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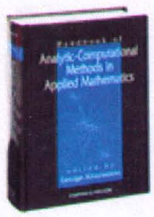

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On Scattered Data Representations Using Bivariate Splines

Abstract
The objective of this paper is to present a study of scattered data representation using bivariate splines. First, we open a discussion with emphasis on the optimal order of approximation. When the polynomial degree is allowed to be sufficiently large as compared to the order of smoothness, it is shown that the spline elements can be used to represent scattered data with the optimal order of approximation over arbitrary triangulations. In real applications, the polynomial degree is required to be lower, it is necessary to find a so-called optimal triangulation so that the spline space can achieve the optimal approximation order. We present an algorithm to transform an arbitrary triangulation of the sample points into an optimal triangulation for representation of the scattered data using C^1 quartic splines. Then, we consider the possibilities to find optimal triangulations for even lower degree spline spaces such as C^1 cubic and C^1 quadratic spaces. Some interpolation schemes and stable local basis construction are also presented. Finally, we mention some recent results on representing scattered data using other spline elements, such as splines on spheres and natural splines.

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Contents

0.1 Introduction

In many applications, it is desirable to approximate a given surface with a high degree of accuracy. Scattered data on the surface may be collected by recording the distance from sample points in a fixed plane to the surface. Once the scattered data has been collected, it is necessary to determine simple functions to interpolate, or best fit, the data. An ideal choice for these simple functions is splines, also called piecewise polynomial (*pp*) functions.

Since a bivariate spline is piecewise-defined over its planar domain, it is necessary to create a partition of the sample points in the plane. One of the most applicable partitions in this case is triangulation.

DEFINITION 0.1
A triangulation of a finite set V of n sample points $\mathbf{v}_i = (x_i, y_i)$, $i = 1, \cdots, n$ in a plane $\mathbb{R}^2$ is defined as a collection Δ of triangles τ satisfying (i) the vertices of the triangles are precisely the sample points $\mathbf{v_i}$; (ii) the union of the triangles in Δ is a connected set, and (iii) the intersection of any two adjacent triangles in Δ is either a common vertex or a common edge. The vertex set of the triangulation Δ will be denoted as V.

In general, for a given set V of data sites, there are many different triangulations with vertex set V. On a triangulation Δ of a polygonal domain $\Omega \subset \mathbb{R}^2$ with vertex set V, one of most important problems in application is to represent scattered data defined on V by C^r smooth spline functions. Of course, one usually wants to find an optimal triangulation of the given sample sites. Though the notion of optimality depends on the desirable properties in the approximation or modeling problems, Here, we are concerned with optimal order of approximation with respect to the given order r of smoothness and degree k of the polynomial pieces of the smooth spline functions. In the study of spline functions on a triangulation Δ, the notation $S_k^r(\Delta)$ is used to denote the subspace of $C^r(\Omega)$ of all *pp* functions with total degree $\leq k$ and with grid lines given by the edges of Δ.

In scattered data representation using splines in the space $S_k^r(\Delta)$, it is critical to answer the question of how well the splines can approximate classes of smooth data. We give the definition of the approximation order of a function space S as follows.

DEFINITION 0.2 *The approximation order of a space S of functions on $\mathbb{R}^2$ is defined to be the largest real number ρ for which*

$$\operatorname{dist}(f, S) \leq \text{Const}\|D^{k+1}f\| \ |\Delta|^{\rho} \tag{1}$$

for any sufficiently smooth function f, with the distance measured in the maximum norm $\| \ \|$, and with the mesh size $|\Delta| := \sup_{\tau\in\Delta} \operatorname{diam} \tau$.

It is clear that the full order of approximation from the spline space $S_k^r(\Delta)$ cannot be better than $k+1$ regardless of r and is trivially $k+1$ in the case $r = 0$.

We use the term *optimal triangulation* of a given set V of data sites to mean that (i) the set V of sample sites is the same as the set of vertices of the triangulation, and (ii) the space of *pp* functions with degree k and smoothness order r on this triangulation achieves the full order of approximation. More precisely, we make the following.

DEFINITION 0.3 *For a given set V of data sites, the degree k and the smoothness order r, any triangulation Δ with vertex set V is called optimal (of type (k, r)) if the spline space $S_k^r(\Delta)$ admits full approximation order $k+1$.*

In addition to the huge volume of research published on representing scattered data using multivariate splines, there are several survey articles that are related to this area (cf. Schumaker [61] and [62], Barnhill [7], Franke [32], Alfeld [1], Dahmen and Micchelli [25], Böhm [8], and Hong [**?**]). Currently, box splines, thin-plate splines, and radial-basis functions are among the most commonly used tools for scattered data interpolation. However, from the computational point of view, a simple and efficient multivariate spline interpolation scheme for scattered data is still not available. This give rise the problem to study lower degree spline spaces and to find locally supported basis elements for the optimal order of approximation.

This problem, however, is extremely complicated, and a general approach does not seem to be feasible. Here, in this paper, we first consider to find optimal triangulations for C^1 quartic spline spaces. We present an algorithm to transform an arbitrary triangulation of the sample points into an optimal triangulation for representation of the scattered data using C^1 quartic splines. Then, we also open the discussions to even lower degree spline spaces such as C^1 cubic and C^1 quadratic spaces. Some interpolation schemes and stable local basis construction are also presented. Finally, we present some recent results on representing scattered data using other spline elements such as splines on sphere and natural splines.

The outline of this paper is as follows. Results on approximation order of spline spaces over arbitrary triangulations will be first discussed. Optimal triangulations and the algorithm to create optimal triangulations based on arbitrarily given data sites for C^1 quartic spline spaces will be introduced in Section 3. The discussion of optimal triangulations for C^1 cubic and quadratic spline spaces and some other spline spaces will be in Section 4. Section 5 will be devoted to the study of existence of local basis and the local linear independence of basis functions. The approaches of scattered data representation using splines on sphere and natural splines are mentioned in the last section.

0.2 Approximation order of spline spaces over triangulations

As usual, let $\mathbb{R}$ be the set of all real numbers and $\mathbb{Z}_+$ the set of nonnegative integers. Thus $\mathbb{R}^3$ denotes the 3-dimensional Euclidean space and $\mathbb{Z}_+^3$ can be used as a multi-index set, while $\pi_k(\mathbb{R}^2)$ is the space of all polynomials of (total) degree $\le k$ in two variables. Let $\tau = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ be a proper triangle with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbb{R}^2$. Then for any $\mathbf{x} \in \mathbb{R}^2$, we have

$$\mathbf{x} = \xi_0\mathbf{v}_0 + \xi_1\mathbf{v}_1 + \xi_2\mathbf{v}_2 \quad \text{with} \quad \xi_0 + \xi_1 + \xi_2 = 1.$$

The 3-tuple $\xi = (\xi_0, \xi_1, \xi_2)$ is called the barycentric coordinate of $\mathbf{x}$ with respect to the triangle τ. For $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}_+^3$, the length of α is defined by $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2$, and the factorial $\alpha!$ is defined as $\alpha_0!\alpha_1!\alpha_2!$. We define the Bernstein Polynomial $B_{\alpha,\delta}$ as

$$B_{\alpha,\tau}(\mathbf{x}) = \binom{|\alpha|}{\alpha}\xi^\alpha,$$

where $\xi^\alpha = \xi_0^{\alpha_0}\xi_1^{\alpha_1}\xi_2^{\alpha_2}$ and

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_0!\alpha_1!\alpha_s!}.$$

Moreover, we define the (domain) points

$$\mathbf{x}_{\alpha,\tau} := \frac{\alpha_0\mathbf{v}_0 + \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2}{|\alpha|}, \quad |\alpha| = k. \tag{2}$$

It is well known that any polynomial $p \in \pi_k$ can be written in a unique way as

$$p = \sum_{|\alpha|=k} b_{\alpha,\tau} B_{\alpha,\tau},$$

where $b_{\alpha,\tau}$ is called the B-net ordinate of p with respect to triangle τ. This gives rise to a mapping $b : \mathbf{x}_{\alpha,\tau} \to b_{\alpha,\tau}$, $|\alpha| = k$. Such a mapping b is called the B-net representation of p with respect to triangle τ.

Now, let us discuss the B-net representation of bivariate splines. Let Δ be a triangulation of a polygonal domain in $\mathbb{R}^2$ and $S_k^0(\Delta)$ the space of all continuous splines of degree k on Δ. Assume $s \in S_k^0(\Delta)$. On each triangle $\tau \in \Delta$, s agrees with some polynomial $p \in \pi_k$. Thus, we have

$$s|_\tau = \sum_{|\alpha|=k} b_{\alpha,\tau} B_{\alpha,\tau}.$$

Let X denote the set of all (domain) points $\mathbf{x}_{\alpha,\tau}$ as defined in (2). Then a mapping can be defined as follows:

$$b_s: \ \mathbf{x}_{\alpha,\tau} \mapsto b_{\alpha,\tau}, \quad |\alpha| = k, \ \tau \in \Delta. \tag{3}$$

Such a mapping b_f is called the B-net representation of the spline s.

It is well-known that to each triangle $\tau \in \Delta$, the matrix

$$(B_{\alpha,\tau}(\mathbf{x}_{\beta,\tau}))_{|\alpha|=k,|\beta|=k}$$

is invertible. Thus, the linear system

$$\sum_{|\gamma|=k} c_{\alpha,\gamma} B_{\beta,\tau}(\mathbf{x}_{\gamma,\tau}) = \delta_{\alpha,\beta} := \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

has a unique solution.

Since this linear system depends only on the barycentric coordinates of $\mathbf{x}_{\alpha,\tau}$, the solution $\{c_{\alpha,\beta}\}$ is independent of τ. Let $[\ \cdot\ ]$ denote the point-evaluation functional, namely:

$$[\mathbf{x}_{\alpha,\tau}]f := f(\mathbf{x}_{\alpha,\tau}).$$

Then it is easy to see that the functionals

$$L_{\alpha,\tau} := \sum_{|\gamma|=k} c_{\alpha,\gamma}[\mathbf{x}_{\gamma,\tau}], \quad \alpha \in \mathbb{Z}_+^3, \quad |\alpha| = k,$$

form a dual basis of $\{B_{\alpha,\tau}; |\alpha| = k\}$ in the sense of

$$L_{\alpha,\tau} B_{\beta,\tau} = \delta_{\alpha,\beta}, \quad |\alpha| = |\beta| = k.$$

Furthermore, there is a positive constant C_k, depending only on the degree k, such that

$$||L_{\alpha,\tau}|| := \sup_{||f||_\infty = 1} ||L_{\alpha,\tau} f||_\infty = \max_{|\beta|=k} |c_{\alpha,\beta}| \leq C_k, \tag{4}$$

for $\alpha \in \mathbb{Z}_+^3$, $|\alpha| = k$. From (4) and the fact that $b_s(\mathbf{x}_{\alpha,\tau}) = L_{\alpha,\tau}\, s$, we have the following.

LEMMA 0.1

If $s \in S_k^0(\Delta)$ and $b_s \in \mathbb{R}^X$ is the B-net representation of s, then

$$||s||_\infty \leq ||b_s||_\infty \leq C_k ||s||_\infty. \tag{5}$$

Now, let $\tau = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ and $\widetilde{\tau} = [\mathbf{v}_0, \mathbf{v}_1, \widetilde{\mathbf{v}}_2]$ be two triangles in Δ with common edge $e = [\mathbf{v}_0, \mathbf{v}_1]$. Let S, S_0, S_1, and S_2 denote the oriented areas of the triangles τ, $[\widetilde{\mathbf{v}}_2, \mathbf{v}_1, \mathbf{v}_2]$, $[\mathbf{v}_0, \widetilde{\mathbf{v}}_2, \mathbf{v}_2]$, and $\tilde{\tau}$, respectively. The following result, which describes C^r-smoothness conditions on a spline function s in terms of its B-net representation, can be found in [44] (see also [10], [16], etc.).

THEOREM 0.1

Suppose that a bivariate spline function s is defined on the union of two triangles $\tau \cup \widetilde{\tau}$ by

$$s|_\tau = \sum_{|\alpha|=k} b(\mathbf{x}_{\alpha,\tau}) B_{\alpha,\tau};$$

$$s|_{\tilde{\tau}} = \sum_{|\alpha|=k} b(\mathbf{x}_{\alpha,\tilde{\tau}}) B_{\alpha,\widetilde{\tau}}.$$

Then $s \in C^r(\tau \cup \widetilde{\tau})$ if and only if for all positive integers $\ell \leq r$ and $\gamma = (\gamma_u, \gamma_v, 0) \in \mathbb{Z}_+^3$ with $|\gamma| = k - \ell$,

$$b(\mathbf{x}_{\gamma+\ell\mathbf{e}^3,\tilde{\tau}}) = \sum_{|\beta|=\ell} \binom{\ell}{\beta} b(\mathbf{x}_{\gamma+\beta,\tau}) \left(\frac{S_0}{S}\right)^{\beta_0} \left(\frac{S_1}{S}\right)^{\beta_1} \left(\frac{S_2}{S}\right)^{\beta_2}, \qquad (6)$$

where $\beta = (\beta_0, \beta_1, \beta_2) \in \mathbb{Z}_+^3$ and $\mathbf{e}^3 = (0,0,1)$.

Let E_I be the set of interior edges of Δ. For $e \in E_I$ and two triangles $\tau = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ and $\widetilde{\tau} = [\mathbf{v}_0, \mathbf{v}_2, \widetilde{\mathbf{v}}_2]$ sharing the common edge $e = [\mathbf{v}_0, \mathbf{v}_1]$, and $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}_+^3$ with $1 \leq \alpha_2 \leq r$, we define the functionals $f_{e,\alpha}$ on $\mathbb{R}^X$, the space of all real functions, by

$$\lambda_{e,\alpha} b = b(x_{\alpha,\tilde{\tau}}) - \sum_{|\beta|=\alpha_{\tilde{w}}} \binom{\alpha_2}{\beta} b(x_{(\alpha_1,\alpha_2,0)+\beta,\tau}) \left(\frac{S_0^{\beta_0} S_1^{\beta_1} S_2^{\beta_2}}{S^{\alpha_2}}\right). \qquad (7)$$

It is clear that the support of the functional $\lambda_{e,\alpha}$ is included in a diamond domain with vertices $\frac{(\alpha_0+\alpha_2)\mathbf{v}_0+\alpha_1\mathbf{v}_1}{k}$, $\frac{\alpha_0\mathbf{v}_0+(\alpha_1+\alpha_2)\mathbf{v}_1}{k}$, $\mathbf{x}_{\alpha,\widetilde{\tau}}$ and $\mathbf{x}_{\alpha,\tau}$.

As an example, we can see that the C^1-smoothness conditions across the edge e for $s \in S_4^1(\Delta)$ is determined by the relation

$$b_{\alpha+\mathbf{e}^3,\widetilde{\tau}} = c_1 b_{\alpha+\mathbf{e}^1,\tau} + c_2 b_{\alpha+\mathbf{e}^2,\tau} + c_3 b_{\alpha+\mathbf{e}^3,\tau}, \qquad (8)$$

where $\alpha = (\alpha_0, \alpha_1, 0) \in \mathbb{Z}_+^3$ with $\alpha_0 + \alpha_1 = 3$, $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ are the standard unit vectors in $\mathbb{R}^3$, $b_{\alpha,\tau} = b_s(\mathbf{x}_{\alpha,\tau})$ is the B-net representation of s, and c_i, $i = 1,2,3$, are the barycentric coordinates of $\tilde{\mathbf{v}}_2$ with respect to τ.

Recall that $\mathbb{R}^X$ is the space of all real functions. Let $|X|$ denote the cardinality of the set X of domain points. Then the spaces $S_k^0(\Delta)$ and $\mathbb{R}^X$ are isomorphism. Therefore, we have

$$\dim(S_k^0(\Delta)) = |X|.$$

Let $\Lambda = \Lambda_k^r$ be the set of all such functionals defined by (7) and

$$\Lambda^\perp = \{b \in R^X;\ \lambda b = 0, \forall \lambda \in \Lambda\}.$$

Then we see that the spaces $S_k^r(\Delta)$ and $(\Lambda_k^r)^\perp$ are isomorphism. Therefore, we have the following.

THEOREM 0.2

$$\dim(S_k^r(\Delta)) = |X| - \dim(\Lambda_k^r).$$

It is certainly not trivial to determine $\dim(\Lambda_k^r)$. However, this theorem is helpful to determine dimension of spline spaces in the case of $k \geq 3r+2$ (see [41]). For some recent progress in the study of dimension problem of spline spaces, please see [42].

B-net representation of splines can also be applied to study approximation order of $S_k^r(\Delta)$. For this purpose, we define L_∞ norm on linear space $S_k^r(\Delta)$ and ℓ_∞ norm on linear space $\mathbb{R}^X$. Then these two norms are equivalent by (5). Since $\mathbb{R}^X$ is finitely dimensional, the dual space of $\mathbb{R}^X$ has norm ℓ_1. For $g \in S_k^0(\Delta)$, using the dual theorem in functional analysis, we have

$$\text{dist}(g, S_k^r(\Delta)) = \sup_{\lambda \in \Lambda} \frac{|\lambda g|}{\|\lambda\|}. \tag{9}$$

If f is a continuous function, then there is a unique function $g \in S_k^0(\Delta)$ such that f and g have the same values at points in X. Then $f \mapsto g$ gives a projection operator P from C to $S_k^0(\Delta)$. From (9), we have

THEOREM 0.3

Let f be a continuous function. Then

$$|\text{dist}(f, S_k^r(\Delta)) - \sup_{\lambda \in \Lambda} \frac{|\lambda P f|}{\|\lambda\|}| \leq \|f - Pf\|.$$

In the above theorem, $\|f - Pf\|$ turns to be a local approximation problem and it is not so difficult to determine its approximation order. Thus, the key point is to determine

$$\sup_{\lambda \in \Lambda} \frac{|\lambda P f|}{\|\lambda\|}.$$

In general, it is well known that the approximation order of $S_k^r(\Delta)$ not only depends on k and r, but also on the geometric structure of the partition Δ. According to the results of finite elements analysis in [68] and [9], it was believed in the past that the full approximation order of $\rho = k+1$ can be obtained from the spline space $S_k^r(\Delta)$ only when the degree of the polynomial k is at least $4r+1$. de Boor and Höllig [14] applied this theorem and proved the following (see also [21] for a constructive proof).

THEOREM 0.4
For $k \geq 3r+2$ and sufficiently smooth function f, there is a constant Const *which only depends on the smallest angle of the partition Δ such that*

$$\mathrm{dist}(f, S_k^r(\Delta)) \leq \mathrm{Const}|\Delta|^{k+1}\|f\|_{k+1,\infty}.$$

Chui, Hong, and Jia in [21] provided a constructive scheme to achieve this optimal approximation order based on a stable basis of $S_k^r(\Delta)$ for the case of $k \geq 3r+2$. Therefore, any triangulation Δ is optimal for the spline space $S_k^r(\Delta)$ as long as $k \geq 3r+2$. It is natural to ask that if the condition $k \geq 3r+2$ is sharp. In other words, is there any pair of integers k and r with $k \leq 3r+1$ such that any triangulation is optimal for $S_k^r(\Delta)$? The first result in this direction was obtained by de Boor and Höllig (see [12]). They proved that $S_3^1(\Delta^{(1)})$ has approximation order 3 instead of 4. Here $\Delta^{(1)}$ stands for a three-directional mesh (also called a type-1 triangulation) which is formed by a uniform rectangular partition plus all northeast diagonals. Later, de Boor and Jia in [15] considered the approximation order of spline spaces over the three-direction mesh $\Delta^{(1)}$ for general smoothness order r. They obtained the following.

THEOREM 0.5
For a three-directional mesh $\Delta^{(1)}$, the approximation order of the space $S_k^r(\Delta^{(1)})$ is at most k provided that $k \leq 3r+1$.

Theorem 4 shows that if k is sufficiently large compared to r, the spline space $S_k^r(\Delta)$ provides the full accuracy expected of piecewise polynomials of degree k. If $k \leq 3r+1$, generally speaking, the space $S_k^r(\Delta)$ does not have full order of approximation as shown in Theorem 5 for Δ is the three-direction mesh $\Delta^{(1)}$. But, it is not known what its generic approximation order is. de Boor conjectured that bivariate C^1 cubic spline space has approximation order 0 generically (see [11], sect. 7.).

Recall that a triangulation formed from a uniform rectangular partition by drawing both northeast diagonals and northwest diagonals are is called a type-2 or four-direction mesh, and it is denoted by $\Delta^{(2)}$. Notice that the intersections of diagonals are also in the vertex set V.

For a type-2 triangulation $\Delta^{(2)}$, Dahmen and Micchelli [26] proved that the space $S_4^1(\Delta^{(2)})$ arrives at the optimal approximation order 5. More general, for a type-2 triangulation $\Delta^{(2)}$, Jia [46] proved the following general result by considering the local approximation order provided by the box splines in $S_k^r(\Delta^{(2)})$.

THEOREM 0.6

The approximation order of $S_k^r(\Delta^{(2)})$ for a four-direction mesh $\Delta^{(2)}$ is $k+1$ if $r \le 1$ and $k \ge r+1$.

Theorem 6 shows that the space $S_k^1(\Delta^{(2)})$ has full order of approximation if $k \ge 2$. It already covered a later result published in [48] on $S_3^1(\Delta^{(2)})$. This gives rise to a question: is there any optimal triangulation for the spline space $S_k^r(\Delta)$ even though $k \le 3r+1$?

It will be very interesting to find some triangulations Δ, which are somewhat specific but more general than the type-2 triangulation, such that the spline space $S_k^r(\Delta)$ still has full approximation order $k+1$. The question was well-answered for C^1 quartic spline spaces and we will discuss this in some details in the next section.

Similar to the structure of type-2 triangulation, a quadrangulation of a connected polygonal domain Ω in $\mathbb{R}^2$ can be defined as follows.

DEFINITION 0.4 *A collection of quadrilaterals $q_i, i = 1, \cdots, N$ is called a quadrangulation diamond of a Ω if (i) $\Omega = \sum_{i=1}^N q_i$; (ii) the intersection of any two quadrilaterals is either empty, a single vertex, or a common edge; (iii) for any two quadrilaterals q_1, q_n, there is a sequence of quadrilaterals $q_1, \cdots, q_n$ in $\Diamond$ such that each pair q_i, q_{i+1} share exactly one edge with each other. $\Diamond$ is called convex if all quadrilaterals are convex, and $\Diamond$ is said to be nondegenerate if none of the quadrilaterals is a triangle.*

For a nondegenerate convex quadrangulation of a polygonal domain Ω in $\mathbb{R}^2$, we use $\diamondplus$ to denote the triangulation obtained by inserting the diagonals of each quadrilateral of $\Diamond$. Spline spaces defined on triangulated quadrangulations have been studied both in the field of finite elements and spline theory. Finite elements in $S_3^1(\diamondplus)$ were constructed in [33] and [60]. The approximation properties of $S_3^1(\diamondplus)$ were investigated in [23] for L_2 norm and in [49] for L_∞ norm. Finite elements spanning a certain subspace of $S_{3r}^r(\diamondplus)$ for odd integer r and $S_{3r_1}^r(\diamondplus)$ for even integer r were constructed in [47] recently. The approximation properties under L_∞ norm were also studied there. In [52], the following result is obtained.

THEOREM 0.7
For integers $r \geq 1$ and $0 \leq m \leq 3r$. There exists a linear quasi-interpolation operator $Q_m : L_1(\Omega) \mapsto S_{3r}^r(\diamondplus)$ such that

$$\|D_x^\alpha D_y^\beta (f - Q_m f)\|_p \leq \mathrm{Const}|\diamondplus|^{m+1-\alpha-\beta}|f|_{m+1,p},$$

for $1 \leq p \leq \infty$, $0 \leq \alpha + \beta \leq m$, and f in the Sobolev space $W_p^{m+1}(\Omega)$. Here $|\diamondplus|$ is the mesh size of $\diamondplus$.

Therefore, the triangulation $\diamondplus$ is optimal for the spline space $S_k^r(\diamondplus)$ if $k \geq 3r$. Notice the similar structures between the triangulation $\diamondplus$ and a four-directional mesh $\Delta^{(2)}$, we would like to make the following conjecture.

Conjecture. *The triangulation $\diamondplus$ is optimal for the spline space $S_k^r(\diamondplus)$ if $k \geq 2r + 1$.*

0.3 Optimal triangulations for lower-degree bivariate spline spaces

Since lower degree spline spaces are preferable for application purposes, it is beneficial to determine optimal triangulations for $S_k^r(\Delta)$ when $k \leq 3r+1$. This section will specifically focus on the spline space of C^1 quartic splines, $S_4^1(\Delta)$.

Recall that de Boor and Jia proved in 1993 in [15] that the bivariate spline space $S_k^r(\Delta^{(1)})$ attains an approximation order of at most k for $k \leq 3r + 1$. So $\Delta^{(1)}$ is not an optimal triangulation for the spline space $S_k^r(\Delta)$ when $k \leq 3r + 1$. In particular, $S_4^1(\Delta^{(1)})$ attains an approximation order of at most 4, but not the optimal approximation order of 5. So $S_4^1(\Delta^{(1)})$ is not optimal for C^1 quartic splines.

A couple of techniques have been implemented in recent years to determine optimal triangulations for C^1 quartic splines. In 1996, Chui and Hong developed in [19] a scheme known as a *Local Clough-Tocher Refinement Scheme* to transform an arbitrary triangulation of data points into an optimal triangulation for C^1 quartic splines. There some triangles are refined into three subtriangles to become a Clough-Tocher cell. Here, locality means that the Clough-Tocher triangle is applied only to some isolated triangles in Δ, and as usual, a triangle is called a *Clough-Tocher triangle*, if it is subdivided, by using an interior point (such as the centroid of the triangle), into three subtriangles. A interpolation scheme was also constructed there by using certain locally supported Hermite elements, which are called star-vertex splines, to achieve this optimal approximation order.

Generation of an optimal mesh is one of the most important facets in finite element modeling. The method of local Clough-Tocher refinement of triangulations can be undertaken without any element distortion, and the local interpolation schemes will help in drastically decreasing the computational complexity as compared with the standard (global) Clough-Tocher scheme.

However, the disadvantage of this scheme is that it requires the inclusion of additional data points and often in applications no scattered data is available for additional data sites. To avoid introducing some new data points in addition to the vertex set of the triangulation Δ, as in the local Clough-Tocher refinement, Hong and Mohapatra later developed in 1997 in [43] a *mixed three-directional mesh* which is an optimal triangulation for C^1 quartic splines on the existing data points. A rectangle with northeast diagonal is called a *NE-rectangle*. Similarly, a rectangle with northwest diagonal is called a *NW-rectangle*. For a triangulation Δ which consists of NE- and NW-rectangles we may call Δ a mixed three- direction mesh and denote it by $\Delta^{(3)}$. In [43], we obtain the following.

THEOREM 0.8

For a mixed three-direction mesh $\Delta^{(3)}$ there is a linear interpolating operator $T: \ f \in C^1(\Delta^{(3)}) \mapsto s \in S_4^1(\Delta^{(3)})$ such that $Tp = p$ for any polynomial $p \in \pi_4$ and such that T achieves the optimal order of approximation; that is,

$$\|Tg - g\| \leq C\|g^{(5)}\||\Delta^{(3)}|^5, \text{ for } g \in C^5(\Delta^{(3)}), \tag{3.1}$$

where $|\Delta^{(3)}|$ is the mesh size of $\Delta^{(3)}$.

Therefore, the mixed three-directional mesh $\Delta^{(3)}$ is optimal for C^1 quartic spline space $S_4^1(\Delta^{(3)})$. And so, the mixed three-directional mesh $\Delta^{(3)}$ is better than the three-directional mesh in the sense that the corresponding spline space has a higher order of approximation. Also the mixed three-directional mesh $\Delta^{(3)}$ is better than local refinements in the sense that the C^1 quartic spline space achieves the optimal approximation order by using a smaller number of data sites in the interpolation. In comparison, the mixed three-directional mesh using the data only at the intersections of rectangle lines and with the optimal-order 5 can also be achieved by the space $S_4^1(\Delta^{(3)})$. Therefore, the mixed three-directional mesh is also better than the four-directional mesh in this point.

The uniform partition certainly restricts the application of the mixed three-directional elements to the arbitrarily given data points. In [20], we considered C^1 quartic spline space over arbitrary triangulations and provided an efficient method, called *Edge Swapping Algorithm*, for triangulating any finite arbitrarily scattered sample sites, such that for any discrete data given at these sample sites, there is a C^1 quartic polynomial spline on this triangulation that interpolates the given data with the optimal order of approximation. The MatLab complementation of the algorithm and numerical examples were given in [30]. To explain the idea of the Edge Swapping Algorithm here, we need to recall some notation from graph theory. The degree of any vertex $\mathbf{v} \in V$, which we will denote by $\deg(\mathbf{v})$, is the number of edges emanating from v. If $\deg(\mathbf{v})$ is an even integer, then we say that $\mathbf{v}$ is an even-degree vertex; otherwise, $\mathbf{v}$ is called an odd-degree vertex. In addition, an interior vertex $\mathbf{v}$ is called a singular vertex if (i) its degree is 4 and (ii) it is the intersection of two straight line segments. If e_{j-1}, e_j, e_{j+1} are three consecutive edges with a common vertex $\mathbf{v}$, then the edge e_j is called degenerate with respect to $\mathbf{v}$, provided that the two edges e_{j-1} and e_{j+1} are collinear. Now, we are ready to introduce the notion of type-O triangulation.

A vertex $\mathbf{u}$ will be called a *type-O vertex* of a triangulation Δ if $\mathbf{u}$ satisfies at least one of the following.
(a) $\mathbf{u}$ is a boundary vertex of Δ.
(b) $\mathbf{u} \in V_I$ with $deg(\mathbf{u}) = 4$.
(c) $\mathbf{u} \in V_I$ and $deg(\mathbf{u})$ is an odd integer.
(d) $\mathbf{u} \in V_I$ and there exists a vertex $\mathbf{v}$ of Δ that satisfies either (i) $\mathbf{v} \in V_I$ and $deg(\mathbf{v}) = 4$ or $deg(\mathbf{v}) =$ an odd integer, or (ii) $\mathbf{v} \in V_b$, such that $[\mathbf{u}, \mathbf{v}]$ is a nondegenerate edge of Δ with respect to $\mathbf{u}$.

We will use V_O to denote the collection of all type-O vertices in V.

DEFINITION 0.5 *A triangulation of V with only type-O vertices (i.e., $V = V_O$) is called a type-O triangulation.*

The reason for introducing the notion of type-O triangulations is the following (see [20]).

THEOREM 0.9
Any type-O triangulation Δ admits the optimal (5th) order of approximation from $S_4^1(\Delta)$.

As a consequence of the above theorem, we have

COROLLARY 0.1
If a triangulation Δ consists only of odd-degree interior vertices, then the spline space $S_4^1(\Delta)$ yields the optimal order of approximation.

To convert any triangulation to be a type-O triangulation, we introduce a so-called edge swapping algorithm. Every interior edge e of a triangulation Δ is the diagonal of a quadrilateral Q_e which is the union of two triangles

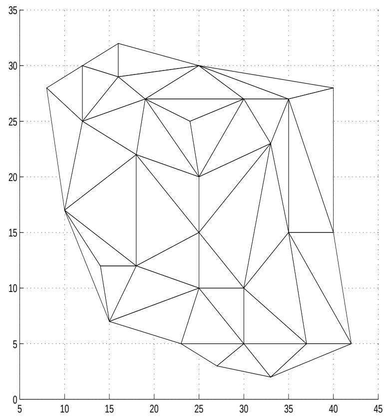

Figure 1. A triangulation Δ

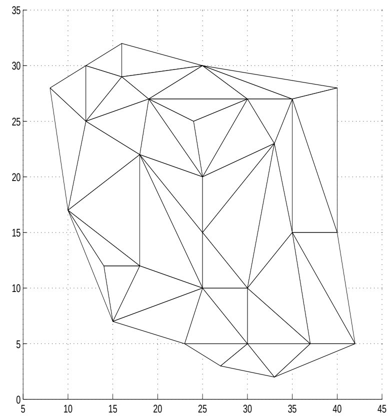

Figure 2. An Optimal triangulation from Δ

of Δ with common edge e. Following [63], we say that e is a swappable edge if Q_e is convex and no three of its vertices are collinear. If an edge e of a triangulation Δ is swappable, then we can create a new triangulation by swapping the edge. That is, if $\mathbf{v}_1, \cdots, \mathbf{v}_4$ are the vertices of Q_e ordered in the counterclockwise direction, and if e has endpoints $\mathbf{v}_1$ and $\mathbf{v}_3$, then the swapped edge has endpoints $\mathbf{v}_2$ and $\mathbf{v}_4$. Two vertices in Δ will be called *neighbors* of each other if they are the endpoints of the same edge in Δ. Hence, while $\mathbf{v}_1$ and $\mathbf{v}_3$ are neighbors in the original triangulation Δ, $\mathbf{v}_2$ and $\mathbf{v}_4$ become neighbors in the new triangulation after the edge e is swapped.

For any given set of sample sites, it is clear that, with the exception of those that are collinear, there is a triangulation with these sample sites as its only vertices. Let Δ be a triangulation associated with the given set V, and let V_O be the set of all type-O vertices in Δ. Set

$$\tilde{V} = V \setminus V_O.$$

If $\mathbf{u} \in \tilde{V}$, then $\mathbf{u}$ and all its neighbors with nondegenerate edges with respect to $\mathbf{u}$ must be even-degree vertices with $deg(\mathbf{u}) \geq 6$. We can see that, for every interior vertex $\mathbf{u}$ with $n := deg(\mathbf{u}) \geq 5$, there is a swappable edge $e \in E_{\mathbf{u}}$. Hence, there is at least one vertex $\mathbf{u}_i$ such that both $\angle \mathbf{u}_{i-1}\mathbf{u}\mathbf{u}_{i+1}$ and $\angle \mathbf{u}_{i-1}\mathbf{u}_i\mathbf{u}_{i+1}$ are less than π. Therefore, the quadrilateral $Q := [\mathbf{u}_{i-1}, \mathbf{u}_i, \mathbf{u}_{i+1}, \mathbf{u}]$ is convex; and hence, the edge $[\mathbf{v}, \mathbf{v}_i]$ is swappable.

Now we are ready to describe our *Edge Swapping Algorithm* for constructing a type-O triangulation $\widehat{\Delta}$, starting with any triangulation Δ.

Swapping Algorithm

Do while ($\tilde{V} \neq \emptyset$)
Pick any vertex $\mathbf{u}$ in $\tilde{V}$ and consider its neighbors.
Pick any neighbor v of $\mathbf{u}$ so that the edge $[\mathbf{u}, \mathbf{v}]$ is swappable.
Swap $[\mathbf{u}, \mathbf{v}]$, yielding a new edge $[\mathbf{u}', \mathbf{v}']$.
Form a subset of $\tilde{V}$ by deleting from $\tilde{V}$ all the neighbors $\mathbf{w}$ of $\mathbf{w}' := \mathbf{u}$, $\mathbf{v}$, $\mathbf{u}'$, or $\mathbf{v}'$, with $[\mathbf{w}, \mathbf{w}']$ being a nondegenerate edge with respect to $\mathbf{w}$.
Call this subset $\tilde{V}$.
Enddo

The new triangulation obtained by applying this Edge Swapping Algorithm is denoted by $\widehat{\Delta}$. It is clear that the triangulations Δ and $\widehat{\Delta}$ have the same number of triangles, singular vertices, interior and boundary vertices, and edges. Hence, it follows that

$$\dim S_4^1(\widehat{\Delta}) = \dim S_4^1(\Delta).$$

$\widehat{\Delta}$ is an optimal triangulation for C^1 quartic spline space. Combining the Edge Swapping Algorithm with Theorem 9, we have the following.

THEOREM 0.10
Every finite set V of sample sites admits an optimal triangulation Δ, such that the C^1 quartic spline space $S_4^1(\Delta)$ has the optimal (fifth) order of approximation.

A MATLAB package which applies the Edge Swapping Algorithm to any triangulation on a finite set of vertices to construct a type-O triangulation of the sample points is described in [30]. The package includes a main function `swap.m` as well as subfunctions `consecv.m, delrow.m, findrow.m, findtri.m, nbors.m` and `trimesh2.m`, a modification of the MATLAB 5.0 function `trimesh.m`.

The `swap` program may be used to effectively implement the Edge Swapping Algorithm on any initial triangulation of sample points for which a triangulation admitting an optimal approximation with C^1 quartic splines is desired. Figure 1 shows a triangulation of some scattered sample points which has been defined in MATLAB using the x and y vectors and the `tri` matrix. This triangulation was transformed by `swap` to the type-O triangulation in Figure 2 with a single edge swap. The first non-type-O vertex encountered by `swap` was located at (25, 15). As the neighbors of this vertex were considered, the neighbor at (18, 22) was the first one found to form a swappable edge. The resulting edge swap was sufficient to create the type-O triangulation in the latter figure.

Recall that the three-directional mesh $\Delta^{(1)}$ is not optimal for C^1 quartic splines. Figure 4 depicts a type-O triangulation resulting from an application of `swap` to the sample $\Delta^{(1)}$ in Figure 3. Since `swap` considers the

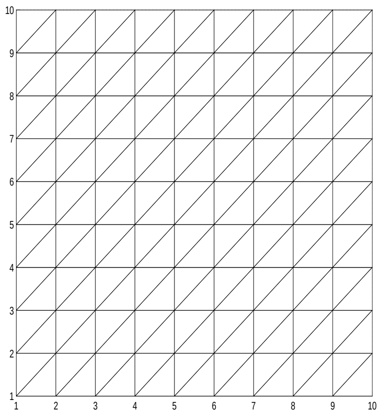

Figure 3. Type-1 triangulation $\Delta^{(1)}$

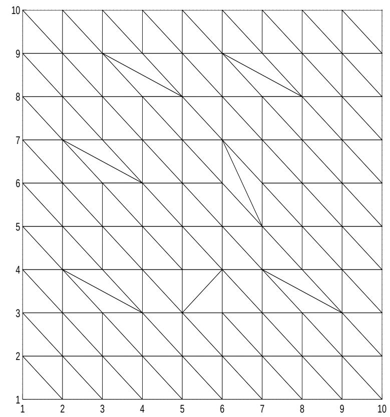

Figure 4. An Optimal triangulation from $\Delta^{(1)}$

vertices of the initial triangulation in sequential order, the type-O triangulation returned by `swap` may be dependent on the order in which the vertices are defined in the x and y vectors. The result in Figure 5 was achieved by ordering the vertices from the bottom to the top of each column, beginning with the leftmost column. Figure 6 depicts a quite different type-O triangulation of this vertex set, where only the direction of the diagonals in the initial triangulation was changed. This illustrates how the output of `swap` on a particular vertex set may be changed, when desirable, by reordering the vertices or altering the initial triangulation.

0.4 Interpolation and Approximation using C^1 cubic and C^1 quadratic splines

In this section, we would like to discuss some possibilities to find the optimal triangulation for C^1 cubic or C^1 quadratic spline functions, and also some results on interpolation using cubic or quadratic spline elements.

As we can see from the previous sections that it is de Boor and Höllig who first applied B-net technique in [12] to study approximation order of C^1 cubic spline space, there they proved that $\Delta^{(1)}$ is not optimal for C^1 cubic splines. We've also seen that type-2 triangulations [46] and the triangulated quadrangulations [50] are optimal triangulations for C^1 cubic splines. From Theorem 6 we know that the type-2 triangulation is also optimal for C^1 quadratic splines. In general, the question of how to find

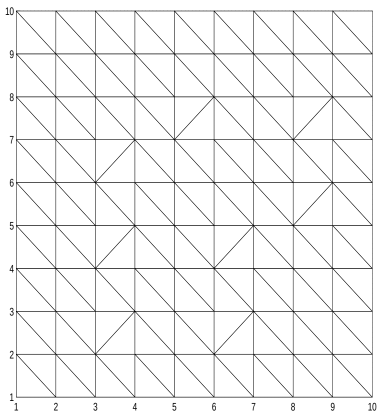

Figure 5. An Optimal triangulation from $\Delta^{(1)}$

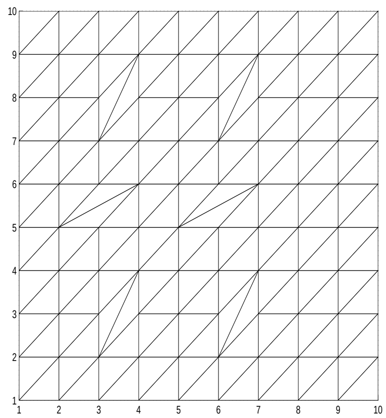

Figure 6. An Optimal triangulation from $\Delta^{(1)}$

an optimal triangulation for the bivariate spline space $S_k^r(\Delta)$ of C^1 cubic or C^1 quadratic is still open and will be a very challenge problem.

It is natural to consider the use of finite elements as interpolant, particularly since there is a large and sophisticated machinery available to handling them. For a piecewise polynomial interpolant to be differentiable globally, its polynomial degree must be at least 5 according to [68]. A well-known technique of reducing that degree, is to subdivide the triangle into three subtriangles. Splitting a triangle about its centroid into three subtriangles and letting the polynomial degree be 3 gives rise to the widely used Clough-Tocher Scheme. Clough-Tocher splits were introduced in [24]. For a given triangulation Δ of a set Ω, we use Δ_{CT} to denote the Clough-Tocher (refinement) triangulation of Δ which is formed by connecting the centroid $\mathbf{v}_\tau$ of each triangle τ in Δ to the three vertices of τ. We can prove the following.

THEOREM 0.11
The Clough-Tocher triangulation Δ_{CT} is optimal for C^1 cubic spline space over Δ_{CT}.

Based on a similar consideration as in [19], we open the following.

Problem 1. *Is there any local Clough-Tocher triangulation Δ_{LCT} so that it is optimal for C^1 cubic spline space over Δ_{LCD}.*

Recall that a mixed three-directional mesh is optimal for C^1 quartic splines. It is certainly interesting to obtain a similar result for C^1 cubic splines.

Problem 2. *Is there any mixed three-directional mesh being optimal for C^1 cubic splines?*

An alternative refinement approach is called Powell-Sabin split, which splits each triangle into six subtriangles. More precisely, we give the following.

DEFINITION 0.6 *Given a triangulation Δ, the Powell-Sabin (refinement) triangulation Δ_{PS} is formed by connecting incenters of triangles of Δ and also by connecting incenters to central points of boundary edges for boundary triangles.*

It is well-known that Powell-Sabin triangulations can be used to study C^1 quadratic splines (see [58]). In [18], some computational schemes and optimization algorithms are introduced for interpolating discrete gridded data by C^1 quadratic spline surfaces that preserve the shape characteristics of the data. Also, some energy functionals are presented there for the characterization of optimal interpolants satisfying the required shape-preservation criteria.

A non-uniform type-1 (three-directional) mesh, we denote it by $\Delta^{(1)}_{MN}$, is constructed from a rectangular grid of a rectangular region $R = [a,b]\times[c,d]$, where

$$a = x_0 < \cdots < x_M = b,\ c = y_0 < \cdots < y_N$$

by drawing in all northeast diagonals. Similarly, $\Delta^{(2)}_{MN}$ will denote the non-uniform type-2 triangulation. It has been attracted attentions to represent scattered data using splines over triangulations $\Delta^{(1)}_{MN}$ and $\Delta^{(2)}_{MN}$. In [65], the following result was obtained.

THEOREM 0.12
For given data $f(\mathbf{v}_{i,j}), f'_x(\mathbf{v}_{i,j}),\ i = 0,1,\cdots,n,\ j = 0,1,\cdots,m;\ f'_y(\mathbf{v}_{i,n}), f''_{xy}(\mathbf{v}_{i,n}),\ i = 2,\cdots,m;\ f'_y(\mathbf{v}_{0,j}),\ f''_{xy}(\mathbf{v}_{1,j}),\ j = 1,\cdots,n;$ and $f'_y(\mathbf{v}_{m,0}), f'_y(\mathbf{v}_{m,1}),\ f'_y(\mathbf{v}_{0,0})$, then there is a unique $s \in S^1_3(\Delta^{(1)}_{MN})$ satisfies the following interpolation conditions:

$$(s(\mathbf{v}_{i,j}), s'_x(\mathbf{v}_{i,j})) = (f(\mathbf{v}_{i,j}), f'_x(\mathbf{v}_{i,j})), i = 0,1,\cdots,n, j = 0,1,\cdots,m;$$

$$(s'_y(\mathbf{v}_{i,n}), s''_{xy}(\mathbf{v}_{i,n})) = (f'_x(\mathbf{v}_{i,n}), f''_{xy}(\mathbf{v}_{i,n})), i = 2,\cdots,m;$$

$$(s'_y(\mathbf{v}_{0,j}), s''_{xy}(\mathbf{v}_{1,j})) = (f'_y(\mathbf{v}_{0,j}), f''_{xy}(\mathbf{v}_{1,j})), j = 1,\cdots,m;$$

$$(s'_y(\mathbf{v}_{m,0}), s'_y(\mathbf{v}_{m,1}), s'_y(\mathbf{v}_{0,0})) = (f'_y(\mathbf{v}_{m,0}), f'_y(\mathbf{v}_{m,1}), f'_y(\mathbf{v}_{0,0})),$$

and also

$$\|f - s\|_\infty \le \mathrm{Const}|\Delta|^2[\omega(D^4 f, |\Delta|) + \|D^4 f\||\Delta|],$$

where $\|D^4 f\| = \|f\|_{4,\infty}$, *and*

$$\omega(D^4 f, |\Delta|) = \max_{0\le i\le 4}\{\omega(\frac{\partial^4 f}{\partial x^i \partial y^{4-i}}, |\Delta|\}.$$

Many authors have also considered scattered data interpolation using bivariate splines over $\Delta_{MN}^{(2)}$. The following results is given by Ye [67] (see also [66]).

THEOREM 0.13
Let $f \in C^3([a,b]\times[c,d])$. *Then there is a quadratic spline function* $s \in S_2^1(\Delta_{MN}^{(2)})$ *satisfying*

$$\|f - s\| \le 5\|D^3 f\||\Delta|^3 + \frac{|\Delta|^2}{16}[\omega_x(f_x^{(3)}, |\Delta|) + \omega_y(f_y^{(3)}, |\Delta|)].$$

Clearly, these interpolation results are not yet ideal since the order of approximation is not optimal due to either the triangulation structure or the interpolation scheme itself. It is interesting to seek both optimal triangulation construction and improving interpolation schemes for optimal order of approximation.

0.5 Stable local basis and local linear independent basis

For applications, one is required to construct an efficient scheme to achieve the full order $k+1$ of approximation. For this purpose, explicit bases for the spaces $S_k^r(\Delta)$ are set up when $k \ge 3r+2$ and an approximation scheme using such bases to achieve the optimal order of approximation was discussed in [39]. However, the bases presented there, as well as in [22], are not stable. For spline space $S_k^r(\Delta)$, a basis $\{B_i\}_i^N$ of $S_k^r(\Delta)$ is said to be a local stable basis if each B_i is locally supported and there exist two positive constants C_1 and C_2, depending only on k and the smallest angle θ of the triangulation Δ, such that

$$C_1 \sup_\ell |a_\ell| \le \left\|\sum_\ell a_\ell B_\ell\right\|_\infty \le C_2 \sup_\ell |a_\ell|.$$

The construction of a local stable basis of a super spline space $S_k^{r,\mu}(\Delta)$ was presented in [21] (see also [40]). The subspace $S_k^{r,\mu}(\Delta)$ of super splines of smoothness r and degree $\leq k$ with enhanced smoothness order $\mu \geq r$ is defined as

$$S_k^{r,\mu}(\Delta) = \{s \in S_k^r(\Delta) : \ s \in C^\mu \text{ at each vertex of } \Delta\}.$$

In [21], the following result is obtained.

THEOREM 0.14
If $k \geq 3r+2$, then there is stable basis $\{B_\ell, \ell = 1, \cdots, N\}$ with $N = \dim(S_k^r(\Delta))\}$, for $S_k^{r,\mu}(\Delta)$. This basis is also local in the sense that, for any ℓ, there exists a vertex $\mathbf{u}$ such that the support of B_ℓ

$$\text{supp } B_\ell \subseteq \overline{St}^{\lfloor r/2 \rfloor + 1}(\mathbf{u}),$$

where the closed star of a vertex $\mathbf{v}$, denoted by $\overline{St}(\mathbf{v}) =: \overline{St}^1(\mathbf{v})$, is the union of all the triangles attached to $\mathbf{v}$, and the m-star of $\mathbf{v}$, denoted by $\overline{St}^m(\mathbf{v})$, is the union of all triangles that intersect with $\overline{St}^{m-1}(\mathbf{v})$, $m > 1$.

The star of a vertex is the set of triangles sharing that vertex. We call splines supported only on the star of a vertex *star-supported* splines.

Recently, Alfeld and Schumaker in [6] proved the following.

THEOREM 0.15
Suppose $r \geq 1$ and $k \leq 3r+1$. Then there are triangulations Δ for which $S_k^r(\Delta)$ does not have a star- supported basis.

The proof of this theorem is based an analysis of spline spaces over a three-directional mesh.

Here, we can make a very easy argument to show the above theorem. Noticing a fact that a spline space over $\Delta^{(1)}$ will have full order of approximation provided that it has a locally supported basis.

By Theorem 5, the spline space $S_k^r(\Delta^{(1)})$ cannot have full order of approximation if $k \leq 3r+1$, therefore, it cannot have locally supported basis and of course, there is no star- supported basis.

Some conjectures were made in [11] on the relations among approximation order of $S_k^r(\Delta)$, that $S_k^r(\Delta)$ contains elements with local support, and that $S_k^r(\Delta)$ contains a local partition of unity. Here, local partition of unity means that a basis $\{B_i\}$ satisfies $\sum_i c_i B_i = 1$ with B_i nonnegative or $\{B_i\}$ is a local supported basis.

DEFINITION 0.7 *A basis $\{B_i\}_{i=1}^N$ of $S_k^r(\Delta)$ is said to be locally linearly independent (LLI) if for every $\tau \in \Delta$, the basis splines $\{B_i\}$ are linear independent on τ.*

Local linear independence was first studied for the integer shift of a box spline (cf. [13], [27], and [45]). Usually, the stability and local linear independence cannot hold simultaneously. An LLI basis for $S_k^r(\Delta)$ when $k \geq 3r+2$ were constructed recently in [29]. Using an LLI basis, a Hermite type interpolation scheme was presented in [28] for $S_k^r(\Delta)$, $k \geq 3r+2$, that possesses optimal approximation order in the same sense as in [21]. That is, the approximation constant does not depend on the geometric structure of Δ. The LLI basis construction is different from the B-net approach. The technique applied in [28] are based on nodal functionals, a common method in finite-element field, and a so-called "weak interpolation" idea introduced in [57]. For comparison, we also mention that an alternative proof for the result of [21] can be found in [51].

0.6 Splines on sphere and natural splines

A special but important and widely encountered problem arises when the data sites lie on a 2-dimensional surface embedded in $\mathbb{R}^3$. The most important instance of such a surface is a sphere. The problems of fitting data on sphere arise in many areas, including for example, geophysics and meteorology where the sphere is taken as a model of the earth. It is often unsatisfactory to project the surface into the plane. Instead, special methods have to be designed. Lawson [54], Renka [59], and Nielson and Ramaraj [56] independently propose schemes based on a triangulation of the surface of a sphere.

Very recently, the spaces of splines defined on triangulations lying on the sphere or on sphere-like surfaces have been discussed in [2] – [5]. These spaces arose out of a new kind of Bernstein-Beźier theory on such surfaces. A constructive theory for such spline spaces analogous to the well-known theory of polynomial splines on planar triangulations has been developed. Formulae for the dimension of such spline spaces, and locally supported bases for them, are given in [3]. Some applications of such spline spaces to fit scattered data on sphere-like surfaces are discussed in [5].

For many years people in the Computer Aided Geometric Design (CAGD) community believed that it was not likely to define barycentric coordinates on a spherical triangle. However, it was recognized in [4] that in fact there is a very natural way to define barycentric coordinates with respect to spherical triangles. It was later discovered that the same coordinates had been introduced and studied by Möbius more that 100 years ago.

The spherical spline space $S_k^r(\Delta)$ is the analog of the space of splines defined over a planar triangulation. As in the planar case, it is possible to identify the dimension of $S_k^r(\Delta)$ and construct locally supported bases for them for all values of $k \geq 3r+2$, see [3]. As shown in [5], the basic interpolation problem can solved as following.

THEOREM 0.16
for given real values $\{f_i\}_{i=1}^n$ at the scattered points $\{\mathbf{v}_i\}_{i=1}^n$ on the unit sphere S, there is a spline function s defined on S such that

$$s(\mathbf{v}_i) = f_i,\ i = 1, \cdots, n.$$

The spherical triangulation Δ has vertices at the given points and the interpolant is local in the sense that the restriction of s to a triangle τ depends only on the data in that triangle.

Here, we need to mention a survey paper [31] of approaches to the interpolation and approximation data on the surface of a sphere. The authors discussed methods based on spherical harmonics, tensor-product spaces on a rectangular map of the sphere, functions defined over spherical triangulations, spherical splines, etc.

A generalized biharmonic spline interpolation scheme for data over sphere was given in [34].

Before we discuss scattered data interpolation using bivariate natural splines, let's recall the definition of the natural spline in the univariate setting at first. For points $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ and integer $m \geq 1$, we define

$$S_m(x_1, \cdots, x_n) = \{s \in C^{m-1}[a,b]; s|_{[x_i,x_{i+1}]} \in \pi_m, i = 0, 1, \cdots, n\}$$

the space of polynomial splines of degree m with n fixed knots $x_1, \cdots, x_n$.

DEFINITION 0.8 *A function $s \in S_{2r+1}(x_1, \cdots, x_n)$, where $r \geq 1$, is called a natural spline of degree $2r+1$ with knots $x_1, \cdots, x_n$, if*

$$s^{(j)}(a) = s^{(j)}(b) = 0,\ j = r+1, \cdots, 2r.$$

Clearly, a natural spline $s \in S_{2r+1}(x_1, \cdots, x_n)$ satisfies that s is a polynomial of degree $2r+1$ over each subinterval (x_i, x_{i+1}) for $i = 1, \cdots, n-1$, a polynomial of degree r over subintervals $[a, x_1)$ and $(x_n, b]$, and that $s \in C^{2r}[a,b]$.

Given a function $f \in C[a,b]$, the natural spline interpolation problem is to determine a natural spline $s \in S_{2r+1}(x_1, \cdots, x_n)$, $r \geq 1$, such that

$$s(x_i) = f(x_i),\ i = 1, \cdots, n.$$

It is well-known that if $n \geq r-1$, then the natural spline interpolation problem has a unique solution from $S_{2r+1}(x_1, \cdots, x_n)$. Furthermore, we have the following optimality properties for the natural interpolating spline function.

THEOREM 0.17
Let $n \geq r-1$, $f \in C^{r+1}[a,b]$ and $s \in S_{2r+1}(x_1,\cdots,x_n)$ be the unique solution of the corresponding natural spline interpolation problem. Then for any natural spline $\widetilde{s} \in S_{2r+1}(x_1,\cdots,x_n)$, we have
(a)
$$\|f^{(r+1)} - s^{(r+1)}\|_2 \leq \|f^{(r+1)} - \widetilde{s}^{(r+1)}\|_2.$$
The equality holds if and only if $s - \widetilde{s} \in \pi_r$.
(b) for any $\mu \in I_f := \{u \in C^{r+1}[a,b];\ u(x_i) = f(x_i),\ i = 1,\cdots,n\}$,
$$\|s^{(r+1)} - \widetilde{s}^{(r+1)}\|_2 \leq \|\mu^{(r+1)} - \widetilde{s}^{(r+1)}\|_2.$$

Let $Q = [a,b]$ and
$$X := H^{r+1}(Q) = \{u(x);\ \frac{d^{r+1}u}{dx^{r+1}} \in L^2(Q),\ \frac{d^\alpha u}{dx^\alpha} \in AC(Q),\ \alpha = 0,\cdots,r\},$$
where $AC(Q)$ is the space of absolutely continuous functions over Q. From the property (b) in Theorem 17, if we choose $\widetilde{s} = 0$, we obtain
$$\|s^{(r+1)}\|_2 \leq \|\mu^{(r+1)}\|_2$$
for any $\mu \in I_f$. Therefore, a natural polynomial spline is the solution of the following problem: given a function $f \in C[a,b]$, find a function $s(x) \in X$ satisfying the interpolation conditions:
$$s(x_i) = f(x_i),\ i = 1,\cdots,n$$
and
$$\int_a^b (s^{(r+1)}(x))^2 dx = \min_{u \in I_f} \int_a^b (u^{(r+1)}(x))^2 dx.$$
Let $Y := L^2(Q)$ and $t : X \mapsto Y$ a linear operator defined by
$$t(u) = u^{(r+1)}(x) = \frac{d^{r+1}u(x)}{dx^{r+1}}.$$
Let $Z := \mathbb{R}^N$ be the N dimensional Euclidean space and $A : X \mapsto Z$ an interpolation operator defined by
$$Au = (u(x_1),\cdots,u(x_N)).$$

Laurent [53] considered the following spline interpolation problem in the Hilbert space $H^{r+1}(Q)$: for a given N scattered data values $\{(x_i, f(x_i)), i = 1,\cdots,N\}$, find a function $s(x) \in X$ such that
$$\|t(s)\|_2^2 = \min_{u \in I_y} \|t(u)\|_2^2,$$

where $y=(f(x_1),\cdots,f(x_n))$ and $I_y=\{u\in X;\ Au=y\}$. The solution to this problem is also called a natural spline.

This natural spline interpolation problem in Hilbert space can be extended to the higher dimensional settings. For $R=[a,b]\times[c,d]$, let $X:=H^{r,s}(R)$ denote the space

$$\{u(x,y);\ \frac{\partial^{r+s}u}{\partial x^r\partial y^s}\in L^2(R),\frac{\partial^{\alpha+\beta}u}{\partial x^\alpha\partial y^\beta}\in AC(R),\alpha=0,\cdots,r-1,\ \beta=0,\cdots,s-1\},$$

where $AC(R)$ is the space of absolutely continuous functions over R. Let

$$Y=L^2(R)\times\prod_{\nu=0}^{s-1}L^2[a,b]\times\prod_{\mu=0}^{r-1}L^2[c,d]$$

and $T:X\mapsto Y$ be a linear operator defined by

$$T=t_0\times\prod_{\nu=0}^{s-1}t_1^{(\nu)}\times\prod_{\mu=0}^{r-1}t_2^{(\mu)},$$

where

$$\begin{aligned}t_0(u)&=u^{(r,s)}(x,y)=\frac{\partial^{r+s}u(x,y)}{\partial x^r\partial y^s};\\ t_1^{(\nu)}(u)&=u^{(r,\nu)}(x,c)=\frac{\partial^{r+\nu}u(x,y)}{\partial x^s\partial y^\nu}|_{y=c},\ \nu=0,...,s-1;\\ t_2^{(\mu)}(u)&=u^{(s,\mu)}(a,y)=\frac{\partial^{s+\mu}u(x,y)}{\partial x^\mu\partial y^s}|_{x=a},\ \mu=0,...,r-1.\end{aligned}$$

Let $Z=\mathbb{R}^N$ and $A:X\mapsto Z$ be an interpolation operator defined by

$$Au=(u(x_1,y_1),...,u(x_N,y_N)).$$

Li and Guan [55] studied such a natural polynomial splines interpolation problem: given N scattered data points and values $\{(x_i,y_i,z_i),i=1,\cdots,N\}$, find a function $\sigma(x,y)\in X$ satisfying

$$\|T\sigma\|^2=\min\{\|Tu\|^2,u\in X,Au=z\},$$

where, $z=(z_1,\cdots,z_N)$ and

$$\begin{aligned}\|Tu\|^2=&\int\int_R(u^{(r,s)}(x,y))^2dxdy+\sum_{\nu=0}^{n-1}\int_a^b(u^{(r,\nu)}(x,c))^2dx,\\ &+\sum_{\mu=0}^{r-1}\int_c^d(u^{(\mu,s)}(a,y))^2dy.\end{aligned}$$

The solution of this bivariate polynomial natural spline interpolation problem in Hilbert spaces $H^{r,s}(R)$ is called a bivariate polynomial natural spline.

The following results are obtained by Li and Guan in [55] (see also [17]) to study the null space of the operator T and to give a closed form of the bivariate natural spline function.

THEOREM 0.18
The null subspace of the operator T is

$$N(T) = P\langle r, s\rangle = \{u; u(x,y) = \sum_{i=0}^{r-1}\sum_{j=0}^{s-1} c_{ij}x^i y^j, c_{ij} \in I\!R\}.$$

THEOREM 0.19
Bivariate natural polynomial spline $\sigma(x,y)$ has the following explicit and closed-form expression:

$$\sigma(x,y) = \sum_{i=1}^{N} \lambda_i g_i(x,y) + \sum_{i=0}^{r-1}\sum_{j=0}^{s-1} c_{ij}x^i y^j$$

where $g_i(x,y) = G(x_i, y_i; x, y), i = 1, \cdots, N$ are $(2r-1, 2s-1)$ natural spline basis functions and

$$G(t,\tau;x,y) = (-1)^{r+s}\frac{(t-x)_+^{2r-1}(\tau-y)_+^{2s-1}}{(2r-1)!(2s-1)!} + \sum_{\nu=0}^{s-1}(-1)^s\frac{(t-x)_+^{2r-1}(y-c)^\nu}{(2r-1)!\nu!}\left[\frac{(\tau-c)^\nu}{\nu!} - (-1)^{s-\nu}\frac{(\tau-c)^{2s-\nu-1}}{(2s-\nu-1)!}\right] + \sum_{\mu=0}^{r-1}(-1)^r\frac{(\tau-y)_+^{2s-1}(x-a)^\mu}{(2s-1)!\mu!}\left[\frac{(t-a)^\mu}{\mu!} - (-1)^{r-\mu}\frac{(t-a)^{2r-\mu-1}}{(2r-\mu-1)!}\right].$$

They also extended the results to general k-dimension setting [17]. Guan [36] considered bivariate natural polynomial splines for smoothing or generalized interpolating of scattered data. In [37], a locally supported basis of bivariate natural polynomial splines were constructed. Recently, Guan and Hong in [38] constructed a locally supported basis of bivariate natural polynomial splines for scattered data on some lines (or for refinement grid points) to address a problem mentioned in [64]. For scattered data in a triangle, there were some similar discussions in [35] comparing with the spline interpolations over triangulations .

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