because of the simplicity in computing with the linear splines and the importance of the orthogonal space decomposition in many applications, we emphasize the construction of piecewise linear wavelets and prewavelets in this paper.

The piecewise linear element is one of the most important and useful elements in solving boundary value problems. Piecewise linear prewavelets with small support have been constructed in [17,20,8–11]. The basis in [17] is over an infinite extended type-1 triangulation, taking advantage of dilation

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and translation over a uniform mesh. The piecewise linear wavelets have 10 nonzero coefficients in the mask, which are minimal support. In [14], a characterization of minimum support piecewise linear prewavelets with only 10 nonzero coefficients in the mask is given on a bounded domain with a type-1 triangulation. A construction of wavelets over arbitrary triangulations is presented in [20] and the wavelets have 23 nonzero coefficients in the mask. This construction is also applicable in higher dimensions. In [9], piecewise linear wavelets over a general triangulation of a bounded domain were constructed under an unusual requirement that the degree of vertices of the triangulation is at most 21. Later, the same authors presented a so-called semi-prewavelet scheme in [8] and constructed piecewise linear wavelets on a bounded type-1 triangulation with 13 nonzero coefficients in the mask. In [10], the restriction on the degree of vertices over an arbitrary triangulation is removed by applying the semi-wavelet approach and using a positive definite matrix. Piecewise linear prewavelets over a type-2 triangulation are constructed in [11] uniquely in the sense that the prewavelets are sum pairs of so-called semi-prewavelets.

On the orthogonal wavelets construction, continuous piecewise linear orthogonal scaling functions are constructed in [6] over type-1 and type-2 triangulations. The corresponding wavelets were first constructed in [4] and can also be found in [7].

In this paper, we present some most recent progress on the construction of continuous piecewise linear prewavelets and orthogonal wavelets over triangulations. On prewavelets construction, we construct piecewise linear prewavelets over a bounded domain with the type-2 triangulation by investigating the orthogonal conditions directly and obtain parameterized prewavelets. Conditions on the parameters for prewavelet basis are also given. On orthogonal wavelet construction, we provide a new "macroelement" construction of the orthogonal scaling functions from [6] using a technique developed in [13]. The paper is organized as follows. Preliminaries are introduced in Section 2. In Section 3, we construct the parameterized wavelets over type-2 triangulations. In the last section, we discuss orthogonal piecewise linear wavelets over triangulations.

## 2. Preliminaries

Let $\Delta = \{\tau_1, \tau_2, \ldots, \tau_M\}$ be a set of triangles and let $\Omega = \bigcup_{i=1}^{M} \tau_i$ be their union.

**Definition 1.** $\Delta$ is a triangulation of $\Omega$ if (i) $\tau_i \cap \tau_j$ is at most a common vertex or a common edge for $i \neq j$, (ii) the number of boundary edges incident on a boundary vertex is two, and (ii) $\Omega$ is simply connected.

If $\Omega$ is a rectangle, say $\Omega = [0,m] \times [0,n]$, and $\Delta$ is formed by grids $x=k$, $k=0,\ldots,m$, $y=\ell$, $\ell = 0,\ldots,n$, and one set of parallel diagonals, say, $y-x=-m+1,\ldots,n-1$, then $\Delta$ is called a type-1 triangulation, denoted by $\Delta^{(1)}$. If $\Delta$ is formed by adding both sets of diagonals, then it is called a type-2 triangulation, denoted by $\Delta^{(2)}$. In order to simplify the discussion we assume $m,n > 1$.

We denote by $V$ the set of all vertices $v$ of triangles in $\Delta$ and by $E$ the set of all edges $e=[v,w]$ of triangles in $\Delta$. For a vertex $v \in V$, the set of neighbors of $v$ in $V$ is $V_v = \{w \in V;\ [v,w] \in E\}$.

As usual, let $S_d^r(\Delta)$ denote the spline space consisting of $C^r$ $pp$ (:= piecewise polynomial) functions of total degree at most $d$ over the triangulation $\Delta$.

Suppose next that $\Delta$ is a triangulation. Given data values $f_v \in \mathbb{R}$ for $v \in V$, there is a unique function $f$: $S_1^0(\Delta)$ which linearly interpolates the data: $f(v) = f_v,\ v \in V$. Clearly, the linear space $S_1^0(\Delta)$ has dimension $|V|$.

For each $v \in V$, let $\phi_v : \Omega \to \mathbb{R}$ be the unique 'hat' or nodal function in $S := S_1^0(\Delta)$ satisfying the interpolation conditions: $\phi_v(w) = \delta_{vw}$, where

$$\delta_{vw} = \begin{cases} 1, & w = v, \\ 0 & \text{otherwise.} \end{cases}$$

The set of functions $\Phi = \{\phi_v\}_{v \in V}$ is a basis for the space $S$ and for any function $f \in S$,

$$f(x) = \sum_{v \in V} f(v)\phi_v(x), \quad x \in \Omega. \tag{2.1}$$

The support of $\phi_v$ is the union of all triangles which contain $v$:

$$\Omega_v = \bigcup_{v \in \tau \in \Delta} \tau.$$

For a given triangulation $\Delta^0 (:= \Delta) = \{\tau_1, \tau_2, \ldots, \tau_n\}$, a refinement triangulation of $\Delta$, denoted by $\mathscr{R}(\Delta)$ is a triangulation such that every triangle in $\Delta$ is a union of some triangles in $\mathscr{R}(\Delta)$. Obviously, there are various kinds of refinements. In this paper, we only consider the following uniform or dyadic refinement: For a given triangle $\tau = [x_1, x_2, x_3]$, let $y_1 = (x_2 + x_3)/2$, $y_2 = (x_1 + x_3)/2$, and $y_3 = (x_1 + x_2)/2$ denote the midpoints of its edges. The set of four triangles

$$\mathscr{R}(\tau) = \{[x_1, y_2, y_3], [y_1, x_2, y_3], [y_1, y_2, x_3], [y_1, y_2, y_3]\}$$

forms a refinement of the coarse triangle $\tau$. The set of triangles $\Delta^1 := \mathscr{R}(\Delta) = \bigcup_{\tau \in \Delta} \mathscr{R}(\tau)$ is evidently a triangulation and a refinement of $\Delta$. Let $\Delta^j := \mathscr{R}^j(\Delta) = \mathscr{R}(\mathscr{R}^{j-1}(\Delta))$ for $j \geqslant 1$. Then, it forms a sequence of refinements of the triangulation $\Delta$: $\Delta^j,\ j = 0, 1, \ldots$ .

In order to discuss some properties of $\Delta^j$ in relation to $\Delta^{j-1}$, let $V^j$ be the set of vertices in $\Delta^j$, and define $E^j$, $\phi_v^j$, $V_v^j$, and $\Omega_v^j$ accordingly. A straightforward calculation shows that

$$\phi_v^{j-1} = \phi_v^j + \frac{1}{2} \sum_{w \in V^j} \phi_w^j, \quad v \in V^{j-1}. \tag{2.2}$$

For $S^0 = S_1^0(\Delta)$, let $S^j = S_1^0(\Delta^j)$. Then we obtain a nested sequence of spaces:

$$S^0 \subset S^1 \subset S^2 \subset \cdots$$

such that $\bigcup_{j \geqslant 0} S^j$ is dense in $L^2(\Omega)$. We call the sequence $\{S^j\}_{j=0}^{\infty}$ of nested spaces a multiresolution approximation.

For the nested vector spaces:

$$S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^j \subset \cdots \to L^2(\Omega),$$

let $W^j$ denote the orthogonal complement of the $S^j$ in the space $S^{j+1}$, that is,

$$S^{j+1} = S^j \oplus W^j.$$

Then $W^j$ can be used to represent the parts of functions in $S^{j+1}$ that cannot be represented in the space $S^j$. We can call $W^j$ the correcting space. Using $j$-step corrections, we have

$$S^j = S^{j-1} \oplus W^{j-1} = S^{j-2} \oplus W^{j-2} \oplus W^{j-1} = \cdots$$
$$= S^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{j-1}.$$

Suppose $\Psi = \{\psi_{j,\ell}\}_{\ell \in L}$ forms a basis of $W^j$. If $\Psi$ is an orthonormal basis of $W^j$, then the elements $\psi_{j,\ell}$ of $\Psi$ are called wavelets, otherwise, they are called prewavelets.

## 3. Piecewise linear prewavelets over type-2 triangulations

Piecewise linear prewavelets over a type-2 triangulation are constructed in [11] uniquely in terms of sum pairs of semi-prewavelets. In this section, we construct piecewise linear prewavelets over a type-2 triangulation by investigating the orthogonal conditions directly and obtain parameterized prewavelets. We also provide conditions on the parameters for prewavelet basis.

It is clear that if $\Delta^0$ is a type-2 triangulation, then its refinement $\Delta^1 = \mathscr{R}(\Delta)$ again is a type-2 triangulation and so is $\Delta^j$ for any positive integer $j$. For $S^0 = S_1^0(\Delta^{(2)})$, let $W^0$ be the orthogonal complement space of $S^0$ in $S^1$, that is, $S^1 = S^0 \oplus W^0$. Similarly, if we define the wavelet space $W^j$ to be the orthogonal complement of $S^j$ in $S^{j+1}$ at every refinement level $j$, that is

$$S^{j+1} = S^j \oplus W^j,$$

then we obtain the decomposition

$$S^j = S^0 \oplus W^0 \oplus W^1 \oplus \cdots \oplus W^{k-1},$$

for any $j \geqslant 1$. We would like to obtain a basis of functions with small support for the purpose of conveniently representing the decomposition of a given function $f^{j+1}$ in $S^{j+1}$ into its two unique components $f^j \in S^j$ and $g^j \in W^j$: $f^{j+1} = f^j \oplus g^j$. Note that the basis elements of any $W^k$ can simply obtain from the basis of $W^0$ using a dilation operator, we can restrict our study only to $W^0$.

The dimension of $W^0$ is $|V^1| - |V^0| = |E^0|$ which is equal to the number of midpoints added to $V^0$ to form $V^1$. Let us simply associate a wavelet $\psi_u$ in $W^0$ with each vertex $u$ in $V^1 \backslash V^0$ and derive a general sufficient condition for the set $\Psi = \{\psi_u\}_{u \in V^1 \backslash V^0}$ to constitute a basis of $W^0$ .

As an element in $S^1$, the function $\Psi_u$ can be written as a linear combination of the basis function $\phi_w^1$, namely

$$\psi_u(x) = \sum_{w \in V^1} q_{w,u} \phi_w^1(x), \tag{3.1}$$

where, the coefficients of $\psi_u$ are $q_{w,u} = \psi_u(w),\ w \in V^1$. The set of nonzero coefficients in (3.1) is called the mask of $\psi_u$. Our aim is to construct a basis of $W^0$ with a small number of coefficients in the mask for each basis element, or equivalently, with a small support of each basis function.

A sufficient condition on these coefficients for $\Psi$ to form a basis of $W^0$ can be derived by evaluating the $\psi_u$ at the vertices in $V^1 \backslash V^0$. Let $u_1, \ldots, u_n,\ n = |E^0|$, be any ordering of the vertices in $V^1 \backslash V^0$. Since any element of $S$ has a unique representation, we see that the element $\psi_u$ of $S^1$

belongs to $W^0$ if and only if

$$\langle \phi_v^0, \psi_u \rangle = \sum_{w \in V^1} \langle \phi_v^0, \phi_w^1 \rangle q_{w,u} = 0, \quad v \in V^0. \tag{3.2}$$

Then it is easy to prove the following (see [8]).

**Theorem 2.** *A set $\Psi$ of functions $\{\psi_{u_1}, \ldots, \psi_{u_n}\}$ in $W^0$ is a basis of $W^0$ if the matrix $Q = (q_{u_i,u_j})_{i,j}$ is nonsingular.*

To construct a small support basis, we follow the idea presented in [9], for a 'new' vertex $u \in V^1 \backslash V^0$, we try to construct a nontrivial wavelet $\psi_u \in W^0$ associated with the vertex $u$, whose support is around $u$, i.e., it has the form

$$\psi_u(x) = \sum_{w \in V(u)} q_{w,u} \phi_w^1(x), \tag{3.3}$$

where $V(u) \subset V^1$ is a small set of vertices of $\Delta^1$ which are near to $u$.

In [11], the authors introduced a so-called semi-wavelets approach to seek prewavelets for the space $W^0$ in terms of sum pairs of elements $\sigma_{v_1,u}$ and $\sigma_{v_2,u}$ (called semi-wavelets) of $S^1$ which have small support and are close to being in the wavelet space $W^0$, in the sense that they are orthogonal to all but two of the nodal functions in the coarse space, where $u$ is the midpoint of the edge $[v_1, v_2]$. Depending on the locations of $v_1$, either $v_1 = (i + 1/2, j + 1/2)$ or $v_1 = (i, j)$, there are three interior semi-wavelets that generate two interior prewavelets $\psi_u$ in the sense that $v_1$ and $v_2$ are both interior vertices of $\Delta^0$ up to rotation and symmetries. Similarly, there are three edge prewavelets $\psi_u$ for which one of $v_1$ and $v_2$ is an interior vertex while the other one lies on the boundary but not the corner. The remaining two prewavelets are corner prewavelets. Requiring the prewavelets being the sum pairs of semi-wavelets, it has been shown in [11] that those seven kinds of prewavelets are uniquely determined. For the second kind of interior prewavelets, there are 13 nonzero coefficients in the masks.

We take the same structures of $V(u)$ as described in [11] and try to construct prewavelets with fewer coefficients in the masks. It turns out that we obtain parameterized prewavelets. For the second kind of interior prewavelets, we have only 11 nonzero coefficients in the masks. In the following, we construct only two interior parameterized prewavelets and provide expressions of three edge prewavelets, and two corner prewavelets.

We label the vertices of $V(u)$ as in Fig. 1. Let $\psi_u^{0,1} \in W^0$ be the prewavelet associated with vertex $u$ having the following expression:

$$\begin{aligned}\psi_u^{0,1} &= a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 \\ &\quad + b_9\phi_9^1 + b_{10}\phi_{10}^1 + b_{11}\phi_{11}^1 + b_{12}\phi_{12}^1 + b_{13}\phi_{13}^1 + b_{14}\phi_{14}^1 + b_{15}\phi_{15}^1 + b_{16}\phi_{16}^1,\end{aligned}$$

where $a$ and $b_i$, $i = 1, \ldots, 16$ are determined by using the orthogonality conditions: $\langle \sigma_u, \phi_{P_i}^0 \rangle = 0$, $i = 1, \ldots, 12$ and $\langle \sigma_u, \phi_1^0 \rangle = 0$, $\langle \sigma_u, \phi_{13}^0 \rangle = 0$. We obtain the following system of equations:

$$M_1 x_1 = 0, \tag{3.4}$$

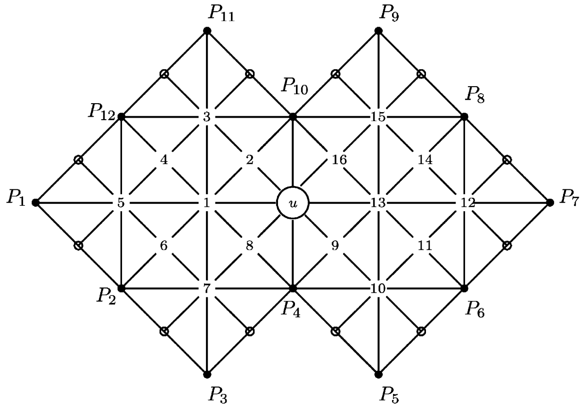


Fig. 1. Support of the first interior prewavelet.

where

$$M_1 = \begin{bmatrix}
4 & 1 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 6 \\
0 & 1 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 6 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 6 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 12 & 1 \\
12 & 24 & 8 & 12 & 8 & 12 & 8 & 12 & 8 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
12 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 12 & 8 & 12 & 24 & 8 & 12 & 8
\end{bmatrix}.$$

and $x_1 = [a, b_1, \ldots, b_{16}]^{\mathrm{T}}$.

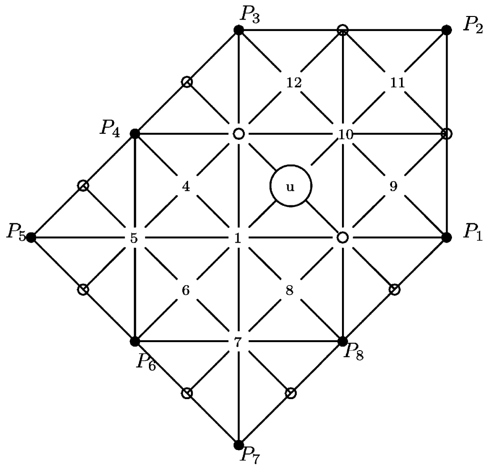


Fig. 2. Support of the second interior prewavelet.

We solve the equations system $M_1x_1 = 0$ by letting $b_{14} = 0$, $b_{11} = 0$, $b_6 = 0$, and $b_4 = 0$ and obtain that

$$x_1 = [38t_1, -12t_1, -12t_1, 2t_1, 0, t_1, 0, 2t_1, -12t_1, -12t_1, 2t_1, 0, t_1, -12t_1, 0, 2t_1, -12t_1]^{\mathrm{T}}$$

for a parameter $t_1 \neq 0$. Therefore, the first interior prewavelet $\psi_u^{0,1} \in W^0$ have the following expression:

$$\psi_u^{0,1} = 38t_1\phi_u^1 - 12t_1\phi_1^1 - 12t_1\phi_2^1 + 2t_1\phi_3^1 + t_1\phi_5^1 + 2t_1\phi_7^1 - 12t_1\phi_8^1 - 12t_1\phi_9^1$$
$$+ 2t_1\phi_{10}^1 + t_1\phi_{12}^1 - 12t_1\phi_{13}^1 + 2t_1\phi_{15}^1 - 12t_1\phi_{16}^1,$$

where $\phi_i^1$, $i = u, 1, \ldots, 16$ are nodal basis functions in $S^1$.

Indexing the vertices of $V(u)$ as in Fig. 2, we assume that the second interior prewavelet associated with $u$ has the following expression:

$$\psi_u^{0,2} = b_1\phi_1^1 + a\phi_u^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1$$
$$+ b_7\phi_7^1 + b_8\phi_8^1 + b_9\phi_9^1 + b_{10}\phi_{10}^1 + b_{11}\phi_{11}^1 + b_{12}\phi_{12}^1,$$

where $\phi_i^1$, $i = u, 1, \ldots, 12$ are nodal basis functions in $S^1$. By the orthogonal conditions, $\langle \phi_u^0, \phi_{P_i}^0 \rangle = 0$, $i = 1, \ldots, 8$ and $\langle \phi_u^0, \phi_1^0 \rangle = 0$, $\langle \phi_u^0, \phi_{10}^0 \rangle = 0$, we obtain the following system of linear equations:

$$M_2x_2 = 0, \tag{3.5}$$

where

$$M_2 = \begin{bmatrix} 24 & 12 & 8 & 12 & 8 & 12 & 8 & 12 & 8 & 1 & 3 & 0 & 1 \\ 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 6 & 20 & 6 & 6 \\ 1 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 8 \\ 1 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 8 & 1 \end{bmatrix}$$

and $x_2 = [b_1, a, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}]^{\mathrm{T}}$.

We solve (3.5) and obtain the following solutions:

$$b_1 = -15t_2, \quad a = \tfrac{253}{6}t_2, \quad b_4 = \tfrac{11}{6}t_2, \quad b_5 = t_2, \quad b_6 = \tfrac{7}{6}t_2, \quad b_7 = t_2$$

$$b_8 = \tfrac{11}{6}t_2, \quad b_9 = t_2, \quad b_{10} = -14t_2, \quad b_{11} = 5t_2, \quad b_{12} = t_2,$$

where $t_2$ is a nonzero arbitrary real number.

This second interior prewavelet only has 11 nonzero coefficients in the mask.

Corresponding to Figs. 3–5, three edge prewavelets and two corner prewavelets, respectively, are given as follows.

The first edge prewavelet is

$$\begin{aligned} \psi_u^{0,3} &= a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 + b_9\phi_9^1 + b_{10}\phi_{10}^1 \\ &= 38t_3\phi_u^1 - 12t_3\phi_1^1 - 12t_3\phi_2^1 + 2t_3\phi_3^1 + t_3\phi_5^1 + -12t_3\phi_6^1 + t_3\phi_7^1 + 2t_3\phi_9^1 - 12t_3\phi_{10}^1, \end{aligned}$$

where $t_3 \neq 0$ and the vertices are labeled as in Fig. 3a.

The second edge prewavelet is

$$\begin{aligned} \psi_u^{0,4} &= a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 \\ &= \tfrac{102}{5}t_4\phi_u^1 - \tfrac{77}{5}t_4\phi_1^1 + \tfrac{3}{5}t_4\phi_2^1 + \tfrac{4}{5}t_4\phi_3^1 + \tfrac{6}{5}t_4\phi_4^1 + t_4\phi_5^1 - \tfrac{32}{5}t_4\phi_6^1 + \tfrac{12}{5}t_3\phi_7^1, \end{aligned}$$

where $t_4 \neq 0$ and the vertices are labeled as in Fig. 3b.

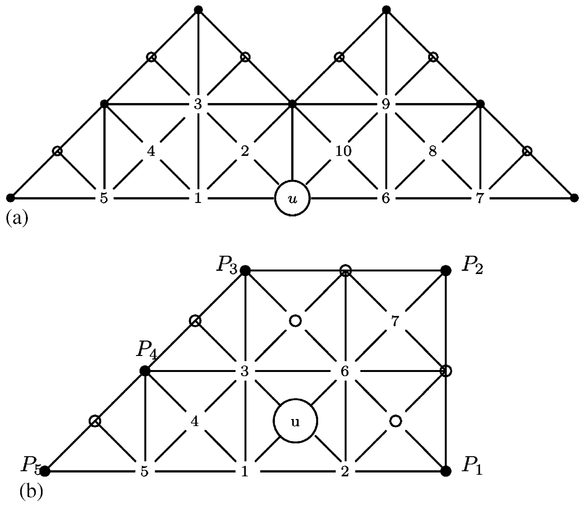


Fig. 3. Support of first and second edge prewavelets.

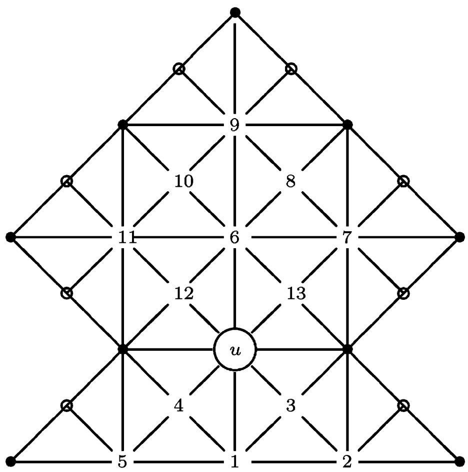


Fig. 4. Support of the third edge prewavelet.

The third edge prewavelet has a support shown in Fig. 4 and is given by

$$
\begin{aligned}
\psi_u^{0,5} &= a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 + b_9\phi_9^1 \\
&\quad + b_{10}\phi_{10}^1 + b_{11}\phi_{11}^1 + b_{12}\phi_{12}^1 + b_{13}\phi_{13}^1 \\
&= 39t_5\phi_u^1 - 24t_5\phi_1^1 + 4t_5\phi_2^1 - 12t_5\phi_3^1 - 12t_5\phi_4^1 + 4t_5\phi_5^1 - 12t_5\phi_6^1 + 2t_5\phi_7^1 + t_5\phi_9^1 \\
&\quad + 2t_5\phi_{11}^1 - 12t_5\phi_{12}^1 - 12t_5\phi_{13}^1,
\end{aligned}
$$

for a parameter $t_5 \neq 0$.

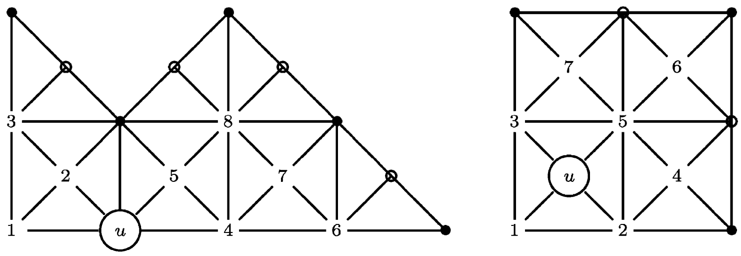


Fig. 5. Support of corner prewavelets.

The support of the remaining two corner prewavelets are shown in Fig. 5 and they have the following expressions:

$$\begin{aligned}\psi_u^{0,6} &= a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 + b_8\phi_8^1 \\ &= 39t_6\phi_u^1 - 24t_6\phi_1^1 - 12t_6\phi_2^1 + 4t_6\phi_3^1 - 12t_6\phi_4^1 - 12t_6\phi_5^1 + t_6\phi_6^1 + 2t_6\phi_8^1\end{aligned}$$

and

$$\begin{aligned}\psi_u^{0,7} &= a\phi_u^1 + b_1\phi_1^1 + b_2\phi_2^1 + b_3\phi_3^1 + b_4\phi_4^1 + b_5\phi_5^1 + b_6\phi_6^1 + b_7\phi_7^1 \\ &= 20t_7\phi_u^1 - 24t_7\phi_1^1 + t_7\phi_4^1 - 6t_7\phi_5^1 + 2t_7\phi_6^1 + t_7\phi_7^1\end{aligned}$$

for parameter $t_6, t_7 \neq 0$.

All prewavelets in $W^0$ can be obtained by rotating or reflecting these seven prewavelets. In particular, setting $t_1 = 1$, $t_3 = 2$, $t_5 = 1$, $t_6 = 2$, and $t_7 = 4$, we obtain the corresponding prewavelets described in [11].

By showing the corresponding matrix $Q$ is diagonal dominant, we obtain the following theorem which provides sufficient conditions on the parameters $t_k$, $k = 1, \ldots, 7$ to ensure that the set of prewavelets $\Psi = \{\psi_u\}_{u \in V^1 \setminus V^0}$ obtained by using symmetries and rotations from $\psi_u^{0,\ell}$, $\ell = 1, \ldots, 7$ becomes a basis of the wavelet space $W^0$.

**Theorem 3.** *The set* $\Psi = \{\psi_u\}_{u \in V^1 \setminus V^0}$ *is a basis of* $W^0$ *if the parameters* $t_\ell \neq 0$, $\ell = 1, \ldots, 7$ *satisfy the following*:

$$\tfrac{144}{149}|t_1| < |t_2| < \min\{7|t_1|, 5|t_7| - 6|t_6|\}$$

$$\tfrac{5}{96}\left(\tfrac{41}{6}|t_2| + 12|t_3| + 12|t_5|\right) < |t_4| < \min\left(\tfrac{5}{8}(18|t_3| - 4|t_5|),\right.$$

$$\left.\tfrac{5}{8}(39|t_5| - 4|t_3| - 5|t_1| - 2|t_2|, \tfrac{1}{2}(35|t_6| - 4|t_5| - 5|t_3|)\right).$$

**Remark 1.** Working with triangulations of arbitrary topology for applications in computer graphics, the approach taken by Lounsbery et al. [18] is based on the use of subdivision schemes to first consider piecewise linear prewavelets with global support. They subsequently truncate them to a small region, and thus producing functions that are no longer elements of the orthogonal complementary wavelet space.

**Remark 2.** The construction of prewavelet systems in the general framework of multiresolution analysis generated by the shifts and dilates of a refinable function, in particular, by box splines, has been extensively studied in [1,16,19]. The theory developed by Jia and Micchelli [16] leads in the piecewise linear case to examples of prewavelets with 69 nonzero coefficients in the masks.

**Remark 3.** Multiresolution approximation using $C^1$ quadratic splines is studied based on Powell–Sabin 6-split in [3]. Very recently, a hierarchical basis for $C^1$ cubic bivariate splines is used for surface compression in [15].

**Remark 4.** Biorthogonal piecewise linear elements can be found in [2].

## 4. Piecewise linear orthogonal wavelets

Two examples of piecewise linear orthogonal continuous scaling functions on triangulations were constructed in [6], one example on a regular type-1 triangulation and the other on a regular type-2 triangulation. The wavelets were constructed in [4]. Both the scaling functions and the wavelets are compactly supported. In this section we review the construction of the type-1 triangulation scaling functions of [6] by first constructing an orthogonal, refinable "macroelement" $\Lambda$ (see [13] for more on constructing refinable macroelements) which may be naturally adapted to an arbitrary triangulation $\Delta$ to get a space $S(\Delta,\Lambda)$. In the case that $\Delta$ is a regular type-1 triangulation of $\mathbb{R}^2$, then we retrieve the scaling vector of [6]. The calculations in this section were all done with the aid of the computer algebra system *Mathematica*. See [12] for a package used to aid with these calculations.

Let $(\Delta^j)_{j\geqslant 0}$ denote the "semi-regular" refinement scheme $\Delta^{j+1} = \mathscr{R}(\Delta^j)$ with $\Delta^0 = \Delta$ and let $S^j := S(\Delta^j,\Lambda)$ for $j=0,1,\ldots$ . Then $S^j \subset S^{j+1}$, $j \geqslant 0$, and the sequence of spaces $(S^j)_{j\geqslant 0}$ forms a multiresolution of $L^2(\Omega)$. Here we restrict our discussion to the construction of $\Lambda$ and to the space $S(\Delta,\Lambda)$. See [13] for a procedure for finding local orthogonal bases (that is, the wavelets) for the spaces $W^j := S^{j+1} \ominus S^j$.

Let $u_0=(0,0)$, $u_1=(1,0)$, and $u_2=(1/2,\sqrt{3}/2)$ and let $\tau^0$ denote the equilateral triangle $[u_0,u_1,u_2]$. We recast the construction of the orthogonal piecewise linear scaling vector given in [6] in terms of a *macroelement* of orthogonal piecewise linear functions on $\tau^0$. These functions will then be pieced together to construct an orthogonal basis for $V_0(\Delta)$ for a given arbitrary triangulation $\Delta$.

### 4.1. $C^0$ symmetric macroelements on $\tau^0$

Let $\Delta^0=\{\tau^0\}$ and recursively define $\Delta^{j+1}=\mathscr{R}(\Delta^j)$ for $j=0,1,2,\ldots$ . The spline spaces $S^j=S_1^0(\Delta^j)$ form a multiresolution of $L_2(\tau)$. For $j=0,1,2,\ldots$ and a vertex $v\in V^j := V(\Delta^j)$, let $\phi_v^j:\mathbb{R}^2 \to \mathbb{R}$ denote the piecewise linear function on the triangulation on $\Delta^j$ taking the value 1 at $v$ and 0 for all other vertices $v'\in V^j$. Let $C(\tau^0)$ denote the collection of continuous real-valued functions on $\tau^0$. If $f$ is a function on a set $A$ and $B\subset A$ we let $f|_B$ denote the restriction of $f$ to $B$ and, for collection of functions $F$ defined on $A$, we let

$$f|_B := \{f|_B\colon\ f\in F\}$$

denote the collection of restrictions of the elements of $F$ to $B$.

If $\Lambda \subset C(\tau^0)$ we partition $\Lambda$ into the following seven subsets:

$$\Lambda^{\tau^0} := \{\lambda \in \Lambda\colon \lambda|_{\partial\tau^0} = 0\},$$

$$\Lambda^{e} := \{\lambda \in \Lambda\colon \lambda|_{e'} = 0 \text{ for } e' \in E(\tau^0) \text{ and } e' \neq e\}\backslash \Lambda^{\tau^0}, \quad (e \in E(\tau^0)),$$

$$\Lambda^{v} := \{\lambda \in \Lambda\colon \lambda|_{e_v} = 0\}\Bigg\backslash\left(\Lambda^{\tau^0} \cup \bigcup_{e\in E(\tau^0)} \Lambda^e\right), \quad (v \in V(\tau^0)),$$

where $e_v$ denotes the edge of $\tau^0$ not containing $v$.

Let $r_0$ denote the rotation taking $u_0 \to u_1$, $u_1 \to u_2$, and $u_2 \to u_0$ and let $r_1$ denote the reflection taking $u_0 \to u_1$, $u_1 \to u_0$, and $u_2 \to u_2$. For a set of functions $\Lambda \subset C(\tau^0)$ and a continuous function $r : \tau^0 \to \tau^0$ let

$$\Lambda \circ r := \{\lambda \circ r\colon \lambda \in \Lambda\} \subset C(\tau^0).$$

**Definition 4** ($C^0$ symmetric macroelement). We call a finite collection of functions $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\lambda_i \in C(\tau^0)$, a $C^0$ *symmetric macroelement* (or just *macroelement*) if

(a) for each $\lambda \in \Lambda$ and $i = 0, 1$, there is some $\varepsilon \in \{-1, +1\}$ such that $\varepsilon\lambda \circ r_i \in \Lambda$,
(b) $\Lambda^v = \{\lambda^v\}$ where $\lambda^v \in \Lambda$ is symmetric about the line from $v$ to the midpoint of the edge $e_v$ opposite $v$, that is, if $r$ is the reflection about this line, then $\lambda_v \circ r = \lambda_v$ and $\lambda^{r_0(v)} \circ r_0 = \lambda^v$ for $v \in E(\tau^0)$,
(c) $\Lambda^{r_0(e)} \circ r_0 = \Lambda^e$ for $e \in E(\tau^0)$, and
(d) $\Lambda^e|_e$ is linearly independent for $e \in E(\tau^0)$.

Suppose $\Lambda$ is a $C^0$ symmetric macroelement. We say $\Lambda$ is an *orthogonal macroelement* if $\Lambda$ is an orthonormal set, that is, if $\langle \lambda, \lambda' \rangle = \delta_{\lambda,\lambda'}$ for $\lambda, \lambda' \in \Lambda$.

Further suppose $\Delta$ is a triangulation. The space $S(\Delta, \Lambda)$ is defined by

$$S(\Delta, \Lambda) := \operatorname{span}\{\lambda \circ \omega_\tau\colon \tau \in \Delta,\ \lambda \in \Lambda\} \cap C(\mathbb{R}^2)$$

where for each $\tau \in \Delta$, $\omega_\tau : \mathbb{R}^2 \to \mathbb{R}^2$ denotes one of the six possible affine mappings such that $\omega_\tau(\tau) = \tau^0$.

We say that a macroelement $\Lambda$ is *refinable* if $S(\Delta^0, \Lambda) \subset S(\Delta^1, \Lambda)$.

**Remark 5.** See [13] for a more detailed development of wavelet constructions based on macroelements. Condition (d) implies that $\Lambda^v$ consists of a single function $\lambda^v$ whose restriction to the boundary of $\tau^0$ is in $\Pi_1$ and is such that $\lambda^v(v') = \delta_{v,v'}$ for $v, v' \in V(\tau^0)$.

**Remark 6.** The symmetry properties of a $C^0$ symmetric macroelement imply that $S(\Delta, \Lambda)$ is independent of the choice of the affine mapping $\omega_\tau$ for each $\tau \in \Delta$.

**Remark 7.** If $\Lambda$ is refinable then it follows that $S(\Delta, \Lambda) \subset S(\Lambda, \mathscr{R}(\Delta))$ for any triangulation $\Delta$.

Let $\Delta$ be an arbitrary triangulation in $\mathbb{R}^2$ with $\Omega = \bigcup_{\tau\in\Delta} \tau$. We next construct a local basis of functions for the space $S(\Delta, \Lambda)$ for an arbitrary triangulation $\Delta$. The basis functions are of three types: triangle functions supported on individual triangles $\tau \in \Delta$, edge functions supported on the union of the triangles containing an edge $e \in E(\Delta)$ (there are either one or two such triangles) and vertex functions supported on the star of a vertex $v \in V(\Delta)$ (the *star of a vertex* $v$ is the union of the triangles containing $v$).

First, for $\tau \in \Delta$, let

$$\Lambda^\tau := \{\lambda \circ \omega_\tau \colon \lambda \in \Lambda^{\tau^0}\}.$$

Second, we define basis functions associated with each edge $e \in E(\Delta)$. If $e$ is a boundary edge, let $\tau$ denote the single triangle in $\Delta$ containing $e$ and then let

$$\Lambda^e := \{\zeta \circ \omega_\tau \colon \zeta \in \Lambda^{\omega_\tau(e)}\}.$$

If $e \in E(\Delta)$ is an interior edge, let $\tau_a$ and $\tau_b$ denote the two triangles in $\Delta$ containing $e$. Let $e_a = \omega_{\tau_a}(e)$ and $e_b = \omega_{\tau_b}(e)$. Then $e_a$ and $e_b$ are edges of $\tau^0$. Let $s$ be the affine mapping such that $s(\tau^0) = \tau^0$ and such that $s \circ \omega_{\tau_b}|_e = \omega_{\tau_a}|_e$. For $\zeta \in \Lambda^{e_a}$ then there is some

$$\lambda^e_\zeta(x) := \begin{cases} \zeta \circ \omega_{\tau_a}(x) & (x \in \tau_a), \\ \zeta \circ s \circ \omega_{\tau_b}(x) & (x \in \tau_b), \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Lambda^e := \{\lambda^e_\zeta \colon \zeta \in \Lambda^{e_a}\}.$$

Note that if the labels of the triangles $\tau_a$ and $\tau_b$ are switched, then we at most introduce a change of sign in the definition of $\lambda^e_\zeta$.

Finally, if $v \in V(\Delta)$, let

$$\lambda^v := \begin{cases} \lambda^{\omega_\tau(v)} \circ \omega_\tau & \text{for } \tau \in \Delta \text{ with } v \in \tau, \\ 0 & \text{otherwise.} \end{cases}$$

One may verify using properties that $\lambda^\tau$, $\lambda^e$, and $\lambda^v$ are well defined and in $S(\Delta, \Lambda)$ where $\tau \in \Delta$, $e \in E(\Delta)$, $v \in V(\Delta)$. Let $\mathscr{T}_\Delta := \Delta \cup V(\Delta) \cup E(\Delta)$. As we verify in [13], the above functions form a local basis of $S(\Delta, \Lambda)$:

**Lemma 5.** *Suppose* $\Lambda$ is a $C^0$ *symmetric macroelement. Then* $\bigcup_{\theta\in\mathscr{T}_\Delta} \Lambda^\theta$ *forms a basis for* $S(\Delta, \Lambda)$.

### 4.2. A piecewise linear orthogonal macroelement

We start with a macroelement

$$\Lambda^0 = (\phi^0_{u_0}, \phi^0_{u_1}, \phi^0_{u_2}, \phi^1_{(u_0+u_1)/2}, \phi^1_{(u_0+u_2)/2}, \phi^1_{(u_1+u_2)/2}).$$

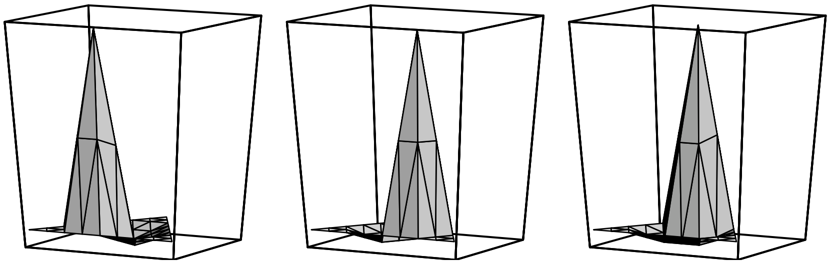

Fig. 6. The components $\gamma_1$, $\gamma_2$ and $\gamma_3$ of the piecewise linear orthogonal macroelement $\Gamma$. Note $\gamma_2=\gamma_1\circ r_0$ and $\gamma_3=\gamma_2\circ r_0$.

Then $S_1^0(\Delta^1)=S(\Delta^0,\Lambda^0)=\operatorname{span}\Lambda^0$. Let $\Lambda^1=\Lambda^0\cup(\phi^2_{(2u_0+u_1+u_2)/4},\phi^2_{(u_0+2u_1+u_2)/4},\phi^2_{(u_0+u_1+u_2)/4})$ (with the implied ordering). Then $\Lambda^1$ satisfies

$$S(\Delta^0,\Lambda^0)\subset S(\Delta^0,\Lambda^1)\subset S(\Delta^1,\Lambda^0)$$

from which it follows that $\Lambda^1$ is refinable.

We proceed to construct an orthogonal macroelement $\Gamma$ such that

$$S(\Delta^0,\Lambda^1)\subset S(\Delta^0,\Gamma)\subset S(\Delta^1,\Lambda^1)$$

from which it follows that $\Gamma$ is refinable and that the components of $\Gamma$ are in $S_0^1(\Delta^2)$, that is, the components are continuous (when restricted to $\tau^0$) and piecewise linear on the triangulation $\Delta^2$. The construction hinges on finding two functions $\mu$ and $v$ in $S(\Delta^1,\Lambda^1)$ satisfying certain conditions that make the construction of an orthogonal macroelement possible.

The first step in constructing $\Gamma$ is to choose an orthonormal basis for the three-dimensional space

$$\operatorname{span}(\Lambda^1)^{\tau^0}=\operatorname{span}(\varphi^2_{(2u_0+u_1+u_2)/4},\phi^2_{(u_0+2u_1+u_2)/4},\phi^2_{(u_0+u_1+u_2)/4})$$

of the form $(\gamma_1,\gamma_1\circ r_0,\gamma_1\circ r_0^2)$ where $\gamma_1=\gamma_1\circ r_1$. Then $\gamma_1$ must be of the form $\gamma_1=\alpha\phi^2_{(2u_0+u_1+u_2)/4}+\beta\phi^2_{(u_0+2u_1+u_2)/4}+\beta\phi^2_{(u_0+u_1+2u_2)/4}$ for some constants $\alpha$ and $\beta$. The conditions $\langle\gamma_1,\gamma_2\circ r_0\rangle=0$ and $\langle\gamma_1,\gamma_1\rangle=1$ are equivalent to the equations

$$\alpha^2+14\,\alpha\,\beta+9\beta^2=0$$

$$3\alpha^2+2\alpha\beta+7\beta^2=64\sqrt{3}.$$

Choosing the most 'localized' (the solution with the largest $|\alpha|$ and smallest $|\beta|$) of the 4 distinct solutions for $\alpha$ and $\beta$ gives $\alpha=3^{-3/4}(4+16\sqrt{2/5})$ and $\beta=3^{-3/4}(4-8\sqrt{2/5})$. Fig. 6 shows the resulting $\gamma_1$.

### 4.2.1. The function $\mu=\gamma_4$

Next we find $\mu\in S(\Delta^1,\Lambda^1)$ such that $\mu$ vanishes on $\partial\Delta=0$, $\mu$ is symmetric (that is, $\mu\circ r=\mu$ for any isometry $r$ leaving $\Delta^0$ invariant), $\mu$ is orthogonal to $\gamma_1$ (and therefore $\mu$ is also orthogonal to $\gamma_2$ and $\gamma_3$ by symmetry) and such that

$$(I-P_\mu)(\Lambda^1)^e\perp(\Lambda^1)^{e'}\cup(\Lambda^1)^v$$

for $e \neq e' \in E(\tau^0)$ and $v$ the vertex of $\tau^0$ opposite of $e$ where $P_\mu$ denotes the orthogonal projection onto the subspace spanned by $\mu$.

The symmetry of $\mu$ implies that $\mu$ must lie in a subspace of $S(\Delta^1, \Lambda^1)$ spanned by the four functions

$$\mu_1 = \phi^3_{(6u_0+u_1+u_2)/8} + \phi^3_{(u_0+6u_1+u_2)/8} + \phi^3_{(u_0+u_1+6u_2)/8}$$

$$\mu_2 = \phi^3_{(5u_0+2u_1+u_2)/8} + \phi^3_{(5u_0+u_1+2u_2)/8} + \phi^3_{(2u_0+5u_1+u_2)/8} + \phi^3_{(2u_0+u_1+5u_2)/8} + \phi^3_{(u_0+5u_1+2u_2)/8} + \phi^3_{(u_0+2u_1+5u_2)/8}$$

$$\mu_3 = \phi^3_{(2u_0+3u_1+3u_2)/8} + \phi^3_{(3u_0+2u_1+3u_2)/8} + \phi^3_{(3u_0+3u_1+2u_2)/8}$$

$$\mu_4 = \phi^2_{(2u_0+u_1+u_2)/4} + \phi^2_{(u_0+2u_1+1u_2)/4} + \phi^2_{(u_0+u_1+2u_2)/4}$$

Expanding $\mu$ in the above basis, we write

$$\mu = \sum_{i=1}^{4} \alpha_i \mu_i.$$

The condition that $\langle \mu, \gamma_1 \rangle = 0$ is equivalent to

$$\alpha_1 + 10\alpha_2 + 11\alpha_3 + 32\alpha_4 = 0.$$

By symmetry, the condition $(I - P_\mu)(\Lambda^1)^e \perp (\Lambda^1)^{e'}$ for $e \neq e' \in E(\tau^0)$ is equivalent to the condition $\langle (I - P_\mu)\phi^2_{(u_0+u_1)/2}, \phi^2_{(u_1+u_2)/2} \rangle = 0$ which is equivalent to finding nonzero $\mu$ such that

$$\langle \phi^2_{(u_0+u_1)/2}, \phi^2_{(u_1+u_2)/2} \rangle = \frac{\langle \phi^2_{(u_0+u_1)/2}, \mu \rangle \langle \mu, \phi^2_{(u_1+u_2)/2} \rangle}{\langle \mu, \mu \rangle}$$

which is equivalent (for $\alpha_1, \ldots, \alpha_4$ not all zero) to the equation

$$-5(37\alpha_1 + 34\alpha_2 - 25\alpha_3)^2 + 192(3\alpha_1^2 + 2\alpha_1\alpha_2 + 7\alpha_2^2 + 4\alpha_3^2 + (\alpha_1 + 10\alpha_2 + 11\alpha_3)\alpha_4 + 16\alpha_4^2) = 0.$$

The final condition that $(I - P_\mu)(\Lambda^1)^e \perp (\Lambda^1)^v$ (where $v$ is the vertex opposite the edge $e$) is equivalent, by symmetry, to the condition $\langle (I - P_\mu)\phi^2_{(u_0+u_1)/2}, \phi^2_{u_2} \rangle = 0$ which is equivalent to

$$-1103\alpha_1^2 - 2172\alpha_1\alpha_2 - 572\alpha_2^2 + 1430\alpha_1\alpha_3 + 1260\alpha_2\alpha_3 - 119\alpha_3^2 + 64\alpha_1\alpha_4 + 640\alpha_2\alpha_4 + 704\alpha_3\alpha_4 + 1024\alpha_4^2 = 0.$$

Solving the above three equations together with the normalization condition $\langle \mu, \mu \rangle = 1$ gives four distinct real solutions, among which we choose

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{2\sqrt{179}}{3^{1/8}(\frac{77005}{3} - 1640\sqrt{17})^{1/4}} \times \left( \frac{8(-81 + 7\sqrt{17})}{537}, \frac{-32(14 + \sqrt{17})}{537}, \frac{8(-137 + 3\sqrt{17})}{537}, 1 \right).$$

We let $\gamma_4 = \mu$. Fig. 7 shows $\mu$ for the above choice of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

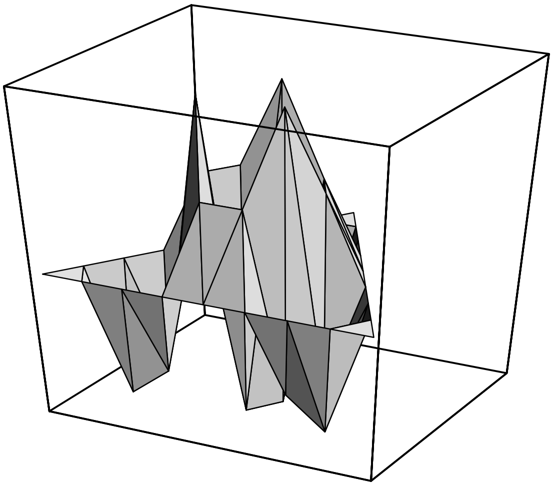

Fig. 7. The component $\gamma_4 = \mu$ of the piecewise linear orthogonal macroelement $\Gamma$.

The function $\gamma_4 = \mu$ is the last of the 'triangle' components of $\Gamma$, that is,

$$\Gamma^{\tau^0} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4 = \mu\}.$$

We next define $\gamma_5 \in \Gamma^{[u_0,u_1]}$ by

$$\gamma_5 := a(I - P_{\Gamma^{\tau^0}})\phi^1_{(u_0+u_1)/2} = a\left(\phi^1_{(u_0+u_1)/2} - \sum_{i=1}^{4} \langle \phi^1_{(u_0+u_1)/2}\gamma_i\rangle\gamma_i\right)$$

where $a$ is chosen so that $\|\gamma_5\| = 1$. Furthermore, let $\gamma_6 := \gamma_5 \circ r_0$, and $\gamma_7 := \gamma_6 \circ r_0$. Then from the construction of $\mu$ it follows that $\{\gamma_1, \ldots, \gamma_7\}$ is an orthonormal set and, furthermore, $\gamma_5 \perp \phi^0_{u_2}$ (and, by symmetry, we also have $\gamma_6 \perp \phi^0_{u_0}$ and $\gamma_7 \perp \phi^0_{u_0}$).

#### 4.2.2. The function $v = \gamma_8$

We next find a function $v \in S(\Delta^1, \Lambda^1)$ that will be the second component of $\Gamma^{[u_0,u_1]}$. We require that (a) $v$ is antisymmetric about with respect to the reflection $r_1$ leaving $\tau^0$ and $e = [u_0, u_1]$ invariant, (b) $v$ is orthogonal to $\gamma_1$, $\gamma_2$, and $\gamma_3$, or equivalently, $v \perp \Gamma^{\tau^0}$ since $v$ is orthogonal to $\gamma_4$ by symmetry, (c) $v$ is orthogonal to $\gamma_6$ (and, hence also $\gamma_7$) (d) $v$ is orthogonal to $v \circ r_0$, and (e) $(I - P_{\mathrm{span}\{v,\gamma_5\}})\phi^0_{u_0} \perp \phi^0_{u_1}$.

First, the antisymmetry of $v \in S(\Delta^1, \Lambda^1)$ implies that $v$ is in a seven dimensional space spanned by the functions

$$v_1 := \phi^2_{(3u_0+u_1)/4} - \phi^2_{(u_0+3u_1)/4} \qquad v_5 := \phi^2_{(2u_0+u_1+u_2)/4} - \phi^3_{(u_0+2u_1+u_2)/4}$$

$$v_2 := \phi^3_{(6u_0+u_1+u_2)/8} - \phi^3_{(u_0+6u_1+u_2)/8} \qquad v_6 := \phi^3_{(3u_0+2u_1+3u_2)/8} - \phi^3_{(2u_0+3u_1+3u_2)/8}$$

$$v_3 := \phi^3_{(5u_0+2u_1+u_2)/8} - \phi^3_{(2u_0+5u_1+u_2)/8} \qquad v_7 := \phi^3_{(2u_0+u_1+5u_2)/8} - \phi^3_{(u_0+2u_1+5u_2)/8}$$

$$v_4 := \phi^3_{(5u_0+u_1+2u_2)/8} - \phi^3_{(u_0+5u_1+2u_2)/8}$$

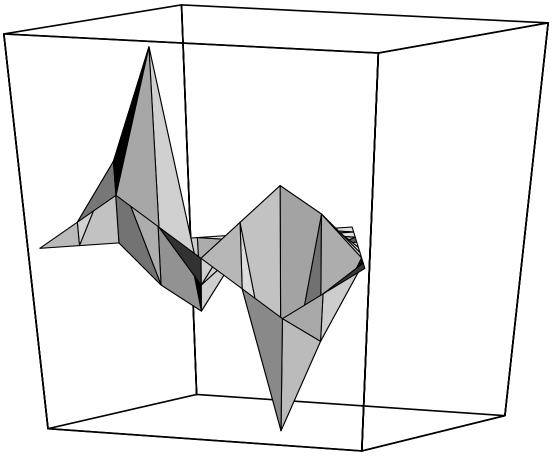

Fig. 8. The component $\gamma_8 = v$ of the piecewise linear orthogonal macroelement $\Gamma$.

Expanding $v$ in terms of $v_1, \ldots, v_7$ we write

$$v = \sum_{i=1}^{7} \beta_i v_i.$$

By symmetry, condition (b) reduces to $v$ to the single linear equation

$$4\beta_1 + \beta_2 + 5(\beta_3 + \beta_4 + 4\beta_5) + 4\beta_6 = 0$$

and the condition (c) reduces to

$$4\beta_1 + 11\beta_2 - 5\beta_3 + 10\beta_4 - \beta_6 + 15\beta_7 = 0.$$

Condition (d) then reduces to the nonlinear equation

$$0 = 4\beta_1^2 + 10\beta_1\beta_2 + 6\beta_2^2 + 2\beta_1\beta_3 + 2\beta_2\beta_3 + \beta_3^2 + 10\beta_1\beta_4 + 2\beta_2\beta_4 + 12\beta_3\beta_4 + \beta_4^2 + 8\beta_1\beta_5$$
$$+2\beta_2\beta_5 + 10\beta_3\beta_5 + 10\beta_4\beta_5 + 20\beta_5^2 + 8\beta_5\beta_6 + 5\beta_6^2 - 8\beta_1\beta_7 - 10\beta_3\beta_7 + 10\beta_4\beta_7$$

and condition (e) reduces to the nonlinear equation

$$0 = 4416\beta_1^2 + 9248\beta_1\beta_2 + 2361\beta_2^2 - 1120\beta_1\beta_3 + 1270\beta_2\beta_3 - 2175\beta_3^2 + 4960\beta_1\beta_4 + 3340\beta_2\beta_4$$
$$+ 100\beta_3\beta_4 - 1500\beta_4^2 - 3200\beta_1\beta_5 - 800\beta_2\beta_5 - 4000\beta_3\beta_5 - 4000\beta_4\beta_5 - 8000\beta_5^2$$
$$- 1728\beta_1\beta_6 - 1242\beta_2\beta_6 - 270\beta_3\beta_6 - 540\beta_4\beta_6 - 3200\beta_5\beta_6 - 1919\beta_6^2 + 2880\beta_1\beta_7$$
$$+ 2070\beta_2\beta_7 + 450\beta_3\beta_7 + 900\beta_4\beta_7 - 270\beta_6\beta_7 - 1775\beta_7^2.$$

Solving these equations numerically we find the numerical solution giving $v$ with $\|v\| = 1$:

$$\beta_1 = 4.012855924 \quad \beta_2 = 3.926833442 \quad \beta_3 = 13.579612984$$
$$\beta_4 = -0.5558588712 \quad \beta_5 = -4.633021348 \quad \beta_6 = 1.8908498152$$
$$\beta_7 = 1.0733941264$$

The resulting $v$ is shown in Fig. 8 and we set $\gamma_8 = v$, $\gamma_9 = \gamma_8 \circ r_0$, and $\gamma_{10} = \gamma_9 \circ r_0$.

Finally, we define $\gamma_{11} = (I - P_{\mathrm{span}\{\gamma_1,\ldots,\gamma_{10}\}})\phi^0_{u_0}$, $\gamma_{12} = \gamma_{11} \circ r_0$, and $\gamma_{13} = \gamma_{12} \circ r_0$. Then it follows that $\Gamma = \{\gamma_1, \ldots, \gamma_{13}\}$ is an orthogonal macroelement with $\Gamma^{\tau^0} = \{\gamma_i\colon i = 1, \ldots, 4\}$, $\Gamma^{[u_0,u_1]} = \{\gamma_i\colon i = 5, \ldots, 10\}$, and $\Gamma^{u_0} = \{\gamma_{11}\}$.

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