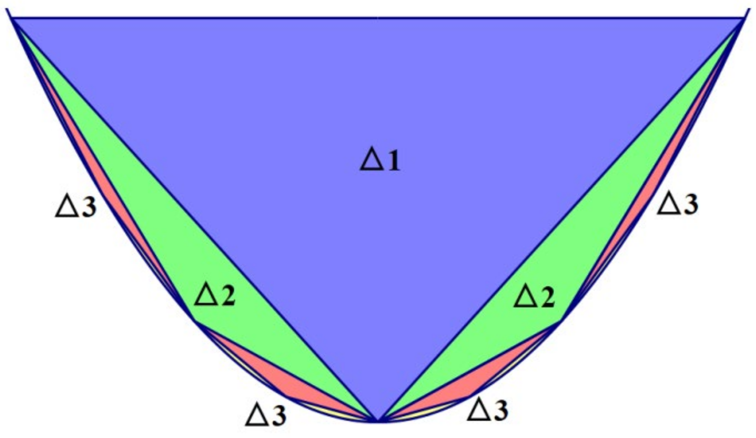
$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Bigg|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} - 0 = \boxed{\frac{1}{3} \textit{ or } 0.333\ldots \textit{ square units}}$$

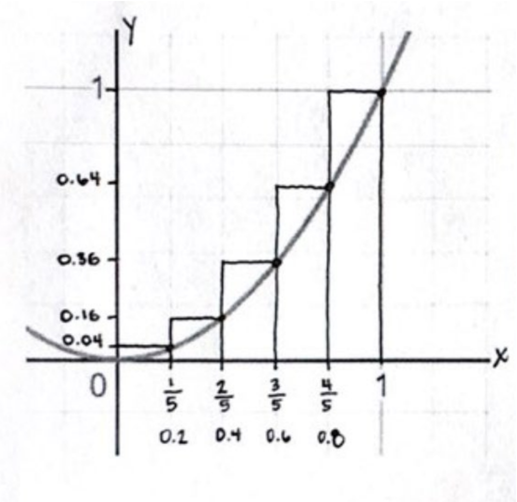
Integration (Calculus) for 8th Grade Math & Algebra I Students



Amara Mattingly

Theory of Calculus

November 14, 2021



Supplies needed for lesson:

Files Provided:

- PowerPoint presentation
- Printable student packet (w/o rectangles on graphs)
- Printable PDF of Elaborate/Evaluate sections w/ rectangles on graphs and middle-school appropriate calculation supports
- Printable PDF of teacher answer key for Elaborate/Evaluate sections

Each student will need:

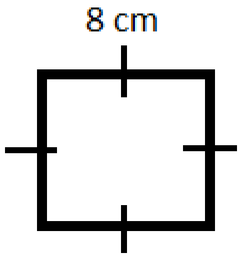
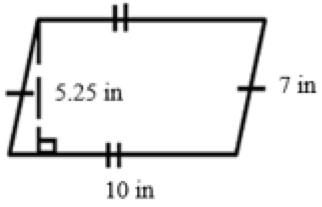
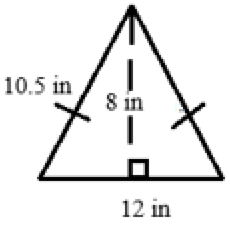
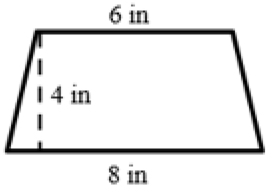
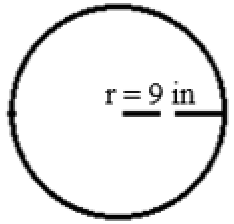
- A printed packet
- Graphing calculator (able to work with decimals and fractions)
- Ruler to measure 2D shape dimensions
- Compass to draw circles
- Protractor to measure angles

Lesson Objectives:

- Engage: Students will be able to calculate the area of 2D shapes.
- Explore/Explain: Students will be able to use the area of 2D shapes to approximate the area under curves.
- **Explain/Elaborate/Evaluate: Students will apply the “triangle method” and “rectangle method” for calculating area inside/under a curve.**
- **Explain/Elaborate: Students will connect the concepts of calculating area of a 2D shape with the concept of integration in calculus.**
- Elaborate: Students will discuss factors impacting the accuracy of area approximations.

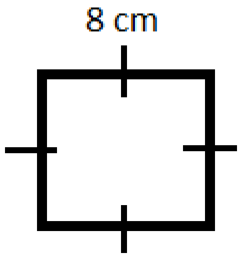
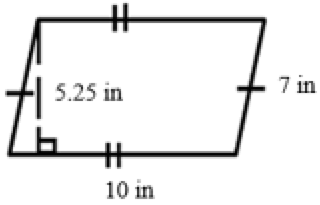
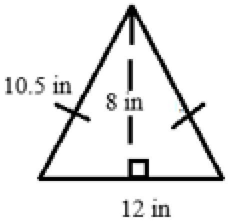
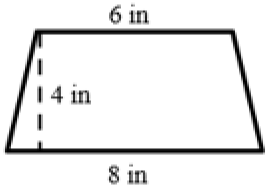
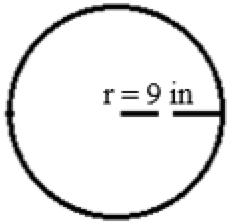
Engage: Find the Area of These 2D Shapes

Use your calculator, but substitute values into the formulas first to show your work!

<table>
<tr><th>Rectangle</th><th>Parallelogram</th><th>Triangle</th><th>Trapezoid</th><th>Circle</th></tr>
<tr><td>$A = lw$</td><td>$A = bh$</td><td>$A = \frac{bh}{2}$</td><td>$A = \frac{(b_1 + b_2)h}{2}$</td><td>$A = \pi r^2$</td></tr>
<tr><td>
</td><td>
</td><td>
</td><td>
</td><td>
</td></tr>
</table>

Engage: Check your work!

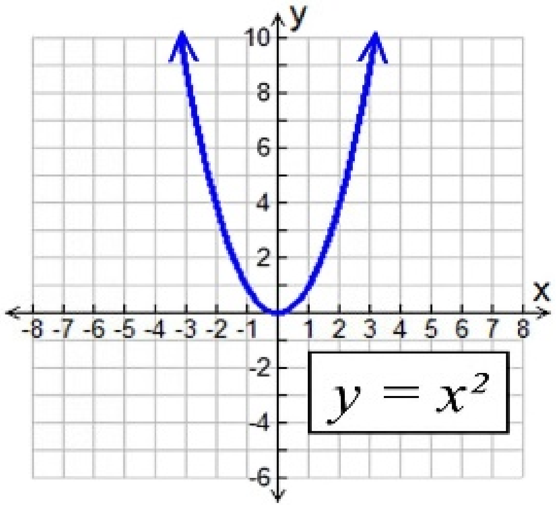
Use your calculator, but substitute values into the formulas first to show your work!

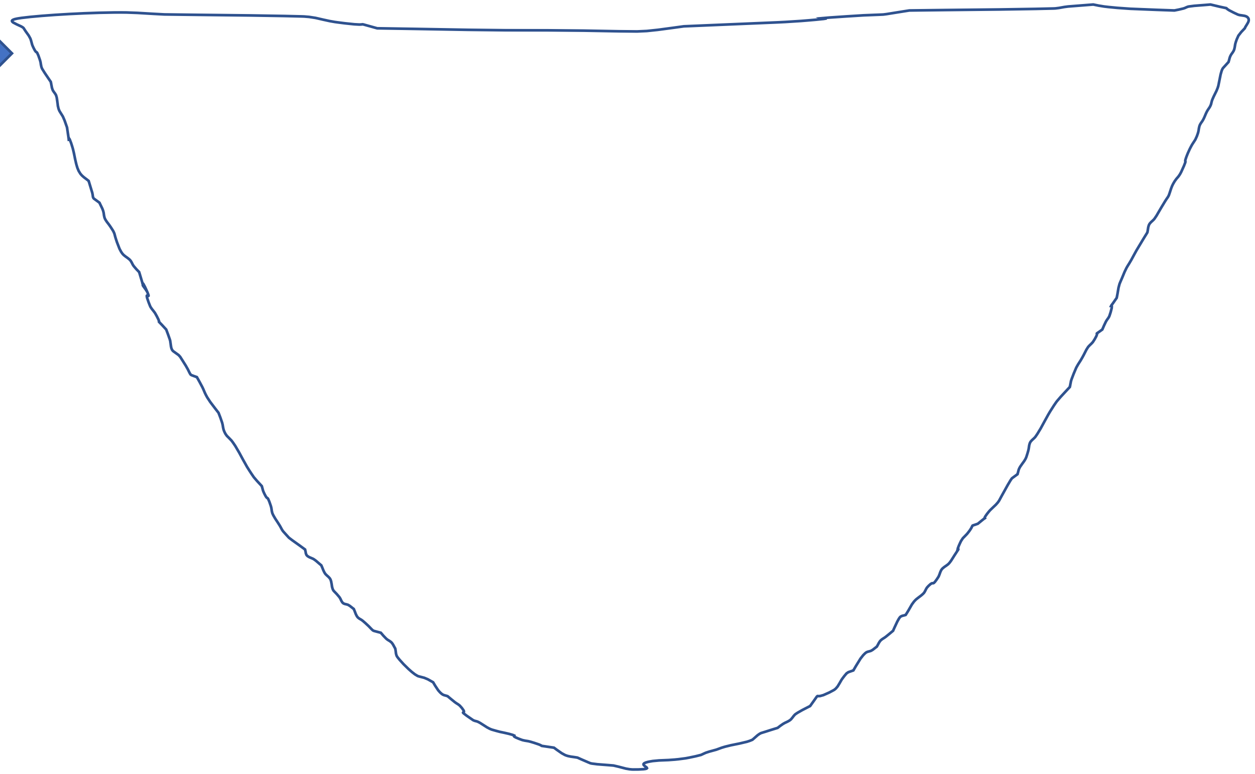
Rectangle	Parallelogram	Triangle	Trapezoid	Circle
$\boldsymbol{A = lw}$ $A = 8(8)$ $A = 64\ sq\ cm$	$\boldsymbol{A = bh}$ $A = 10(5.25)$ $A = 52.25\ sq\ in$	$\boldsymbol{A = \frac{bh}{2}}$ $A = \frac{12(8)}{2}$ $A = \frac{96}{2}$ $A = 48\ sq\ in$	$\boldsymbol{A = \frac{(b_1 + b_2)h}{2}}$ $A = \frac{(8+6)4}{2}$ $A = \frac{(14)4}{2}$ $A = 28\ sq\ in$	$\boldsymbol{A = \pi r^2}$ $A = \pi 9^2$ A = 81 π sq in A ~ 254.34 sq in
				

Explore: Class Discussion

How can we approximate the area inside this "parabola" using our skills for calculating the area of 2D shapes?

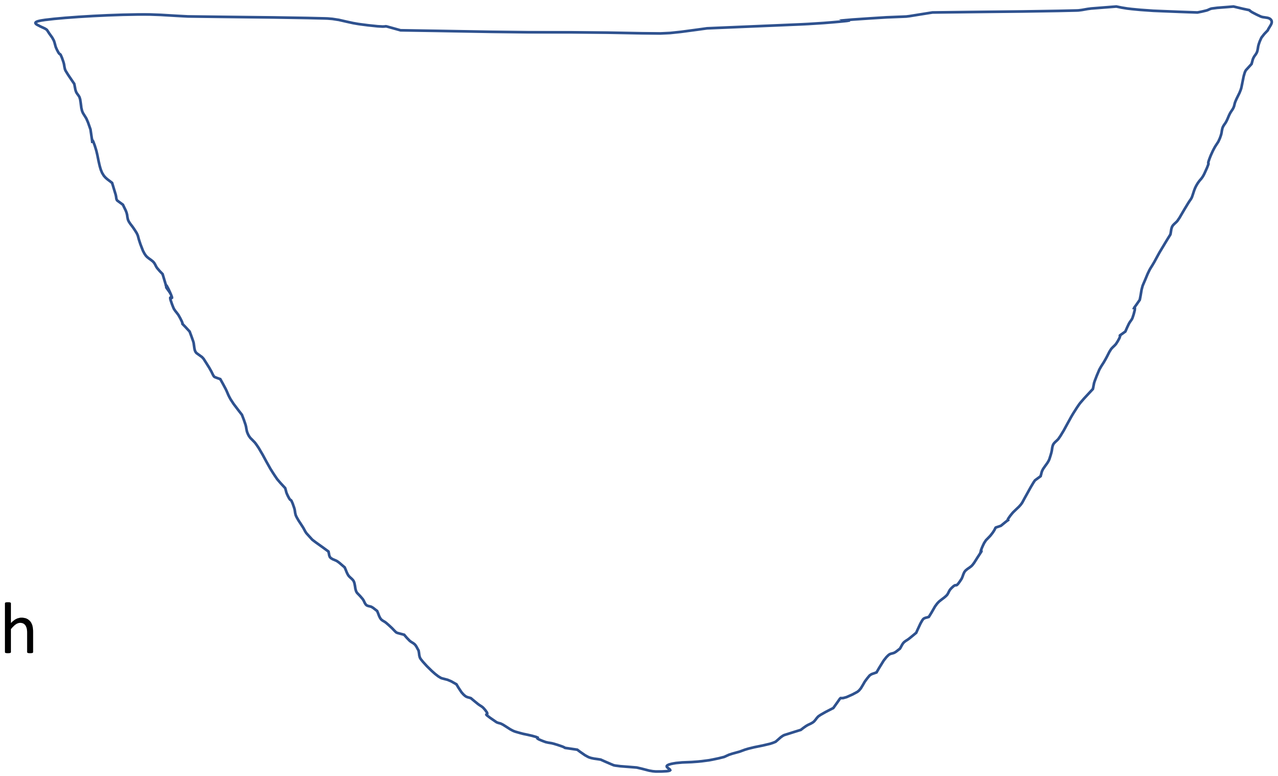
(Note: when measuring your shapes, use centimeters.)





Explore: Class Discussion.

What answers did we get?

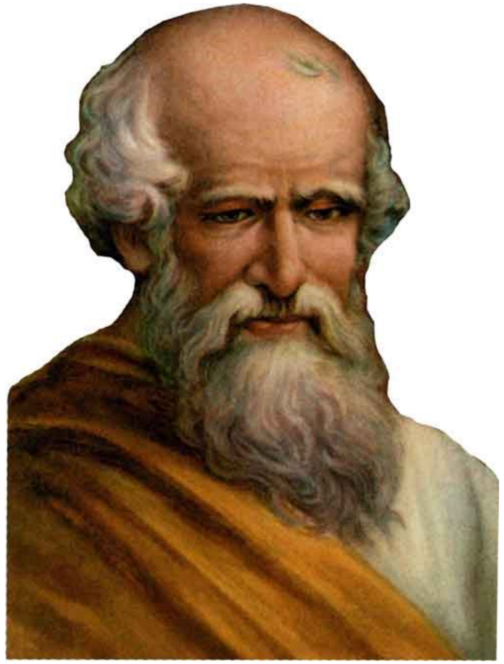


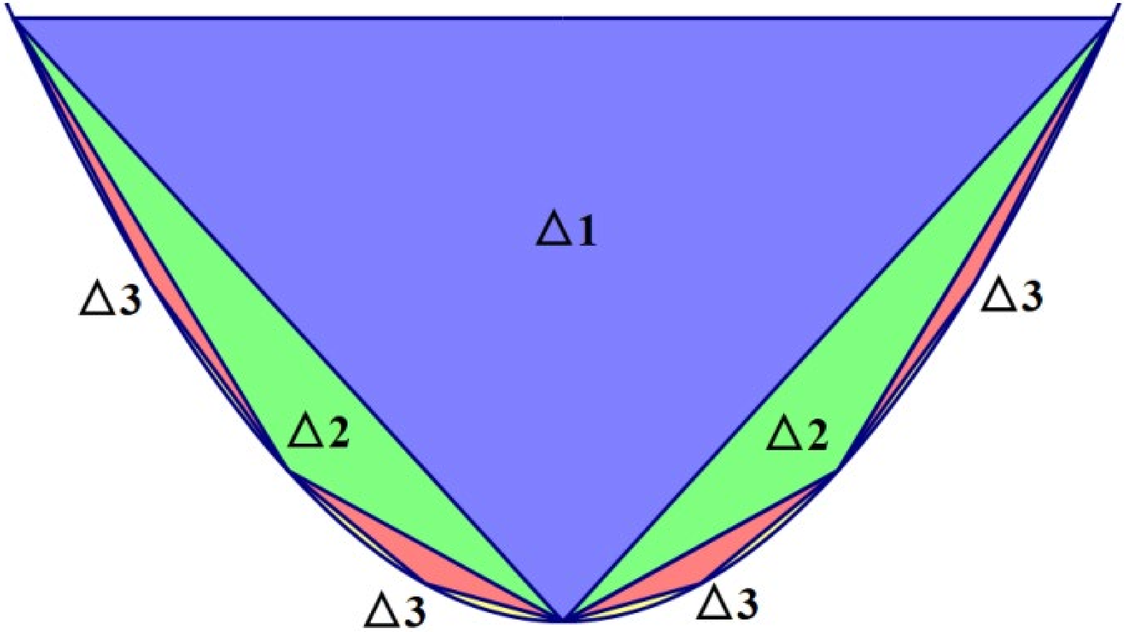
How do our answers compare with the actual answer?

The actual answer is 80 square cm.

Explain: How did Archimedes calculate the area inside this parabola?

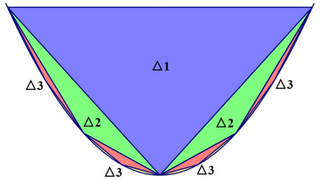
Archimedes is a Greek mathematician who lived from 287 BCE – 212 BCE. He is considered one of the most important mathematicians of all time.





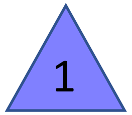
Note: Archimedes' method is more complicated than this, but this is the basic idea!

Explain: Calculate the approximate area inside the parabola using Archimedes' triangles.

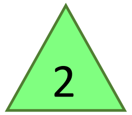


**** You do not need to measure the triangle dimensions yourself. Use the values given to you.****

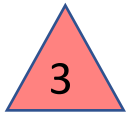
Step 1: Calculate the area of each triangle.



Purple area $= \dfrac{bh}{2} =$



Green area $= \dfrac{bh}{2} =$

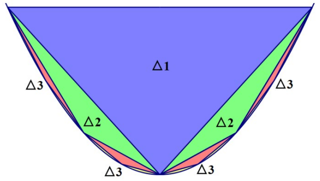


Red area $= \dfrac{bh}{2} =$

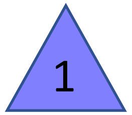
Step 2: Fill in the table and do some calculations to approximate the area inside the parabola.

Areas of Triangles	**x num of triangles**	**Area covered inside parabola**
Purple triangle area: ______	x 1 =	______
Green triangle area: ______	x 2 =	______
Red triangle area: ______	x 4 =	______
Approximate area inside parabola:		______

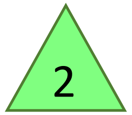
Explain: Check your work!



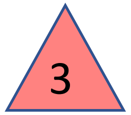
Step 1: Calculate the area of each triangle.



$$\text{Purple area} = \frac{bh}{2} = \frac{15 * 8}{2} = 60\ sq\ cm$$



$$\text{Green area} = \frac{bh}{2} = \frac{10 * 1.5}{2} = 7.5\ sq\ cm$$



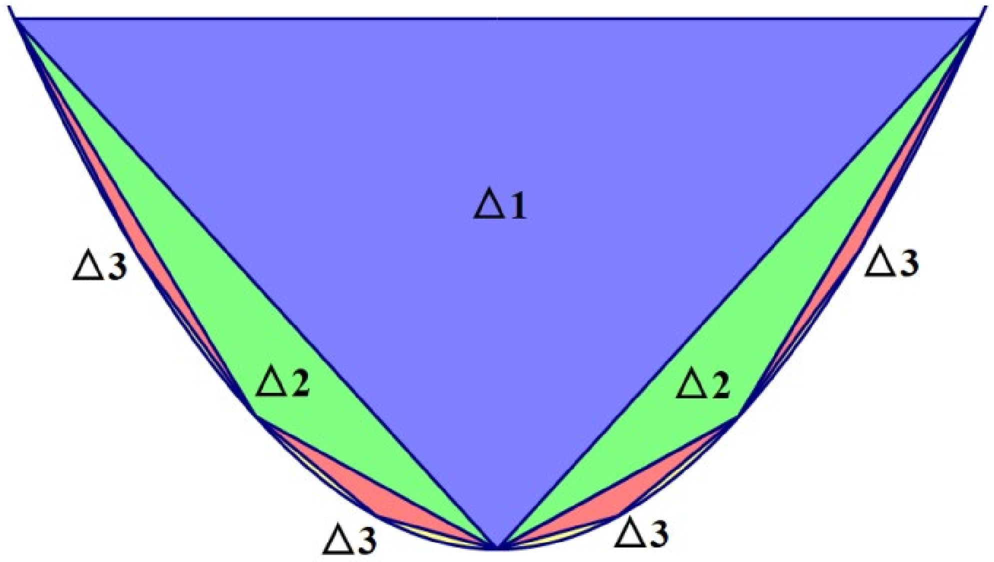
$$\text{Red area} = \frac{bh}{2} = \frac{5 * 0.375}{2} = 0.9375\ sq\ cm$$

Step 2: Fill in the table and do some calculations to approximate the area inside the parabola.

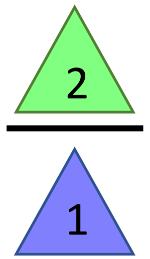
Areas of Triangles		x num of triangles	Area covered inside parabola
Purple triangle area:	60	x 1 =	60.00
Green triangle area:	7.5	x 2 =	15.00
Red triangle area:	0.9375	x 4 =	3.75
Approximate area inside parabola:			78.75

Explain: Class Discussion

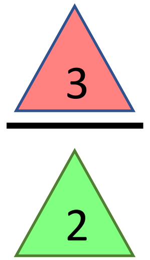
- Why isn't our answer accurate (80 sq cm)?
- What could we do to make our answer more accurate?
- Why is it difficult to get an accurate answer?



Explain: Archimedes found an accurate answer!
How did he do it? He noticed a pattern.
Can you find it? (Hint: simplify the fractions or convert them to decimals.)

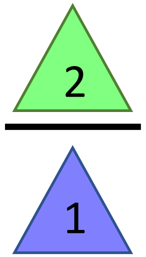


$$= \frac{Area\ of\ green\ triangle}{Area\ of\ purple\ triangle} = \frac{7.5}{60}$$

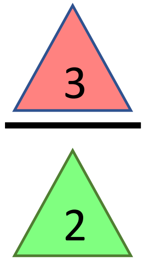


$$= \frac{Area\ of\ red\ triangle}{Area\ of\ green\ triangle} = \frac{0.9375}{7.5}$$

Explain: Check your work!



$$= \frac{Area\ of\ green\ triangle}{Area\ of\ purple\ triangle} = \frac{7.5}{60} \color{red}{= \frac{1}{8} = 0.125}$$



$$= \frac{Area\ of\ red\ triangle}{Area\ of\ green\ triangle} = \frac{0.9375}{7.5} \color{red}{= \frac{1}{8} = 0.125}$$

Explain: Archimedes' "Proof by Exhaustion" (Jun, 2021) & (Hooper, 2021)

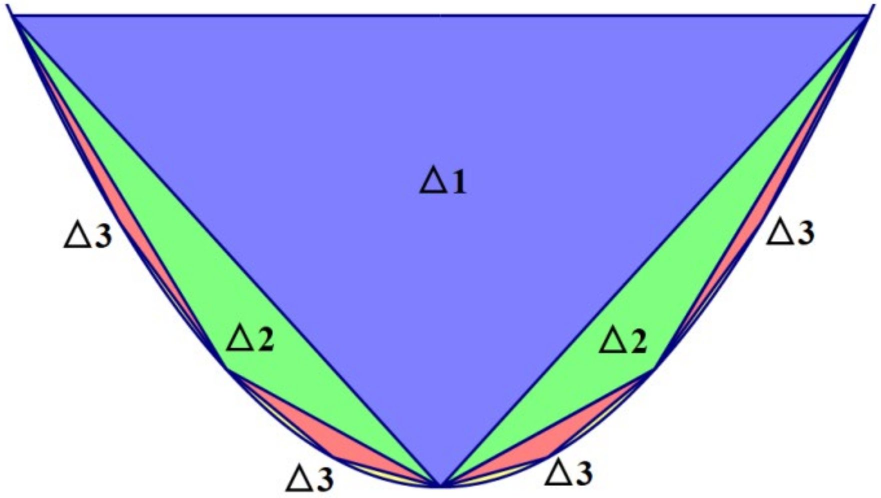
$$\Delta 1 + 2\Delta 2 + 4\Delta 3 + 8\Delta 4 \ldots = \text{Area of parabolic segment}$$

$$\Delta 1 + 2\left(\frac{1}{8}\right)\Delta 1 + 4\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)\Delta 1 + 8\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)\left(\frac{1}{8}\right)\Delta 1 + \ldots$$

$$\Delta 1 + \frac{1}{4}\Delta 1 + \frac{1}{16}\Delta 1 + \frac{1}{64}\Delta 1 + \ldots = \text{Area of parabolic segment}$$

$$\vdots$$

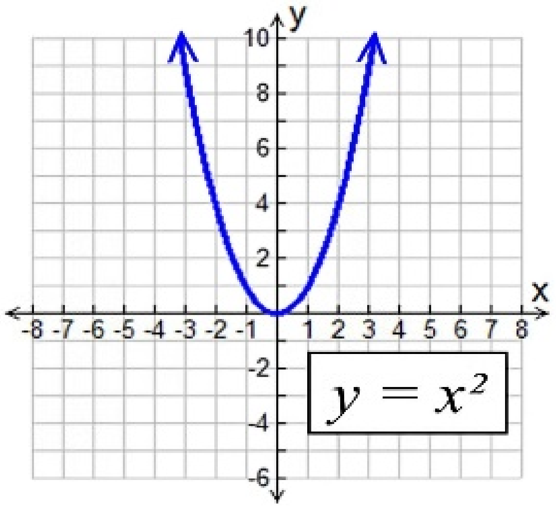
$$\boxed{\frac{4}{3}\Delta 1 = \text{Area of parabolic statement}}$$

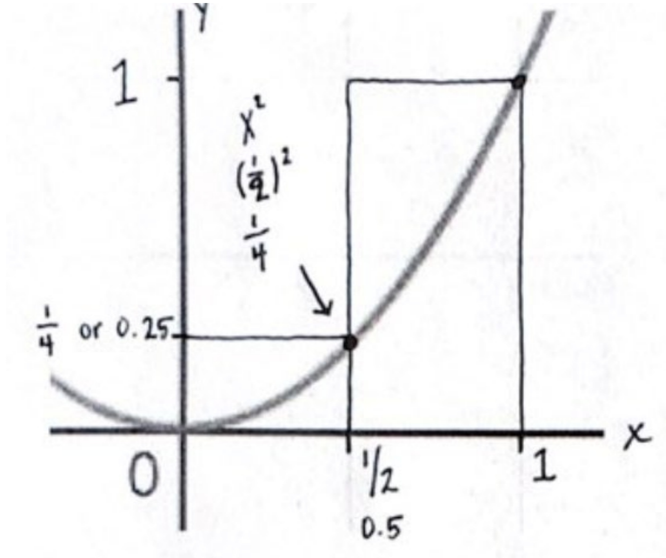


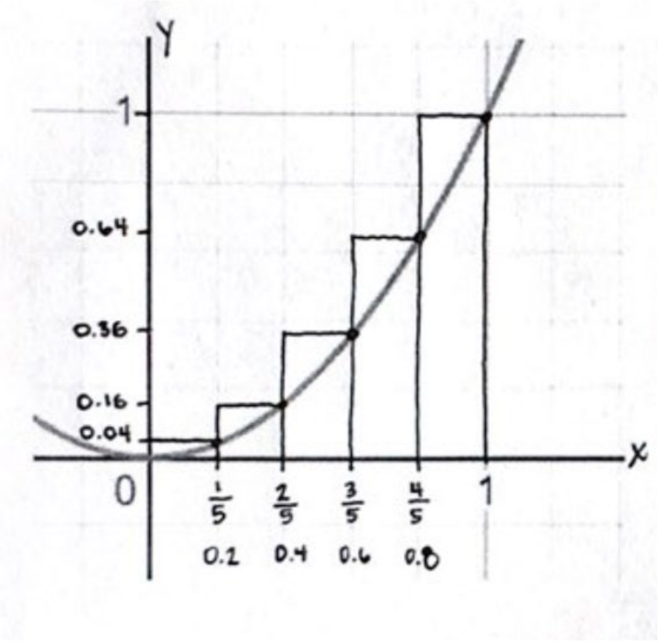
$$\textit{Area of the parabola} = \frac{4}{3}\left(\frac{b1 * h1}{2}\right) = \frac{4}{3}\left(\frac{15 * 8}{2}\right) = \frac{4}{3}\left(\frac{120}{2}\right) = \left(\frac{480}{6}\right) = 80\ sq\ cm$$

Explain: How does this relate to calculus?

In calculus, we use **INTEGRATION** to calculate the area inside (or "under" curved shapes.

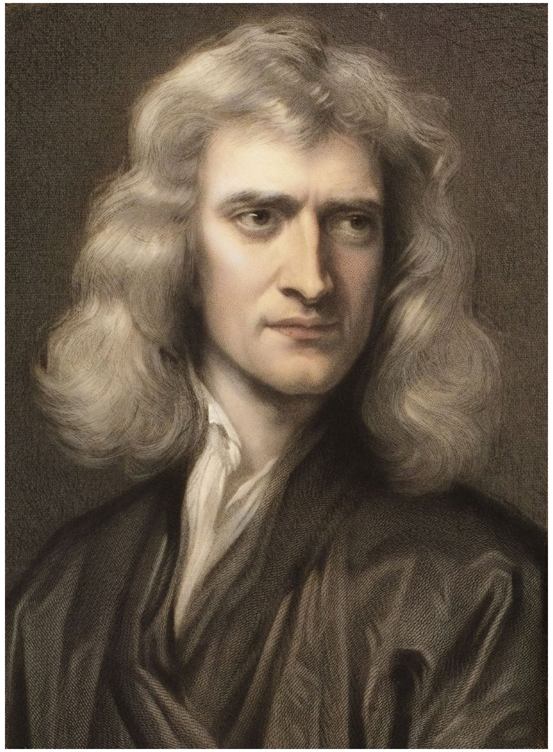






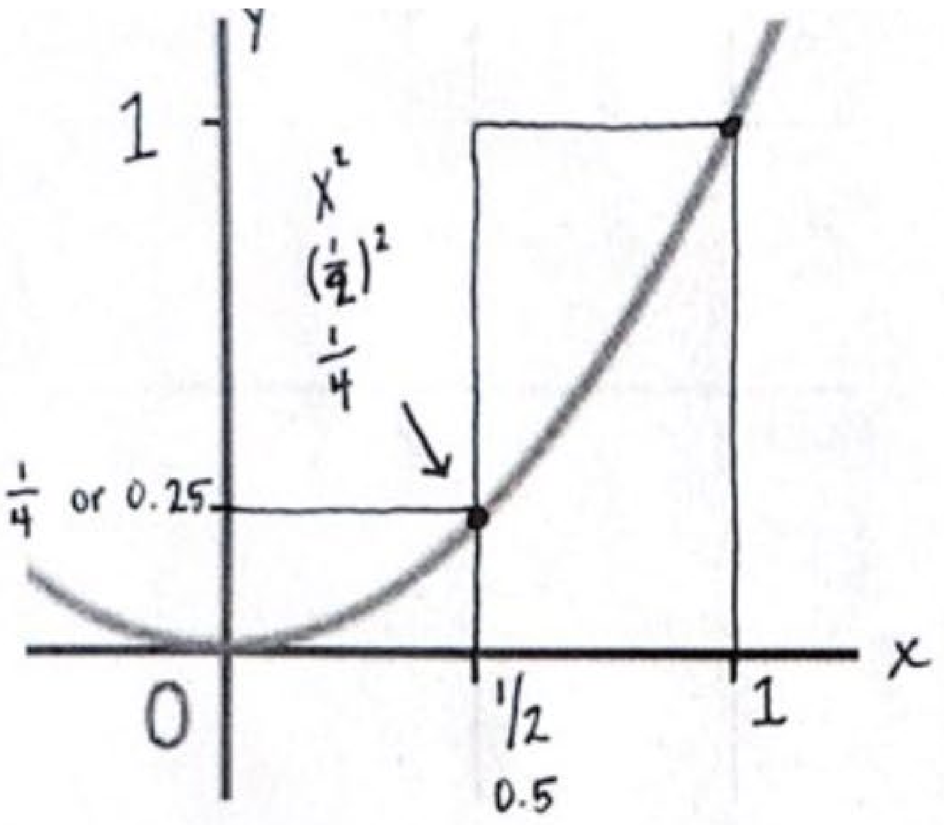
Elaborate: In addition to Archimedes' "triangle method," there are other ways to approximate the area inside shapes with curves. Let's learn about Sir Isaac Newton's "rectangle method."

Sir Isaac Newton is an English mathematician who lived from 1643 - 1727. He is an inventor of calculus and considered one of the most important mathematicians of all time.



$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} - 0 = \boxed{\frac{1}{3} \text{ or } 0.333\ldots \text{ square units}}$$

Elaborate: Take notes while your teacher models the “rectangle method”



1. What is the length of the base of each rectangle?
   ________ (<- in calculus, this is called a partition)
2. Calculate the area of each rectangle, then sum the areas to approximate the area under the curve.

   A = (____ * ____) + (____ * ____)

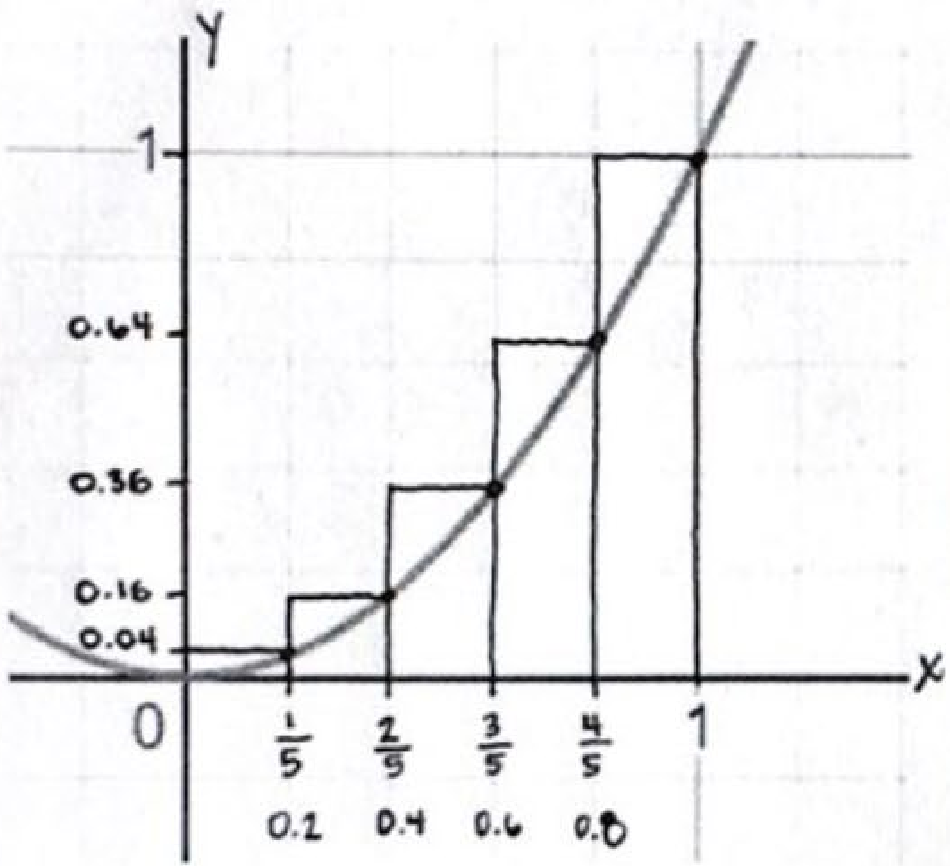
   A = (__________) + (__________)

   A = ______________________________ square units

3. Is the area approximation greater or less than 1/3 or 0.333...? (Circle one.)

   Greater than 1/3 or 0.333...   Less than 1/3 or 0.333...

Elaborate: Take notes while your teacher models the “rectangle method”



4. What is the length of the base of each rectangle?

   ________ (<- in calculus, this is called a partition)

5. Calculate the area of each rectangle, then sum the areas to approximate the area under the curve.

   A = (__ * __) + (__ * __) + (__ * __) + (__ * __) + (__ * __)

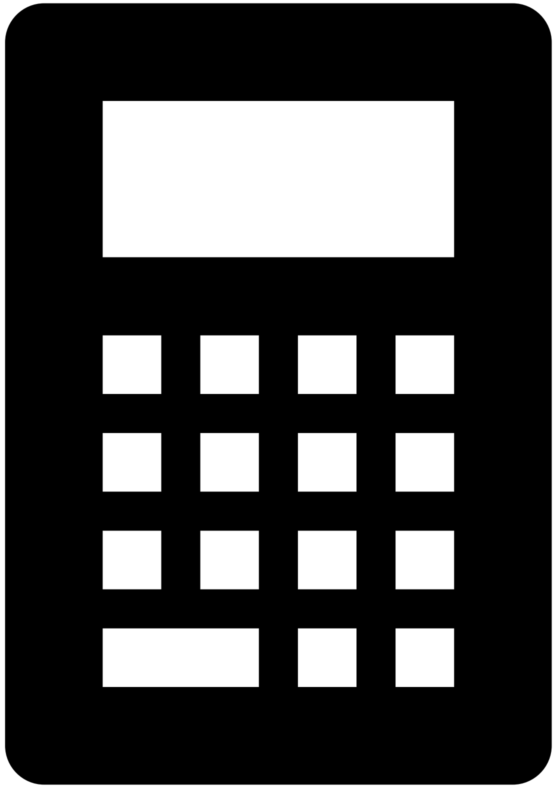
   A = (______) + (______) + (______) + (______) + (______)

   A = ______________________________ square units

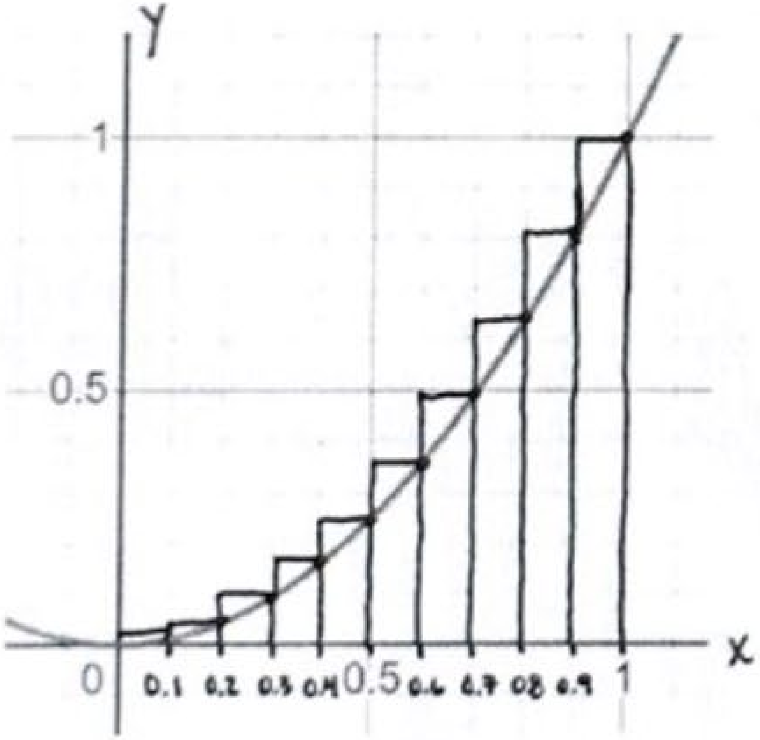
6. Is the area approximation greater or less than 1/3 or 0.333...? (Circle one.)

   Greater than 1/3 or 0.333...   Less than 1/3 or 0.333...

Elaborate: Now finish Example 1 (#7 – 11) and doing all of Example 2!



Elaborate: Check your work on Example 1!



7. What is the length of the base of each rectangle?
   1/10 or 0.1 (<- in calculus, this is called a partition)
8. Calculate the area of each rectangle, then sum the areas to approximate the area under the curve.

   A = (.1 * .01) + (.1 * .04) + (.1 * .09) + (.1 * .16) + (.1 * .25)

   + (.1 * .36) + (.1 * .49) + (.1 * .64) + (.1 * .81) + (.1 * 1)

   A = (.001)+(.004)+(.009)+(.016)+(.025)+(.036)+(.049)+(.064)+(.081)+(.1)

   A = 0.385 square units

9. Is the area approximation greater or less than 1/3 or 0.333...? (Circle one.)

   Greater than 1/3 or 0.333... Less than 1/3 or 0.333...

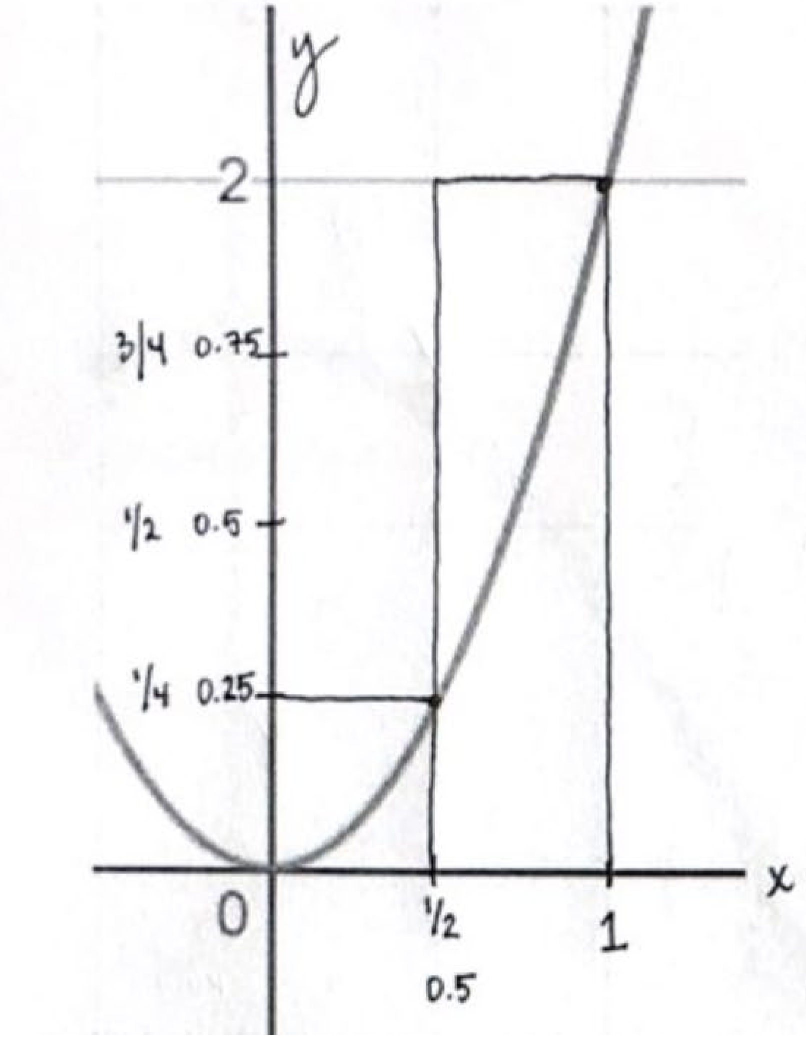
Elaborate: Class Discussion

Example 1

10. Which rectangle method's approximate area under the curve was closest to the accurate area?

11. All of the area approximations were greater than the accurate answer. Why do you think that is?

Elaborate: Check your work on Example 2!



1. What is the length of the base of each rectangle?
   1/2 or 0.5 (<- in calculus, this is called a partition)
2. Calculate the area of each rectangle, then sum the areas to approximate the area under the curve.

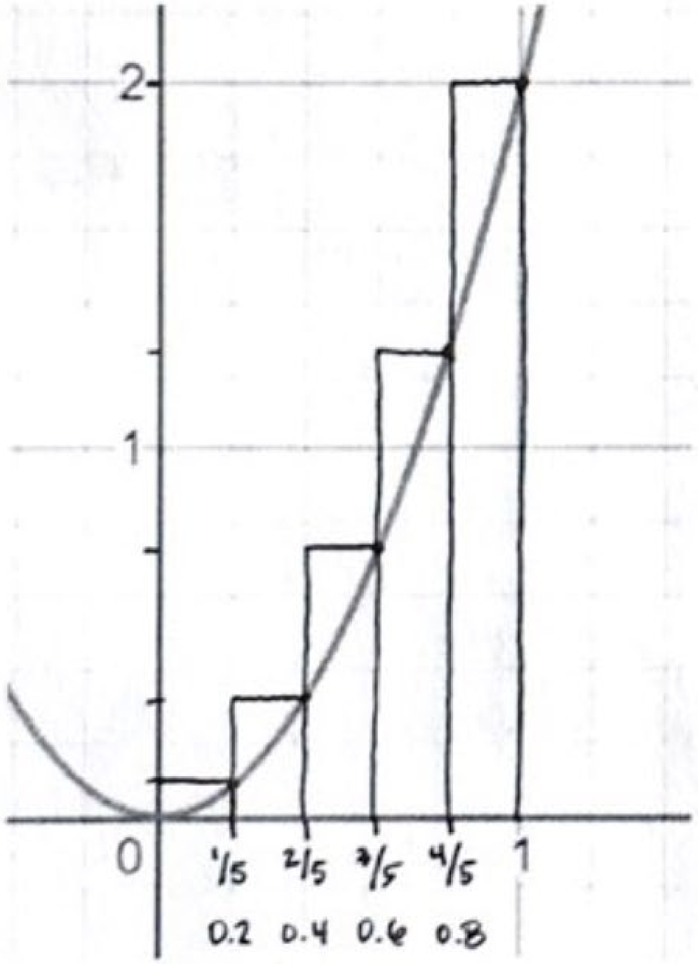
   A = (0.5 * .25) + (0.5 * 2)

   A = (0.125) + (1)

   A = 1.125 square units
3. Is the area approximation greater or less than 2/3 or 0.666...? (Circle one.)

   Greater than 2/3 or 0.666... Less than 2/3 or 0.666...

Elaborate: Check your work on Example 2!



4. What is the length of the base of each rectangle?
   1/5 or 0.2 (<- in calculus, this is called a partition)
5. Calculate the area of each rectangle, then sum the areas to approximate the area under the curve.

   A = (.2 * .08) + (.2 * .32) + (.2 * .72) + (.2 * 1.28) + (.2 * 2)

   A = (.016) + (.064) + (.144) + (.256) + (0.4)

   A = 0.88 square units
6. Is the area approximation greater or less than 2/3 or 0.666...? (Circle one.)

   Greater than 2/3 or 0.666...   Less than 2/3 or 0.666...

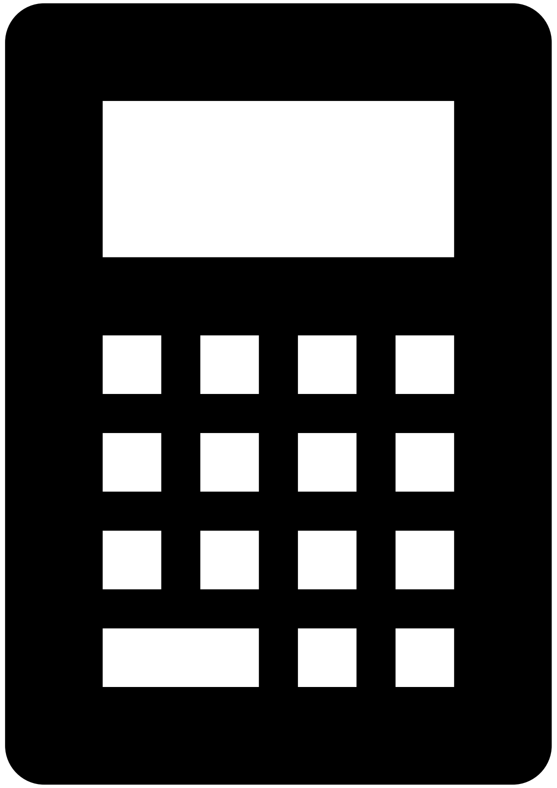
Elaborate: Class Discussion

Example 2

7. How does the size of the partition change the accuracy of the approximation of the area under the curve? Why?

8. What similarities and differences do you notice between the “triangle method” and the “rectangle method”?

Evaluate: Now complete Example 3 on your own!



Sources:

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Hooper, Wyatte C. “Archimedes of Syracuse and Sir Isaac Newton: On the Quadrature of a Parabola.” *Journal of Humanistic Mathematics* 11, no. 2 (2012). Retrieved on November 14, 2021 from https://scholarship.claremont.edu/cgi/viewcontent.cgi?article=1597&context=jhm

Jun, Nagao. “Calculus you can understand, in-depth interpretation of 30,000 words” [translated from Chinese into English via Google Translate]. September 25, 2021. Retrieved on November 14, 2021 from https://mp.weixin.qq.com/s/tYOt1D9iBKGMQasGNxWX6g