An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 40 (2000) 1153–1169
www.elsevier.nl/locate/camwa

Orthogonal Multiwavelets of Multiplicity Four

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(Received May 1999; revised and accepted May 2000)

Abstract—We consider solutions of a system of refinement equations with a 4×1 function vector and three nonzero 4×4 coefficient matrices. We give explicit expressions of coefficient matrices such that the refinement function vector and the corresponding wavelet vector have properties of short support $[0, 2]$, symmetry or antisymmetry, and orthogonality. The properties of convergence of the subdivision scheme, approximation order, and smoothness of the refinement functions are also discussed.

Keywords—Wavelets, Multiwavelets, Approximation order, Refinement equations, Subdivision.

1. INTRODUCTION

Wavelet theory is based on the idea of multiresolution analysis (MRA). Usually, an MRA is generated by one scaling function. However, such wavelets cannot possess the properties of short support, symmetry or antisymmetry, and orthogonality simultaneously. The study of multiwavelets was initiated by Goodman, Lee and Tang in [1]. Then Goodman and Lee in [2] gave a characterization of scaling functions and wavelets. Multiwavelets open new possibilities in the construction of wavelets with those properties based on multiscaling functions. Multiwavelets have more freedom in their construction. Therefore, they can have shorter support with more vanishing moments than a single wavelet, and they may have orthogonality and symmetry at the same time. These properties are very desirable in many applications. Thus, multiwavelets can be very useful for various practical problems (see [3,4], for examples). The literature on this subject is growing rapidly (see [5] and the references therein). In this paper, we consider solutions of a

The authors are very grateful to the anonymous referees for their valuable comments and suggestions. They also would like to thank L. Guan and P. Xiao for their valuable discussions during the revision of the paper and for the numerical computation about the joint spectral radius. The research was supported in part by an RDC grant of ETSU under Grant #00-007/m.

PII: S0898-1221(00)00229-7

system of refinement (scaling) equations in the form

$$\phi(x) = \sum_{k=0}^{2} h_k \phi\,(2x - k)\,,$$

where ϕ is a 4×1 vector function, and h_0, h_1, and h_2 are 4×4 matrices (the sequence $\langle h_k \rangle$ of matrices is called a refinement mask). We give explicit expressions of h_k, $k = 0, 1, 2$ such that the refinement (scaling) function vector ϕ has properties of short support $[0, 2]$, symmetry or antisymmetry and orthogonality. The multiwavelets with the same properties as the scaling functions are also constructed. The paper is organized as follows. In the remainder of this section, we recall some results on multiwavelets construction. In Section 2, we construct a scaling function vector with multiplicity 4 such that it has support on $[0, 2]$, symmetry or antisymmetry, and orthogonality. The multiwavelets with the same properties as the scaling functions are constructed in Section 3. We discuss the properties of approximation order and analyze the smoothness of the refinement functions in the final section.

In general, we are concerned with the system of refinement equations

$$\phi(x) = \sum_{k=0}^{N} h_k \phi\,(2x - k)\,, \tag{1.1}$$

where h_k is an $r \times r$ matrix, and $\phi = (\phi_1, \phi_2, \dots, \phi_r)^\top$ is a vector of functions. The Fourier transform of the above refinement equation is

$$\hat{\phi}(\xi) = H\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right),$$

where

$$H\,(\xi) = \frac{1}{2} \sum_{k=0}^{N} h_k e^{-ik\xi}.$$

We denote

$$M = H(0) = \frac{1}{2} \sum_{k=0}^{N} h_k. \tag{1.2}$$

In the Fourier domain, L^2-stability of ϕ is ensured if and only if the sequences $\{\hat{\phi}_k(\omega + 2\pi\ell)\}_{\ell \in \mathbb{Z}}$, $k = 1, \dots, r$ are linearly independent for each $\omega \in \mathbb{R}$ (see [6]).

If $\phi_1, \dots, \phi_r$ are functions in $L^1(\mathbb{R})$ with stable shifts, it was proved by Dahmen and Micchelli in [7] that the matrix M has a simple eigenvalue 1 and all other eigenvalues of M are less than 1 in modulus. Therefore, we may assume that the $r \times r$ matrix M has the following form:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \qquad \text{and} \qquad \lim_{n \to \infty} \Lambda^n = 0. \tag{1.3}$$

Under assumption (1.3), it was proved by Heil and Colella in [8] that there exists a unique vector ϕ of compactly supported distributions such that ϕ satisfies the refinement equation (1.1) and $\hat{\phi}(0) = (1, 0, \dots, 0)^\top$. Such a solution is called a normalized solution. If ψ is another distribution solution, then we must have $\psi = C\phi$ for some constant C.

Let

$$Qf = \sum_{k=0}^{N} h_k f\,(2 \cdot - k)\,, \qquad f \in L^p(\mathbb{R}).$$

Then Q is a linear operator on $(L^p(\mathbb{R}))^r$ $(1 \le p \le \infty)$. If f is an $r \times 1$ initial vector of compactly supported functions in $L^p(\mathbb{R})$ such that $Q^n f$ converges to the normalized solution ϕ of (1.1) in the L^p-norm ($1 \le p \le \infty$), then the subdivision scheme associated with the mask $\langle h_k \rangle$ is said to be convergent in the L^p-norm.

If the subdivision scheme associated with the mask $\{h_k\}$ converges in the L^p-norm for some p ($1 \le p \le \infty$), it was proved in [9] that $\{h_k\}$ satisfies

$$e_1^\top \sum h_{2k} = e_1^\top \sum h_{2k+1} = e_1^\top, \tag{1.4}$$

where e_1 is the first column of the $r \times r$ identity matrix I_r.

In [10,11], the smoothness conditions of refinement functions were discussed. The method presented in [10] is to connect the optimal smoothness $\nu(\phi)$ to the p-norm joint spectral radius of the block matrices A_ϵ, $\epsilon = 0, 1$, given by $A_\epsilon = (h_{k+2\alpha-\beta})_{\alpha,\beta}$ over a certain finite-dimensional common invariant subspace. When $p = 2$, the optimal smoothness is also given in terms of the spectral radius of the transition matrix associated with the refinement mask.

We call a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ an orthogonal multiresolution analysis (OMRA) of multiplicity r if the following conditions are satisfied.

(a) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$.
(b) $\cup_{j=-\infty}^{\infty} V_j = L_2(\mathbb{R})$, $\cap_{j=-\infty}^{\infty} V_j = \{0\}$.
(c) There exists a vector ϕ of functions in $L^2(\mathbb{R})$ such that $\{2^{j/2}\phi_k(2^j \cdot -\ell);\ k = 1, 2, \dots, r,\ \ell \in \mathbb{Z}\}$ forms an orthonormal basis of V_j.

A function vector ϕ that generates an OMRA with multiplicity $r \ge 1$ is called an orthogonal multiscaling vector. If ϕ is an orthogonal multiscaling vector, then

$$H(\xi) H^*(\xi) + H(\xi+\pi) H^*(\xi+\pi) = I_r,$$

or equivalently,

$$\sum_k h_k h_{k+2m} = 2\delta_{m,0} I_r,\ m \in \mathbb{Z}. \tag{1.5}$$

Let W_j be the orthogonal complements of V_j in V_{j+1}. If there is a function vector $\psi = (\psi_1, \dots, \psi_r)^\top$ such that $\{\psi_\nu(\cdot - \ell);\ \nu = 1, 2, \dots, r,\ \ell \in \mathbb{Z}\}$ forms an orthonormal basis of W_0, then we call functions $\psi_1, \dots, \psi_r$ orthonormal multiwavelets. Then ψ can be expressed as

$$\psi = \sum_k g_k \phi(2 \cdot -k)$$

and $\{g_k\}$ satisfies the following equations:

$$\sum_k h_k g_{k+2m} = 0, \qquad m \in \mathbb{Z}, \tag{1.6}$$

$$\sum_k g_k g_{k+2m} = 2\delta_{m,0} I_r, \qquad m \in \mathbb{Z}. \tag{1.7}$$

2. MULTISCALING FUNCTIONS

Let ϕ be the normalized solution of (1.1). Then ϕ has support $[0, N]$. If ϕ_i is centrally symmetric on the support $[0, N]$ for any even number i and centrally antisymmetric on $[0, N]$ for any odd number i, then

$$\phi(x) = S\phi(N - x), \tag{2.1}$$

where

$$S = \begin{pmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & (-1)^{r+1} \end{pmatrix}_{r \times r}$$

Using this in the refinement equation (1.1), we obtain

$$\begin{aligned} \phi = S\phi(N - \cdot) &= \sum_k Sh_k \phi(2N - 2 \cdot -k) \\ &= \sum_k Sh_k S\phi(2 \cdot +k - N) = \sum Sh_{N-k} S\phi(2 \cdot -k). \end{aligned}$$

Thus, ϕ will have the desired symmetric property provided that

$$Sh_{N-k}S = h_k, \qquad k \in \mathbb{Z}. \tag{2.2}$$

We hope to find scaling functions with properties of short support, symmetry, and orthogonality. In [12], orthogonal multiscaling functions with multiplicity 2 and support $[0, 2]$ (i.e., $r = 2$ and $N = 2$) were constructed. However, the fact that the multiscaling functions ϕ_1 and ϕ_2 are in $L^2(\mathbb{R})$ with orthogonal shifts were not verified in the paper. Jia, Riemenschneider and Zhou gave the verification in [9] and constructed an entire family of orthogonal double multiwavelets that are continuous and have symmetry. In this paper, we consider the case of $r = 4$ and $N = 2$.

From (1.3), we have

$$h_0 + h_1 + h_2 = 2\begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}, \tag{2.3}$$

where Λ is a 3×3 matrix and all its eigenvalues are less than 1 in modulus. By (1.4), we need

$$e_1^\top (h_0 + h_2) = e_1^\top h_1 = e_1^\top. \tag{2.4}$$

Combining (2.2)–(2.4), we obtain that

$$h_0 = \begin{pmatrix} \frac{1}{2} & t_1 & 0 & t_3 \\ r_1 & h_{22} & h_{23} & h_{24} \\ 0 & h_{32} & h_{33} & h_{34} \\ r_3 & h_{42} & h_{43} & h_{44} \end{pmatrix} =: \begin{pmatrix} \frac{1}{2} & t \\ r & h \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & u \\ 0 & 0 & d_3 & 0 \\ 0 & v & 0 & d_4 \end{pmatrix},$$

and

$$h_2 = Sh_0S.$$

From (1.5), we have

$$\begin{aligned} h_0h_0^\top + h_1h_1^\top + h_2h_2^\top &= 2I_4, \qquad (2.5) \\ h_0h_2^\top &= 0. \qquad (2.6) \end{aligned}$$

Let $\tilde{S} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$. Then $S = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{S} \end{pmatrix}$ and

$$h_2 = Sh_0S = \begin{pmatrix} \frac{1}{2} & t\tilde{S} \\ \tilde{S}r & \tilde{S}h\tilde{S} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -t \\ -r & \tilde{S}h\tilde{S} \end{pmatrix}.$$

Then (2.6) implies

$$\begin{aligned} \frac{1}{4} - tt^\top &= 0, \\ -\frac{1}{2}r^\top + t\tilde{S}h_0^\top \tilde{S} &= 0, \\ \frac{1}{2}r - ht^\top &= 0, \\ -rr^\top + h\tilde{S}h^\top \tilde{S} &= 0. \end{aligned} \tag{2.7}$$

Noticing that $t\tilde{S} = -t$, $\tilde{S}r = -r$, we find that the second and the third equations are equivalent in (2.7), and thus,

$$t_1^2 + t_3^2 = \frac{1}{4}.$$

$$\begin{aligned} h_{22}t_1 + h_{24}t_3 &= \frac{1}{2}r_1, \\ h_{32}t_1 + h_{34}t_3 &= 0, \\ h_{42}t_1 + h_{44}t_3 &= \frac{1}{2}r_3, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} h_{22}^2 - h_{23}^2 + h_{24}^2 &= r_1^2, \\ h_{22}h_{32} - h_{23}h_{33} + h_{24}h_{34} &= 0, \\ h_{22}h_{42} - h_{23}h_{43} + h_{24}h_{44} &= r_1r_3, \\ h_{32}^2 - h_{33}^2 + h_{34}^2 &= 0, \\ h_{32}h_{42} - h_{33}h_{43} + h_{34}h_{44} &= 0, \\ h_{42}^2 - h_{43}^2 + h_{44}^2 &= r_3^2. \end{aligned} \tag{2.9}$$

Now let us consider equation (2.5). Suppose h_1 is a diagonal matrix. Then

$$h_0h_0^\top + h_2h_2^\top = h_0h_0^\top + Sh_0h_0^\top S = 2I_4 - h_1h_1^\top$$

is again a diagonal matrix. Noticing that the matrix $h_0h_0^\top + Sh_0h_0^\top S$ has the form of

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix},$$

we obtain a diagonal matrix by setting the $(2,4)$ entry to zero. This implies

$$r_1r_3 + h_{22}h_{42} + h_{23}h_{43} + h_{24}h_{44} = 0. \tag{2.10}$$

Together with the third equation of (2.9), we have

$$\begin{aligned} h_{22}h_{42} + h_{24}h_{44} &= 0, \\ r_1r_3 + h_{23}h_{43} &= 0. \end{aligned} \tag{2.11}$$

Solving this set of equations, we have

$$\begin{aligned} h_{23} &= r_1\beta, & h_{43} &= -\frac{r_3}{\beta}, \\ h_{22} &= \sigma h_{24}, & h_{42} &= -\frac{h_{44}}{\sigma}, \end{aligned} \tag{2.12}$$

where β, σ are constant. If we choose $t_3 = \alpha t_1$, then $(1+\alpha^2)t_1^2 = 1/4$. From (2.8), we have

$$h_{24} = \frac{1}{2}\frac{r_1}{t_1(\sigma+\alpha)}, \qquad h_{32} = -\alpha h_{34}, \qquad \text{and} \qquad h_{44} = \frac{r_3}{2t_1(\alpha - 1/\sigma)} = \frac{\sigma r_3}{2t_1(\sigma\alpha - 1)}.$$

From the fourth equation of (2.9), we have

$$h_{33}^2 = \left(\alpha^2+1\right) h_{34}^2 = \frac{h_{34}^2}{4t_1^2},$$

or equivalently, $h_{33} = h_{34}/2t_1$. By solving other equations of (2.9), we obtain

$$\beta = \frac{1-\sigma\alpha}{\sigma+\alpha}.$$

Finally, h_1 can be determined by using the equation $h_1h_1^{\mathrm{T}} = 2I_4 - h_0h_0^{\mathrm{T}} - Sh_0h_0^{\mathrm{T}}S$. Therefore, we have the following.

THEOREM 2.1. *Suppose that β and $\sigma+\alpha$ are nonzero and $\beta = (1-\sigma\alpha)/(\sigma+\alpha)$, $(\alpha^2+1)t^2 = 1/4$. Let*

$$h_0 = \begin{pmatrix} \frac{1}{2} & t & 0 & \alpha t \\ r_1 & \frac{\sigma r_1}{2t(\sigma+\alpha)} & r_1\beta & \frac{r_1}{2t(\sigma+\alpha)} \\ 0 & -\alpha r_2 & \frac{r_2}{2t} & r_2 \\ r_3 & \frac{r_3}{2t(\sigma+\alpha)\beta} & -\frac{r_3}{\beta} & -\frac{\sigma r_3}{2t(\sigma+\alpha)\beta} \end{pmatrix}, \tag{2.13}$$

$$h_1 = \begin{pmatrix} 1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{pmatrix}, \qquad \text{and} \qquad h_2 = Sh_0S, \tag{2.14}$$

for

$$d_2^2 = 2 - 4\left(\beta^2+1\right) r_1^2, \qquad d_3^2 = 2 - \frac{r_2^2}{t^2}, \qquad \text{and} \qquad d_4^2 = 2 - 4\left(\beta^{-2}+1\right) r_3.$$

Then h_0, h_1, h_2 satisfy the orthogonal conditions (2.5) and (2.6).

REMARK. If $\beta = 0$ or $\sigma + \alpha = 0$, there also exist solutions for (2.5) and (2.6). For example, if $\beta = 0$, we can follow a similar procedure as above and find a solution as follows.

$$h_0 = \begin{pmatrix} \frac{1}{2} & t & 0 & \alpha t \\ r_1 & 2tr_1 & 0 & 2t\alpha r_1 \\ 0 & -\alpha r_2 & \frac{r_2}{2t} & r_2 \\ 0 & -\alpha r_3 & \frac{r_3}{2t} & r_3 \end{pmatrix},$$

$$h_1 = \operatorname{diag}(1, d_2, d_3, d_4), \qquad \text{and} \qquad h_2 = Sh_0S,$$

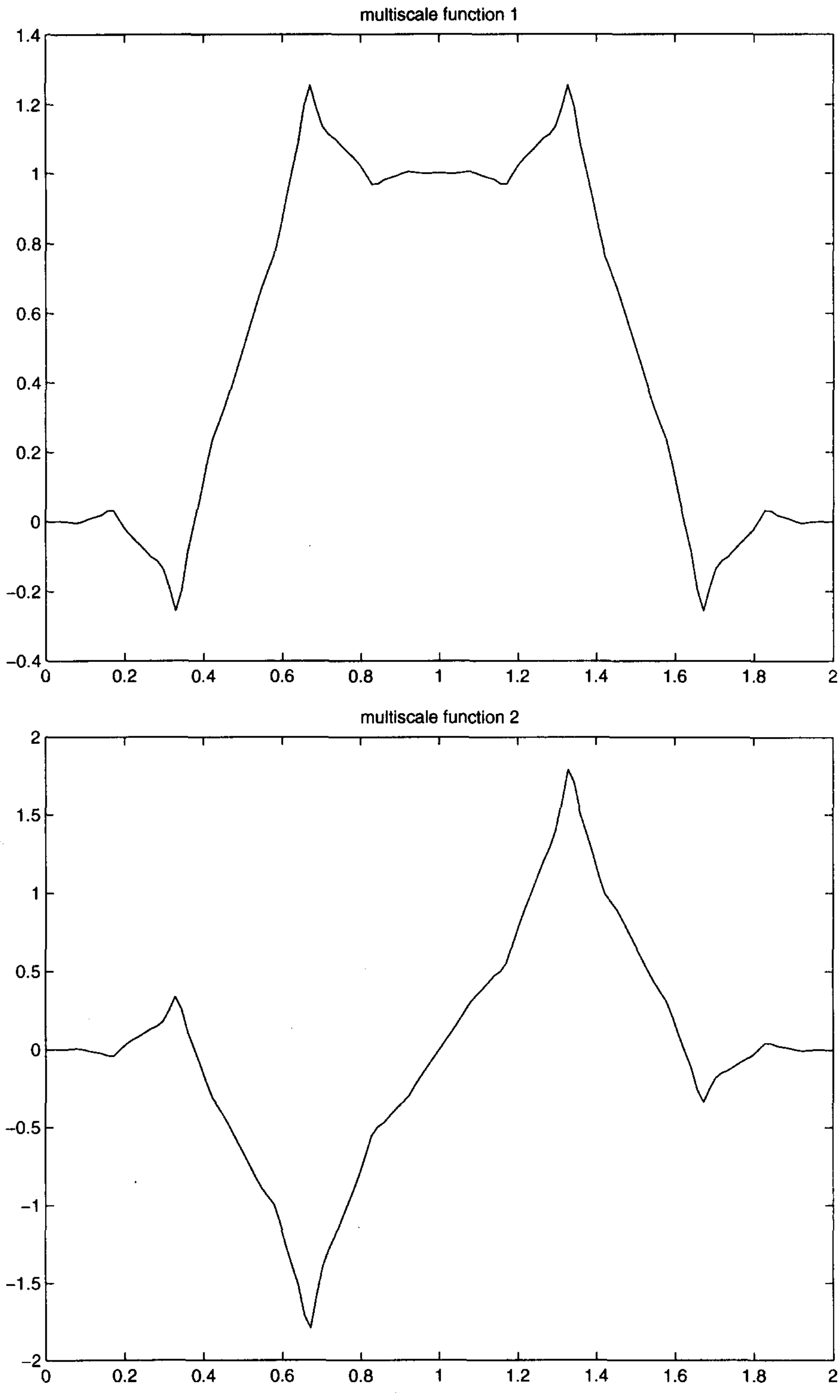

Figure 1. Two of the four multiscaling functions.

for

$$d_2^2 = 2 - 4r_1^2, \qquad d_3^2 = 2 - \frac{r_2^2}{t^2}, \qquad \text{and} \qquad d_4^2 = 2 - \frac{r_3^2}{t^2}.$$

Figures 1 and 2 show a set of multiscaling functions for a choice of free parameters. It is clear that the multiscaling functions are orthogonal, either symmetric or antisymmetric, and having support on $[0, 2]$.

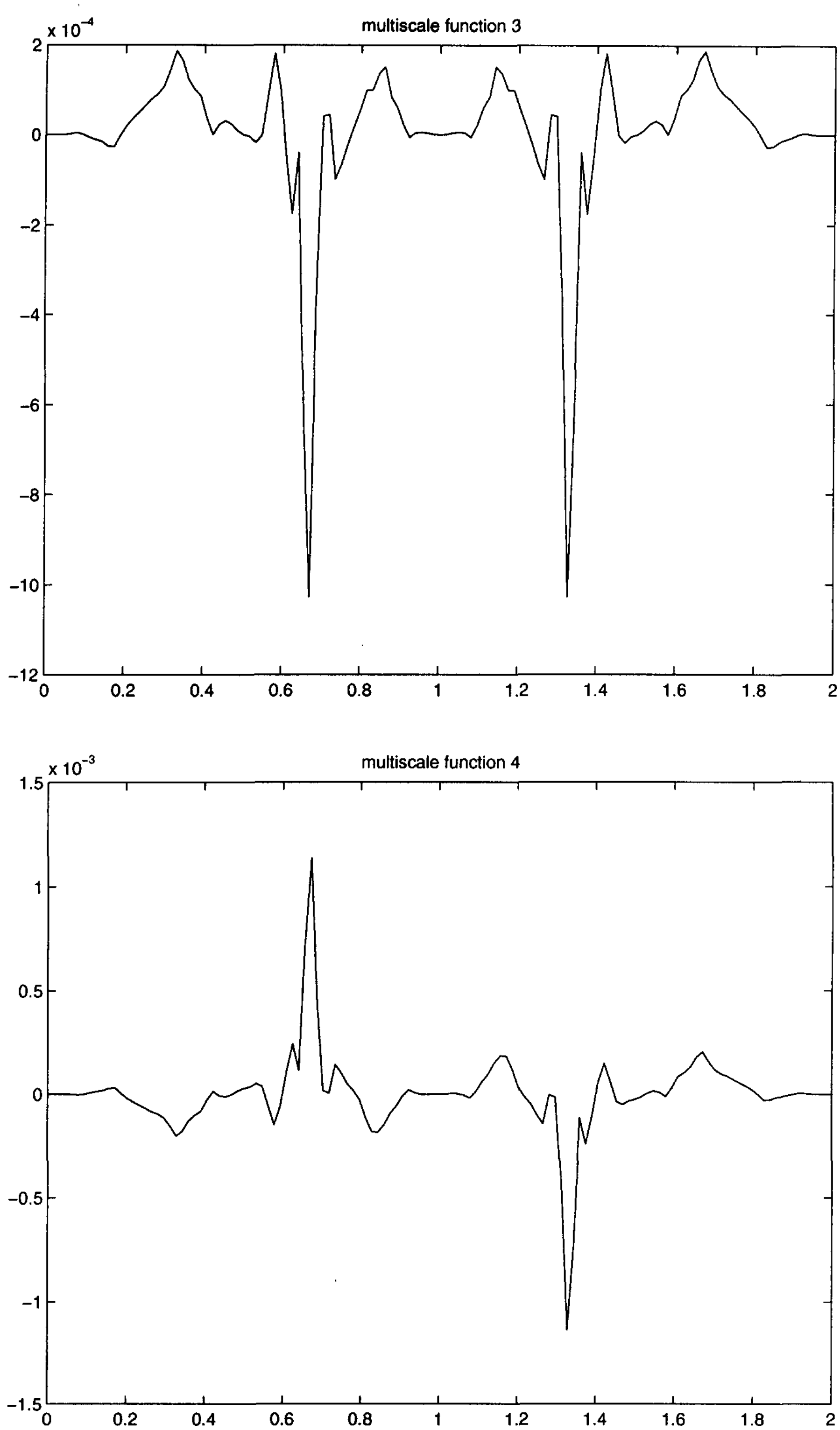

Figure 2. Two of the four multiscaling functions.

The following result (see [9, Theorem 8.1]) gives a necessary and sufficient condition that the system of scaling functions and their shifts becomes an orthonormal system in $L^2(\mathbb{R})$.

THEOREM 2.2. *The set of scaling functions $\phi_1, \ldots, \phi_4$ and their shifts becomes an orthonormal system in $L^2(\mathbb{R})$ if and only if*

(i) *the discrete orthogonal conditions in (2.5) and (2.6) are satisfied for the mask* $\langle h_k \rangle$, $k = 0, 1, 2$, *and*
(ii) *the corresponding subdivision scheme converges in the* L^2*-norm.*

Therefore, we'd like to check the convergence of the corresponding subdivision scheme associated with the mask $\langle h_k \rangle$, $k = 0, 1, 2$ constructed above. This can be done by using the following result corresponding to Theorem 7.1 in [9].

THEOREM 2.3. *Let* $h_0 = (h_{ij})_{i,j=1}^{4,4}$, $h_1 = (d_{i,j})_{i,j=1}^{4,4}$, *and*

$$h_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then the subdivision scheme associated with the mask $\langle h_k \rangle$, $k = 0, 1, 2$ *converges in the* L^2*-norm if and only if the following conditions are satisfied.*

(i) $h_{21} = h_{31} = h_{41} = 0$,
(ii) $d_{32} = d_{42} = h_{32} = h_{42} = 0$ *or* $d_{23} = d_{24} = h_{23} = h_{24} = 0$,
(iii) $d_{43} = h_{43} = 0$ *or* $d_{34} = h_{34} = 0$, *and*
(iv) *if* $C_i = \begin{pmatrix} d_{ii} & 0 \\ h_{ii} & h_{ii} \end{pmatrix}$, $B_i = \begin{pmatrix} h_{ii} & h_{ii} \\ d_{ii} & 0 \end{pmatrix}$, *and the 2-norm joint spectral radius* $\rho_i = \rho_2(B_i, C_i)$ *for* $i = 2, 3, 4$, *then* $\rho_i < \sqrt{2}$ *for* $i = 2, 3, 4$.

3. MULTIWAVELETS FUNCTIONS

Let h_0, h_1, and h_2 be 4×4 matrices which satisfy the orthogonal conditions (2.5) and (2.6), and

$$\phi(x) = \sum_{k=0}^{2} h_k \phi(2x - k).$$

Suppose ϕ generates an MORA and ψ is the corresponding orthogonal multiwavelet vector of the form

$$\psi(x) = \sum_{k=0}^{2} g_k \phi(2x - k). \tag{3.1}$$

Then g_0, g_1, and g_2 must satisfy

$$\begin{aligned} g_0 g_0^\top + g_1 g_1^\top + g_2 g_2^\top &= 2I, & g_0 g_2^\top &= 0, \\ g_0 h_0^\top + g_1 h_1^\top + g_2 h_2^\top &= 0. & g_0 h_2^\top &= 0. \end{aligned} \tag{3.2}$$

Suppose that S_0, S_1, and S_2 are diagonal matrices and $g_i = S_i h_i$ for $i = 0, 1, 2$. Then we have

$$g_0 g_2^\top = S_0 h_0 h_2^\top S_2 = 0$$

and

$$g_0 h_2^\top = S_0 h_0 h_2^\top = 0.$$

In order to ensure that the multiwavelet functions are symmetric, we need $S_0 = S_2$. This gives

$$g_2 = S_2 h_2 = S_0 S h_0 S = S S_0 h_0 S = S g_0 S.$$

By (2.5), we have

$$h_0 h_0^\top + h_2 h_2^\top = 2I - h_1 h_1^\top.$$

Substituting into (3.2), we obtain

$$\begin{aligned} S_1 h_1 h_1^\top S_1 &= 2I - S_2 \left(2I - h_1 h_1^\top\right) S_2, \\ S_1 h_1 h_1^\top &= -S_2 \left(2I - h_1 h_1^\top\right). \end{aligned} \tag{3.3}$$

Solving for S_1 from the second equation of (3.3), we have

$$S_1 = -S_2 \left(2I - h_1 h_1^\top\right) \left(h_1 h_1^\top\right)^{-1}. \tag{3.4}$$

Then solving for S_2^2 from the first equation of (3.3), we obtain that

$$S_2^2 = \left(h_1 h_1^\top\right) \left(2I - h_1 h_1^\top\right)^{-1}. \tag{3.5}$$

Hence, we obtain the following.

THEOREM 3.1. *Suppose h_i, $i = 0,1,2$ satisfy (2.5) and (2.6) and $h_1 h_1^\top$, $2I - h_1 h_1^\top$ are invertible diagonal matrices. Let $g_i = S_i h_i$ for $i = 0,1,2$ and S_2^2 and S_1 satisfy (3.5) and (3.4), respectively. Then g_i, $i = 0,1,2$ satisfy (3.2) and (3.3).*

For h_i, $i = 0,1,2$ described in Theorem 2.1, we have

$$S_2^2 = \begin{pmatrix} 1 & & & \\ & d_2^2 & & \\ & & d_3^2 & \\ & & & d_4^2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 2-d_2^2 & & \\ & & 2-d_3^2 & \\ & & & 2-d_4^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & & & \\ & \dfrac{d_2^2}{2-d_2^2} & & \\ & & \dfrac{d_3^2}{2-d_3^2} & \\ & & & \dfrac{d_4^2}{2-d_4^2} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \dfrac{d_2^2}{4\left(\beta^2+1\right) r_1^2} & & \\ & & \dfrac{d_3^2 t^2}{r_2^2} & \\ & & & \dfrac{d_4^2}{4\left(\beta^{-2}+1\right) r_3^2} \end{pmatrix}.$$

Thus,

$$S_2 = \begin{pmatrix} 1 & & & \\ & \dfrac{d_2}{2r_1\sqrt{\beta^2+1}} & & \\ & & \dfrac{d_3 t}{r_2} & \\ & & & \dfrac{d_4}{2r_3\sqrt{\beta^{-2}+1}} \end{pmatrix}.$$

Then, we have

$$S_1 = -S_2 \left(2I - h_1 h_1^\top\right) \left(h_1 h_1^\top\right)^{-1} = -S_2 \begin{pmatrix} 1 & & & \\ & \dfrac{2-d_2^2}{d_2^2} & & \\ & & \dfrac{2-d_3^2}{d_3^2} & \\ & & & \dfrac{2-d_4^2}{d_4^2} \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & & & \\ & \dfrac{2r_1\sqrt{\beta^2+1}}{d_2} & & \\ & & \dfrac{r_2}{d_3 t} & \\ & & & \dfrac{2r_3\sqrt{\beta^{-2}+1}}{d_4} \end{pmatrix}.$$

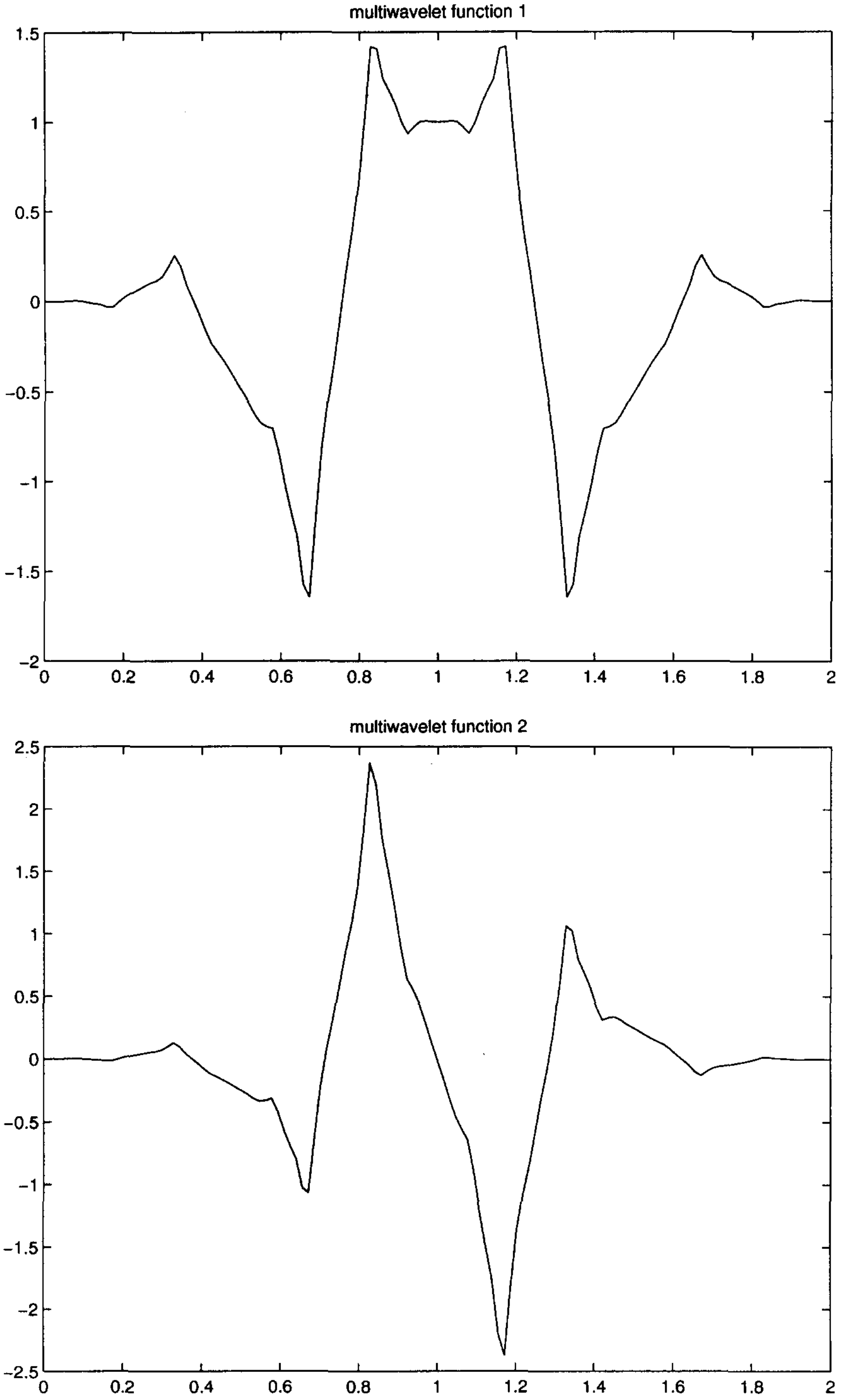

Figure 3. Two of the four multiwavelet functions.

The following result [1, Theorem 8.2] gives a characterization on $\langle g_k \rangle$, $k = 0, 1, 2$ for the orthonormality of the shifts of $\psi_1, \dots, \psi_4$.

THEOREM 3.2. *Let $\langle h_k \rangle$, $k = 0, 1, 2$ be chosen such that the matrix $M = \sum_{k=0}^{2} h_k/2$ satisfies (1.3), and let $\phi = (\phi_1, \dots, \phi_4)^{\mathsf{T}}$ be the normalized solution of the refinement equation*

$$\phi = \sum_{k=0}^{2} h_k \phi (2 \cdot -k),$$

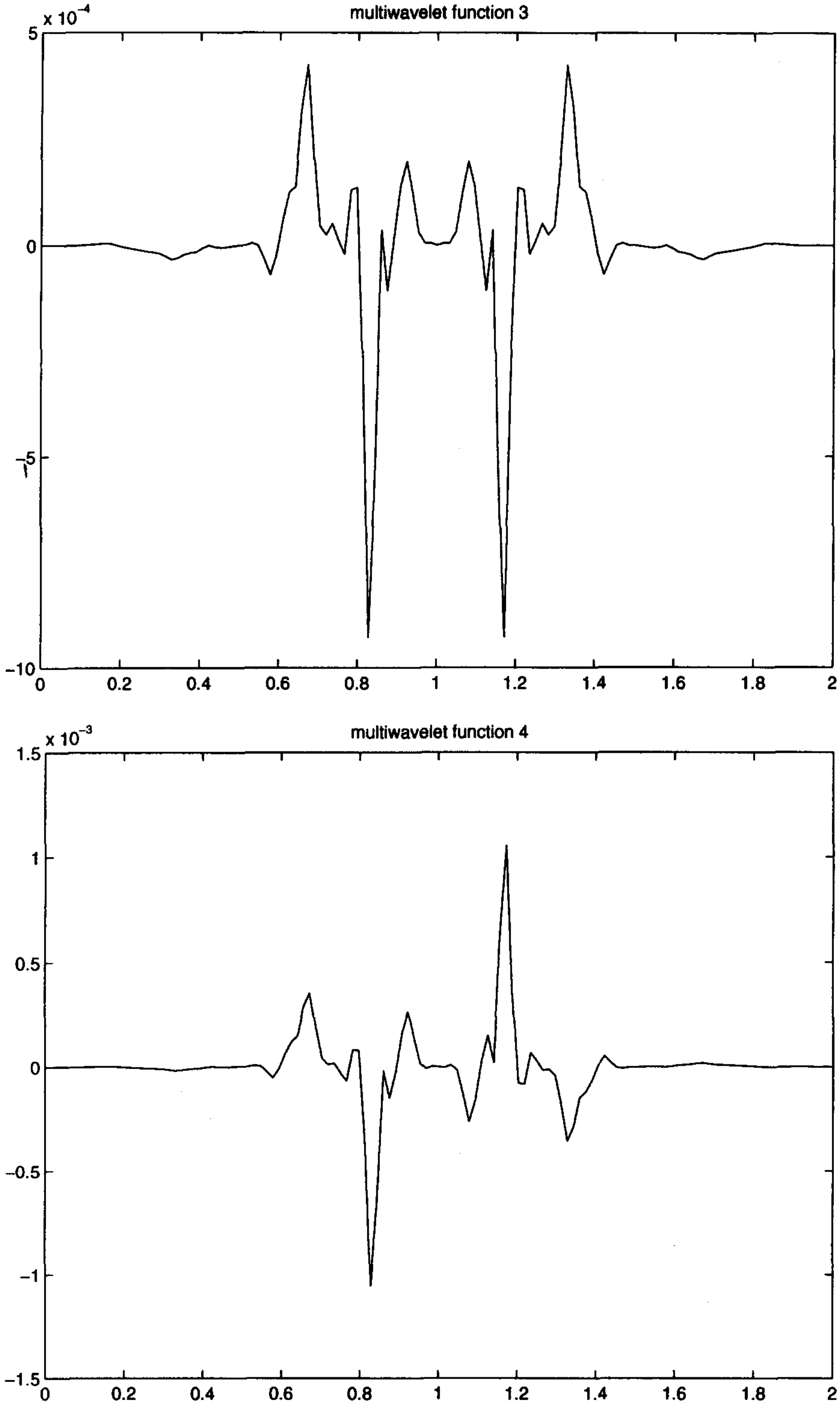

Figure 4. Two of the four multiwavelet functions.

for which $\{\phi_j(\cdot - k);\ j = 1,\dots,4, k = 0,1,2\}$ *forms an orthonormal system in* $L^2(\mathbb{R})$. *Let* $\psi = (\psi_1,\dots,\psi_4)$ *be the vector given by (3.1). Then* ψ *becomes a vector of multiwavelets if and only if condition (3.2) is satisfied.*

Figures 3 and 4 show the multiwavelets by a choice of a set of parameters.

4. APPROXIMATION ORDER OF REFINEMENT FUNCTIONS

A characterization of approximation order of multiscaling (refinement) vectors using symbol and mask was given as follows in [13].

PROPOSITION 4.1. *Assume that ϕ is an integrable solution of the matrix refinement equation (1.1) such that the integer translates of $\phi_1, \phi_2, \dots, \phi_r$ are linearly independent. Then ϕ has approximation order p if and only if there are p vectors $U_0, U_1, \dots, U_{p-1}$ satisfying the following equations:*

$$\sum_k Y_k^{(j)} h_{2k+1} = 2^{-j} \sum_{m=0}^{j} \binom{j}{m} Y_0^{(m)} \tag{4.1}$$

and

$$\sum_k Y_k^{(j)} h_{2k} = 2^{-j} U_j, \tag{4.2}$$

for $j = 0, 1, \dots, p-1$. Here

$$Y_\ell^{(j)} = \sum_{m=0}^{j} \binom{j}{m} (-\ell)^{j-m} U_m. \tag{4.3}$$

In this case, we have

$$t^j = \sum_k Y_k^{(j)} \phi(t+k).$$

In the situation discussed in the previous two sections, the proposition can be rewritten as the following.

THEOREM 4.2. *Let*

$$h_0 = \begin{pmatrix} \frac{1}{2} & t_1 & 0 & t_3 \\ r_1 & h_{22} & h_{23} & h_{24} \\ 0 & h_{32} & h_{33} & h_{34} \\ r_3 & h_{42} & h_{43} & h_{44} \end{pmatrix}, \qquad h_1 = \begin{pmatrix} 1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{pmatrix},$$

and $h_2 = S h_0 S$ for $S = \operatorname{diag}(1, -1, 1, -1)$ and $d_3 = 1/4$. Then the corresponding multiscaling vector ϕ has approximation order three if and only if one of the following conditions are satisfied.

(1) $d_2 = 1/2,\ t_3(2h_{22} - 1/2) = 2h_{24}t_1,\ 4r_1t_3 = h_{24}$.
(2) $d_4 = 1/2,\ t_1(2h_{44} - 1/2) = 2h_{42}t_3,\ 4r_3t_1 = h_{42}$.
(3) $d_2 = d_4 = 1/2,\ (r_1\ r_2)\, A^{-1} \binom{t_1}{t_2} = 1/8$.

Here,

$$A = \begin{pmatrix} 2h_{22} - \dfrac{1}{2} & 2h_{42} \\ 2h_{24} & 2h_{44} - \dfrac{1}{2} \end{pmatrix}$$

is invertible.

PROOF. We derive only the first condition. Conditions (2) and (3) can be derived similarly. In the case we are discussing, equations (4.1) and (4.2) become

$$U_0 h_1 = U_0 \qquad \text{and} \qquad U_0(h_2 + h_0) = U_0,$$

for $j = 0$ and the solution is

$$U_0 = (a_0, 0, 0, 0)^\top, \tag{4.4}$$

for some $a_0 \in \mathbb{R}$.

If $j = 1$, then the corresponding equations are

$$\begin{aligned} U_1 h_1 &= \frac{1}{2}(U_0 + U_1), \\ U_1 h_0 + (U_1 - U_0) h_2 &= \frac{1}{2} U_1. \end{aligned} \tag{4.5}$$

Let $U_1 = (\, u_{11} \; u_{12} \; u_{13} \; u_{14} \,)^{\mathsf{T}}$. Then from the first equation of (4.5), we have

$$u_{11} = a_0 \quad \text{and} \quad \left(d_i - \frac{1}{2}\right) u_{1i} = 0, \qquad \text{for } i = 2, 3, 4. \tag{4.6}$$

By using the second equation of (4.5), we have

$$\begin{aligned} \left(2h_{22} - \frac{1}{2}\right) u_{12} + 2h_{42} u_{14} &= -a_0 t_1, \\ \left(2h_{33} - \frac{1}{2}\right) u_{13} &= 0, \\ 2h_{24} u_{12} + \left(2h_{44} - \frac{1}{2}\right) u_{14} &= -a_0 t_3. \end{aligned} \tag{4.7}$$

Combining (4.6) and the second equation of (4.7), we obtain

$$u_{13} = \begin{cases} 0, & d_3 \neq \dfrac{1}{2} \text{ or } h_{33} \neq \dfrac{1}{4}, \\ \text{an arbitrary number}, & \text{otherwise}. \end{cases}$$

Clearly, $u_{12}^2 + u_{14}^2 \neq 0$, since $t_1^2 + t_3^2 \neq 0$. We divide the discussion into the following three cases.

CASE 1. If $u_{12} \neq 0$ but $u_{14} = 0$, then by (4.6) and (4.7), we have

$$d_2 = \frac{1}{2}, \qquad t_3 \left(2h_{22} - \frac{1}{2}\right) = 2h_{24} t_1, \tag{4.8}$$

and

$$u_{12} = -\frac{a_0 t_3}{2h_{24}}. \tag{4.9}$$

CASE 2. If $u_{12} = 0$ but $u_{14} \neq 0$, then from (4.6) and (4.7) again, we obtain

$$d_2 = 4 = \frac{1}{2}, \qquad t_1 \left(2h_{44} - \frac{1}{2}\right) = 2h_{42} t_3, \tag{4.10}$$

and

$$u_{14} = -\frac{a_0 t_1}{2h_{42}}. \tag{4.11}$$

CASE 3. If $u_{12} \neq 0$ and $u_{14} \neq 0$, then equations in (4.6) and (4.7) imply that

$$\begin{gathered} d_2 = \frac{1}{2}, \qquad d_4 = \frac{1}{2}, \\ \begin{pmatrix} u_{12} \\ u_{14} \end{pmatrix} = \begin{pmatrix} 2h_{22} - \dfrac{1}{2} & 2h_{42} \\ 2h_{24} & 2h_{44} - \dfrac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} -a_0 t_1 \\ -a_0 t_3 \end{pmatrix}. \end{gathered} \tag{4.12}$$

Now, let us consider the vector U_2. In this case ($j = 2$), equations (4.4) and (4.5) become

$$U_2 h_1 = \frac{1}{4}(U_0 + 2U_1 + U_2) \tag{4.13}$$

and

$$U_2 h_0 + (U_0 - 2U_1 + U_2)\, h_2 = \frac{1}{4} U_2. \tag{4.14}$$

Denote $U_2 = (\, u_{21}\ u_{22}\ u_{23}\ u_{24}\,)^\top$. Then (4.13) implies that

$$u_{12} = a_0 \quad \text{and} \quad \left(d_i - \frac{1}{4}\right) u_{2i} = \frac{1}{2} u_{1i}, \qquad \text{for } i = 2,3,4. \tag{4.15}$$

Consequently,

$$u_{22} = 2u_{12}, \qquad u_{24} = 2u_{14}. \tag{4.16}$$

Equation (4.14) can be rewritten as

$$U_2 \left(h_0 + h_2 - \frac{1}{4} I\right) = (2U_1 - U_0)\, h_2.$$

That is,

$$\begin{aligned} \frac{3}{4} a_0 &= \frac{1}{2} a_0 - 2r_1 u_2^{(1)} - 2r_3 u_4^{(1)}, \\ \left(2h_{22} - \frac{1}{4}\right) 2u_{12} + 4h_{42} u_{14} &= -a_0 t_1 + 2h_{22} u_{12} + 2h_{42} u_{14} - 2h_{32} u_{13}, \\ \left(2h_{33} - \frac{1}{4}\right) u_{23} &= -2h_{23} u_{12} + 2h_{33} u_{13} - 2h_{43} u_{14}, \\ 4h_{24} u_{12} + \left(2h_{44} - \frac{1}{4}\right) 2u_{14} &= -a_0 t_3 + 2h_{24} u_{12} + 2h_{44} u_{14} - 2h_{34} u_{13}. \end{aligned} \tag{4.17}$$

If $u_{13} = 0$, then from (4.7), we see that the second and the fourth equations are satisfied. By (4.15), we have $d_3 = 1/4$.

u_{23} can be determined by the third equation of (4.17). Therefore, as long as the first equation is satisfied, the vector U_2 will satisfy equations (4.13) and (4.14). For this purpose, we repeat the discussion of the three cases again. Since the discussions are similar, we provide only the discussion of Case 1. In this case, equations (4.9) and (4.17) yield that

$$4r_1 t_3 = h_{24}.$$

Therefore, if one of the Conditions (1)–(3) is satisfied, then there are vectors U_0, U_1, and U_2 satisfying (4.1) and (4.2). Therefore, ϕ has approximation order of three according to Proposition 4.1. On the other hand, if the vectors U_0, U_1, and U_2 satisfy (4.5), (4.13), and (4.14), then h_i must satisfy the first equation of (4.17) and so satisfy one of Conditions (1)–(3). This completes the proof. ∎

As to the approximation order of the multiscaling functions we considered in the Section 2, we have the following.

THEOREM 4.3. *Let h_i, $i = 0,1,2$ be determined as in Theorem 2.1 and ϕ be the corresponding scaling vector of functions. Then ϕ has approximation order of only two.*

PROOF. Clearly, the approximation order of ϕ is at least two from Proposition 4.1 and the proof of Theorem 4.2. In the following, we apply Theorem 4.2 to show that the approximation order of ϕ cannot be three. For this purpose, we need to check that none of the three conditions in Theorem 4.2 is satisfied. We check only Condition (1) here. The fact that ϕ does not satisfy the other two conditions can be similarly verified.

Clearly, Condition (1) in Theorem 4.2 can be rewritten as

$$
\begin{aligned}
2-4\left(\beta^{2}+1\right) r_{1}^{2} &= \frac{1}{4}, \\
\alpha h_{22}-h_{24} &= \frac{\alpha}{4}, \\
h_{24} &= 4 \alpha r_{1} t.
\end{aligned}
\tag{4.18}
$$

Noticing that $h_{24}=r_{1}/2t(r+\alpha)$, $h_{22}=\sigma r_{1}/2t(\sigma+\alpha)$, we have

$$
\begin{aligned}
r_{1}^{2}\left(\beta^{2}+1\right) &= \frac{7}{16}, \\
\frac{\beta r_{1}}{t} &= -\frac{\alpha}{2}, \\
\frac{1}{2(\sigma+\alpha)} &= 4 \alpha t^{2}.
\end{aligned}
\tag{4.19}
$$

Combining the third equation of (4.19) with $4t^{2}=1/(\alpha^{2}+1)$, we have

$$
\sigma=-\frac{\alpha^{2}-1}{2 \alpha}.
$$

Substituting it into $\beta=(1-\alpha\sigma)/(\sigma+\alpha)$, we obtain that $\beta=\alpha$. Together with the second equation of (4.19), we have $r_{1}=-t/2$, and thus,

$$
\left(\beta^{2}+1\right) r_{1}^{2}=\frac{1}{16}.
$$

This contradicts with the first equation of (4.19). ∎

In [10], a characterization of smoothness of the refinement functions is given in terms of the corresponding refinement mask. The authors there use the generalized Lipschitz space to measure smoothness of a given function. As usual, the difference operator ∇_{y} for function f and $y\in\mathbb{R}$ is defined by $\nabla_{y} f=f(\cdot)-f(\cdot-y)$, and the modulus of continuity of f in $L^{p}(\mathbb{R})$ $(1\leq p\leq\infty)$ is defined by

$$
\omega(f, t)_{p}=\sup_{|y| \leq t}\left\|\nabla_{y} f\right\|_{p}, \qquad t \geq 0.
$$

For a positive integer k, the k^{th} modulus of smoothness of $f\in L^{p}(\mathbb{R})$ is defined by

$$
\omega_{k}(f, t)_{p}=\sup_{|y| \leq t}\left\|\nabla_{y}^{k} f\right\|_{p}, \qquad t \geq 0.
$$

Let $\nu>0$. The *generalized Lipschitz space* $\operatorname{Lip}^{*}(\nu, L^{p}(\mathbb{R}))$ is the collection of those functions $f\in L^{p}(\mathbb{R})$ for which

$$
\omega_{k}(f, t)_{p} \leq C t^{\nu}, \qquad \forall t>0 \quad \text{and for some} \quad k>\nu,
$$

where C is a positive constant independent of t.

By $(\operatorname{Lip}^{*}(\nu, L^{p}(\mathbb{R})))^{r}$, we denote the linear space of all vectors $f=(f_{1},\ldots,f_{r})^{\top}$ such that $f_{1},\ldots,f_{r}\in\operatorname{Lip}^{*}(\nu, L^{p}(\mathbb{R}))$. The optimal smoothness of a vector $f\in(L^{p}(\mathbb{R}))^{r}$ in the L^{p}-norm is described by its critical exponent $\nu_{p}(f)$ defined by

$$
\nu_{p}(f):=\sup\left\{\nu ;\ f \in\left(\operatorname{Lip}^{*}\left(\nu, L^{p}(\mathbb{R})\right)\right)^{r}\right\}.
$$

For the refinement functions constructed in Section 2, we followed the iteration steps described in [10] and obtained that $\nu_{2}(\phi)\geq 1/2-\log_{2} a\approx 0.01462$, where the mask $a(k)=h_{k}$ for $k=0,1$, and 2.

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