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Modeling the transmission dynamics of a two-strain dengue disease with infection age

Xiaoguang Li*, Liming Cai*,‡ and Wandi Ding†
**College of Mathematics and Statistics*
Xinyang Normal University, Xinyang, P. R. China
†*Department of Mathematical Sciences*
Middle Tennessee State University
Murfreesboro, TN, USA
‡*limingcai@amss.ac.cn*

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In this paper, we introduce a partial differential equation (PDE) model to describe the transmission dynamics of dengue with two viral strains and possible secondary infection for humans. The model features the variable infectiousness during the infectious period, which we call the infection age of the infectious host. We define two thresholds $\mathscr{R}_1^j$ and $\mathscr{R}_2^j, j = 1, 2$, and show that the strain j can not invade the system if $\mathscr{R}_1^j + \mathscr{R}_2^j < 1$. Further, the disease-free equilibrium of the system is globally asymptotically stable if $\max_j\{\mathscr{R}_1^j + \mathscr{R}_2^j\} < 1$. When $\mathscr{R}_1^j > 1$, strain j dominance equilibrium $\mathscr{E}_j$ exists, and is locally asymptotically stable when $\mathscr{R}_1^j > 1$, $\mathscr{R}_1^i < \varsigma\mathscr{R}_1^j, i, j = 1, 2, i \neq j$, $\varsigma \in (0, 1)$. Then, by applying Lyapunov–LaSalle techniques, we establish the global asymptotical stability of the dominance equilibrium corresponding to some strain j. This implies strain j eliminates the other strain as long as $\mathscr{R}_1^i/\mathscr{R}_1^j < b_i/b_j < 1, i \neq j$, where b_j denotes the probability of a given susceptible mosquito being transmitted by a primarily infected human with strain j. Finally, we study the existence of the coexistence equilibria under some conditions.

Keywords: Dengue; two strains; stability; competitive exclusion; backward bifurcation

Mathematics Subject Classification 2020: 92D30, 35B35, 35Q92

1. Introduction

Dengue has become an important public health problem worldwide. It is affecting tropical and subtropical regions of the world. It is estimated that in 2021, 390 million people at risk. About 50–100 million symptomatic cases per year, in the African continent [39]. The dengue virus (DENV) is transmitted mainly by

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[‡] Corresponding author.

female mosquitoes, *Aedes aegypti*. DENV has four different strains [37]. It has been shown that humans who have experienced a primary infection with one dengue strain develop lifelong immunity against this strain, but still keep susceptible to a different dengue strain. Secondary infections may lead to health complications such as dengue hemorrhagic fever (DHF) or dengue shock syndrome [13, 31]. Secondary infections have posed a major threat to global public health. Mathematical modeling on dengue epidemiology has been motivated by its the recently frequent outbreaks.

Mathematical modeling of mosquito-born diseases dates back to the work of Sir Ronald Ross [32, 33]. The Ross–Macdonald model shows that the persistence and the prevalence of infections in humans and mosquitos are determined by the basic reproductive number [8, 9]. In recent years, based on the Ross–Macdonald model, many authors have proposed models to investigate mosquito-borne diseases and have designed sorts of control strategies [1, 11, 15, 17, 20, 28, 30, 34, 35, 41, 42]. Noticing that many diseases are caused by more than one antigenically different strains of the causative agent [4, 25]. Mathematical models for dengue transmission with multiple strains were formulated by the ODEs in [10, 14, 30] and PDEs in [7, 21]. The dynamics of epidemiological models with multiple strains have attracted many authors interesting for a long time (see [11, 12, 19, 29]). One of the important phenomena is found that the competitive exclusion principle occurs, which means that when multiple strains circulate in the populations, only the strains with the largest reproduction number persist and the strains with suboptimal reproduction numbers are eliminated [26]. In paper [3], authors first show that the competitive exclusion principle holds. It has been recognized that the humans' infectivity is varying during the infectious period, which may have a significant impact on the occurrence of the secondary infections. In recent years, many authors [18, 22, 27, 38] are interested in modeling mosquito-borne diseases with infection age. However, the effect of the strains infection duration (*called infection age*) on dengue strains circulation in the populations is unclear.

In this paper, according to dengue epidemiology, we formulate a novel two-strain dengue model with the secondary infection and the infection age. The dynamics of the model are investigated. The phenomena of competitive exclusion and backward bifurcation for dengue strains are found. This paper is organized as follows. In Sec. 2, we introduce a two-strain, infection-age structured epidemic model with possible secondary infection. In Sec. 3, we define a threshold reproduction number $\mathscr{R}_0^j$, and show the strain j will go extinct when $\mathscr{R}_0^j < 1$. In Sec. 4, we investigate the local and global stability of the disease-free equilibrium. In Sec. 5, we investigate the existence of dominant equilibria and its stability. In Sec. 6, we pay attention to the uniform persistence of the strains. In Sec. 7, by constructing Lyapunov functions and applying Lyapunov–LaSalle techniques, we show the global stability of the strain dominance equilibrium. In Sec. 8, we study the existence of the coexistence equilibria. At last, a brief discussion is given in Sec. 9.

2. Formulation of the Model

In this section, we formulate a two-strain dengue model describing the transmission of DENV between humans and mosquitoes. The human population is classified into three epidemiological classes, the susceptible, the infectious and the recovered class. The mosquito populations are divided into two classes, the susceptible and infectious mosquitoes. The biological descriptions of the state variable are summarized in Table 1. The total population sizes $N_h(t)$ and $N_v(t)$ are, respectively

$$N_h(t) = S_h(t) + \sum_{j=1}^{2} \int_0^\infty I_h^j(a,t)da + \sum_{j=1}^{2} R_h^j(t) + \sum_{j=1, j\neq i}^{2} \int_0^\infty I_h^{ji}(a,t)da + R_h(t),$$

$$N_v(t) = S_v(t) + I_v^1(t) + I_v^2(t).$$

Susceptible humans $S_h(t)$ increase at rate Λ_h. Being bitten by an infected mosquito with the strain i, the susceptible humans may enter the ith $(i = 1, 2)$ infectious strain class $I_h^i(a,t)$. We define $\widetilde{\lambda}_h^i(t)$ the force of infection primarily from the infected mosquitoes with the strain i to humans, as the product of the number of mosquito bites that one human has per unit time, $b_h(N_h(t), N_v(t))$, the probability of the disease transmission from the mosquito with the strain i to human $\widetilde{\beta}_h^i$ and the probability that the biting mosquito with infectious strain i per unit mosquito, $I_v^i(t)/N_v(t)$. The infected humans $I_h^i(a,t)$ recover via the natural recovery or the treatment, and enter the recovered class $R_h^i(t)$ at the rate $\gamma_i(a)$. The recovered humans $R_h^i(t)$ have life-long immunity against the strain i, but still are susceptible to the different strain j $(j = 1, 2, i \neq j)$. In this case, the class $R_h^i(t)$ may be regarded as a 'semi-susceptible' class while the class $S_h(t)$ can be regarded as a completely susceptible class. Hence, when the humans in recovered class $R_h^i(t)$ are bitten again by an infected mosquito with the strain $j, j \neq i$, they may enter the infectious class $I_h^{ij}(a,t)$. $\sigma_j \widetilde{\lambda}_h^j(t)$ is the infection force of humans reinfection, where the factor σ_j shows the difference in the transmission likelihood from the infected mosquitoes with the strain i. After humans in infected class $I_h^{ij}(a,t)$ stay a duration time, they enter the recovery class $R_h(t)$ at the rate $\gamma_j(a)$.

Similarly, for mosquito populations, the susceptible mosquitoes $S_v(t)$ are born at rate Λ_v, at time t. After bitting the infected humans with the strain i, the susceptible mosquitoes get infected with the strain i. The force of infection from humans

Table 1. Descriptions of the state variables in system (2.1).

State variables	Biological meaning
$S_h(t)$	Number of susceptible humans
$I_h^i(a,t)$	Number of infective humans with the infection age a and primarily strain i
$R_h^j(t)$	Number of recovery humans from primarily infection strain j
$I_h^{ij}(a,t)$	Number of secondary infective humans with the infection age a and strain j
$R_h(t)$	Number of recovery humans from secondary infection
$S_v(t)$	Number of susceptible mosquitoes
$I_v^i(t)$	Number of infective mosquitoes with strain i

to mosquitoes $\tilde{\lambda}_v^i(t)$ is defined as the sum of the force of infections from the infected humans primarily $I_h^i(a,t)$ and the secondarily infected humans $I_h^{ji}(a,t)$. The force of infections from the infected humans primarily (or secondarily) is defined as the product of the number of human bites one mosquito, $b_v(N_h(t), N_v(t))$, the probability of the disease transmission from the primarily (or secondarily) infectious infectious human to the mosquitoes $\tilde{\beta}_v^i(a)$ (or $\phi\tilde{\beta}_v^i(a)$) and the probability that the bitten humans is infected with the strain i, $I_h^i(a,t)/N_h(t)$ (or $I_h^{ji}(a,t)/N_h(t)$). ϕ may characterize the difference in transmission likelihood from humans with the primary or secondary infections to mosquito populations. A flowchart illustrating the transitions between the humans and mosquitoes is shown in Fig. 1. The biological meanings of the parameters in (2.1) are summarized in Table 2. Let σ_v be the number of times that one mosquito would bite humans per unit time, and σ_h be the maximum number of mosquito bites a human can have per unit time. With similar ideas to paper [6], the total number of mosquito bites on humans is

$$b(N_h(t), N_v(t)) = \frac{\sigma_v N_v(t)\sigma_h N_h(t)}{\sigma_v N_v(t) + \sigma_h N_h(t)} = \frac{\sigma_v \sigma_h}{\sigma_v (N_v(t)/N_h(t)) + \sigma_h} N_v(t).$$

The total number of mosquito–human contacts depends on the population sizes of both species. Thus $b(N_h(t), N_v(t)) = b_h(N_h(t), N_v(t))N_h(t)$ can be regarded as the number of bites humans per unite time, and $b(N_h, N_v) = b_v(N_h(t), N_v(t))N_v(t)$ is the number of bites mosquitoes per unite time. The total number of bites received by all humans should keep same as the number of bites per mosquito per unite time, that is

$$b_h(N_h(t), N_v(t))N_h(t) = b_v(N_h(t), N_v(t))N_v(t).$$

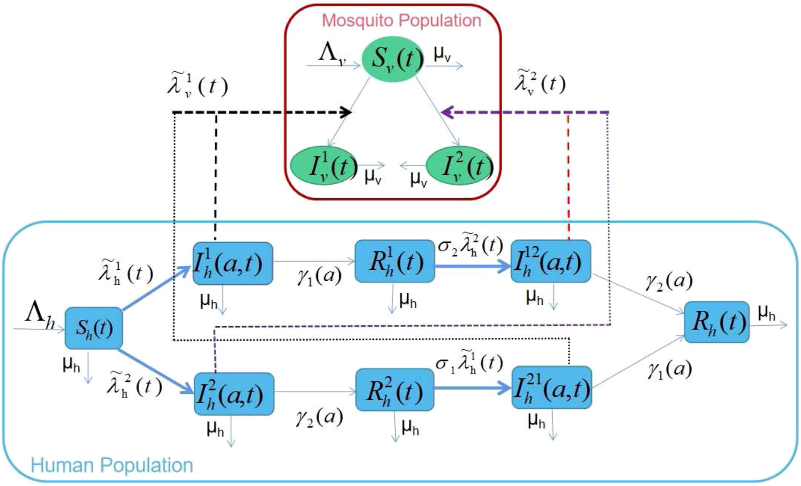

Fig. 1. A flowchart for a two-strain dengue model with infection age and secondary infection.

Table 2. Descriptions of parameters in system (2.1).

Parameters	Biological meaning
Λ_v/Λ_h	The birth or recruitment rates of mosquitoes/humans
μ_v/μ_h	The natural death rates of mosquitoes/humans
$\gamma_j(a)$	The recovery rate of the infected humans of infection age a with strain j
$\widetilde{\beta}_v^j(a)$	The transmission coefficient of the infected humans with strain j
$\widetilde{\beta}_h^j$	The adequate bitting rate from an infected mosquito with strain j to the susceptible humans
ϕ	The factor characterizing the difference in transmission likelihood from humans with primary or secondary infections to mosquitoes
σ_j	The factor showing the difference in transmission likelihood from mosquitoes infected by strain j to humans with or without a previous dengue infection

Obviously, if N_v/N_h is small, then factoring out N_h from the denominator and setting N_v/N_h to 0, we obtain

$$\widetilde{\lambda}_h^j(t) = \sigma_v \widetilde{\beta}_h^j \frac{I_v^j(t)}{N_h(t)}, \quad \widetilde{\lambda}_v^j(t) = \int_0^\infty \sigma_v \widetilde{\beta}_v^j(a) \frac{I_h^j(a,t) + \phi I_h^{ij}(a,t)}{N_h(t)} da.$$

According to the flowchart (Fig. 1), a two-strain dengue epidemic model with infection-age and secondary infection can be described in the following:

$$\begin{aligned}
&\frac{dS_v(t)}{dt} = \Lambda_v - \sum_{j=1}^{2} \widetilde{\lambda}_v^j(t) S_v(t) - \mu_v S_v(t),\\
&\frac{dI_v^j(t)}{dt} = \widetilde{\lambda}_v^j(t) S_v(t) - \mu_v I_v^j(t),\\
&\frac{dS_h(t)}{dt} = \Lambda_h - \sum_{j=1}^{2} \widetilde{\lambda}_h^j(t) S_h(t) - \mu_h S_h(t),\\
&\frac{\partial I_h^j(a,t)}{\partial a} + \frac{\partial I_h^j(a,t)}{\partial t} = -\left(\mu_h + \gamma_j(a)\right) I_h^j(a,t),\\
&I_h^j(0,t) = \widetilde{\lambda}_h^j(t) S_h(t),\\
&\frac{dR_h^j(t)}{dt} = \int_0^\infty \gamma_j(a) I_h^j(a,t) da - \sigma_i \widetilde{\lambda}_h^i(t) R_h^j(t) - \mu_h R_h^j(t),\\
&\frac{\partial I_h^{ji}(a,t)}{\partial a} + \frac{\partial I_h^{ji}(a,t)}{\partial t} = -\left(\mu_h + \gamma_i(a)\right) I_h^{ji}(a,t),\\
&I_h^{ji}(0,t) = \sigma_i \widetilde{\lambda}_h^i(t) R_h^j(t),\\
&\frac{dR_h(t)}{dt} = \int_0^\infty \gamma_1(a) I_h^{21}(a,t) da + \int_0^\infty \gamma_2(a) I_h^{12}(a,t) da - \mu_h R_h(t).
\end{aligned} \tag{2.1}$$

From (2.1), we have

$$\frac{dN_h(t)}{dt} \le \Lambda_h - \mu_h N_h(t).$$

Thus it follows that

$$\limsup_{t\to\infty} N_h(t) \le \frac{\Lambda_h}{\mu_h} \triangleq N_h.$$

Similarly we can yield

$$\frac{d}{dt}N_v(t) \le \Lambda_v - \mu_v N_v(t).$$

Therefore, we have

$$\limsup_{t\to\infty} N_v(t) \le \frac{\Lambda_v}{\mu_v} \triangleq N_v.$$

Let $N_h(t) = N_h$, $N_v(t) = N_v$. To simplify our notations, we rewrite $\widetilde{\lambda}_h^j(t)$ and $\widetilde{\lambda}_v^j(t)$ in system (2.2), as $\lambda_h^j(t)$ and $\lambda_v^j(t)$, respectively. Then, let $\beta_h^j = \sigma_v\tilde{\beta}_h^j$ and $\beta_v^j(a) = \sigma_v\tilde{\beta}_v^j(a)/N_h$. Notice that the equation for $R_h(t)$ is decoupled from the system. Thus, we can investigate the following dynamical system:

$$\begin{aligned}
&\frac{dS_v(t)}{dt} = \Lambda_v - \sum_{j=1}^{2}\lambda_v^j(t)S_v(t) - \mu_v S_v(t),\\
&\frac{dI_v^j(t)}{dt} = \lambda_v^j(t)S_v(t) - \mu_v I_v^j(t),\\
&\frac{dS_h(t)}{dt} = \Lambda_h - \sum_{j=1}^{2}\lambda_h^j(t)S_h(t) - \mu_h S_h(t),\\
&\frac{\partial I_h^j(a,t)}{\partial a} + \frac{\partial I_h^j(a,t)}{\partial t} = -\left(\mu_h + \gamma_j(a)\right) I_h^j(a,t),\\
&I_h^j(0,t) = \lambda_h^j(t)S_h(t),\\
&\frac{dR_h^j(t)}{dt} = \int_0^\infty \gamma_j(a)I_h^j(a,t)da - \sigma_i\lambda_h^i(t)R_h^j(t) - \mu_h R_h^j(t),\\
&\frac{\partial I_h^{ji}(a,t)}{\partial a} + \frac{\partial I_h^{ji}(a,t)}{\partial t} = -(\mu_h + \gamma_i(a))\, I_h^{ji}(a,t),\\
&I_h^{ji}(0,t) = \sigma_i\lambda_h^i(t)R_h^j(t).
\end{aligned} \tag{2.2}$$

System (2.2) is equipped with the following initial conditions:

$$S_v(0) = S_{v0},\quad I_v^j(0) = I_{v0}^j,\quad S_h(0) = S_{h0},\quad I_h^j(a,0) = \psi_j(a),\quad R_h^j(0) = R_{h0}^j,$$
$$I_h^{12}(a,0) = \psi_{12}(a),\quad I_h^{21}(a,0) = \psi_{21}(a).$$

All the parameters are nonnegative and $\Lambda_v > 0, \Lambda_h > 0$, $\phi > 0$, $\sigma_j > 0$, $\mu_v > 0$ and $\mu_h > 0$. In order to perform mathematical analysis, we make the following assumptions.

Assumption 2.1. The parameter-functions in system (2.2) satisfy the following conditions

(1) The functions $\beta_v^j(a)$ are bounded, uniformly continuous and $\inf\{\beta_v^j(a)\} > 0$ for every j;
(2) The functions $\gamma_j(a) \in L^\infty(0,\infty)$ for every j;
(3) For each i, the functions $\psi_j(a)$ and $\psi_{ji}(a)$ all are integrable for every j.

Let us define

$$X = \prod_{j=1}^{3} \mathbb{R} \times \mathbb{R} \times \prod_{j=1}^{2} L^1(0,\infty) \times \prod_{j=1}^{2} \mathbb{R} \times \prod_{j=1}^{2} L^1(0,\infty).$$

By applying similar methods in paper [18], it is easily verified that solutions of system (2.2) with the initial conditions in the positive cone of X, are denoted by X_+, for $t \geq 0$. The following set is positively invariant for system (2.2):

$$\Omega = \Bigg\{ (S_v, I_v^1, I_v^2, S_h, I_h^1, I_h^2, R_h^1, R_h^2, I_h^{12}, I_h^{21}) \in X_+ \,\Big|\, (S_v + I_v^1 + I_v^2) \leq \frac{\Lambda_v}{\mu_v},$$

$$S_h(t) + \sum_{j=1}^{2} \int_0^\infty I_h^j(a,t)da + \sum_{i=1}^{2} R_h^i(t) + \sum_{\substack{j,i=1\\ j\neq i}}^{2} \int_0^\infty I_h^{ji}(a,t)da \leq \frac{\Lambda_h}{\mu_h} \Bigg\}.$$

In the following, we investigate the dengue epidemiology of system (2.2) in the set Ω.

3. The Reproduction Numbers and the Threshold Dynamics

We give the reproduction numbers for the two strains in the system. Noting that the exit rate of the individual infected with the strain j from the infective compartment is given by $\mu_h + \gamma_j(a)$, then the probability of still being infectious after a time units, denoted by $\pi_j(a)$, is given by

$$\pi_j(a) = e^{-\int_0^a (\mu_h + \gamma_j(\sigma))d\sigma}.$$

Let

$$\mathscr{R}_{v1}^j = \frac{\beta_h^j}{\mu_v}, \quad \mathscr{R}_{h1}^j = \frac{\Lambda_v}{\mu_v} \int_0^\infty \beta_v^j(a)\pi_j(a)da.$$

Here $\mathscr{R}_{v1}^j$ represents the number of infected humans with the strain j produced by an infected mosquito with the strain j during its whole infectious period in a completely susceptible human population, $\mathscr{R}_{h1}^j$ gives the number of infected mosquitoes with the strain j by a primarily infected human with the strain j during its whole infectious period in a completely susceptible mosquito population. Let

$$\mathscr{R}_1^j = \mathscr{R}_{h1}^j \mathscr{R}_{v1}^j = \frac{\Lambda_v \beta_h^j}{\mu_v^2} \int_0^\infty \beta_v^j(a)\pi_j(a)da.$$

Then the expression for $\mathscr{R}_1^j$ gives the number of humans infected primarily by the strain j produced in an entirely susceptible population (including humans and mosquitoes) by a typical primarily infected human with the strain j during its entire infectious period. Similarly, let

$$\mathscr{R}_2^j = \left(\frac{\sigma_j \beta_h^j}{\mu_v}\right)\left(\frac{\Lambda_v}{\mu_v}\int_0^\infty \phi\beta_v^j(a)\pi_j(a)\mathrm{da}\right) = \phi\sigma_j\mathscr{R}_1^j.$$

The expression for $\mathscr{R}_2^j$ gives the number of secondary infected humans by the strain j produced in a population, where all humans are in semi-susceptible class R_h^i and the mosquitoes are susceptible, by a typical human with the secondary infection the strain j during its entire infectious period. Let

$$\mathscr{R}_0^j = \mathscr{R}_1^j + \mathscr{R}_2^j = (1+\phi\sigma_j)\mathscr{R}_1^j.$$

Then $\mathscr{R}_0^j$ characterizes the overall infectivity of strain j. We define the disease reproduction number, $\mathscr{R}_0$, by

$$\mathscr{R}_0 = \max\{\mathscr{R}_1^1,\ \mathscr{R}_1^2\}.$$

Specially, when all humans are semi-susceptible, we define it as $\mathscr{R}_c$ by

$$\mathscr{R}_c = \max\{\mathscr{R}_2^1,\ \mathscr{R}_2^2\}.$$

To avoid confusion, the above expressions are summarized in Table 3.

Now we first give the results on the threshold dynamics of strain j.

Table 3. The biology explanations for the threshold expressions.

$\mathscr{R}_{v1}^j = \frac{\beta_h^j}{\mu_v}$	The number of infected humans with strain j produced by a infected vector with strain j during its whole infectious period in a completely susceptible humans population
$\mathscr{R}_{h1}^j = \frac{\Lambda_v}{\mu_v}\int_0^\infty \beta_v^j(a)\pi_j(a)da$	The number of infected vectors with strain j by a primarily infected human with strain j during its whole infectious period in a completely susceptible vectors population
$\mathscr{R}_1^j = \frac{\Lambda_v\beta_h^j}{\mu_v^2}\int_0^\infty \beta_v^j(a)\pi_j(a)da$	The number of human infected primarily by strain j produced in an entirely susceptible population (including humans and vectors) by a typical primarily infected human with strain j during its entire infectious period
$\mathscr{R}_2^j = \frac{\sigma_j\Lambda_v\beta_h^j}{\mu_v^2}\int_0^\infty \phi\beta_v^j(a)\pi_j(a)\mathrm{da}$	The number of human infected secondarily by strain j produced in a population, where all humans are in semi-susceptible class R_h^i and vectors are susceptible, by a typical secondarily infected human with strain j during its entire infectious period
$\mathscr{R}_0^j = \mathscr{R}_1^j + \mathscr{R}_2^j$	The overall infectivity of strain j
$\mathscr{R}_0 = \max\{\mathscr{R}_1^1,\ \mathscr{R}_1^2\}$	The disease reproduction number
$\mathscr{R}_c = \max\{\mathscr{R}_2^1,\ \mathscr{R}_2^2\}$	The disease reproduction number when all the humans are semi-susceptible
$\mathscr{R}_1$	$\mathscr{R}_1^1$ or $\mathscr{R}_1^2$ when $\beta_h^1 = \beta_h^2$, $\beta_v^1(a) = \beta_v^2(a)$ and $\gamma_1(a) = \gamma_2(a)$

Theorem 3.1. *If* $\mathscr{R}_0^j < 1$, *the strain* j *in system* (2.2) *will die out.*

Proof. We only prove the case for one strain infection, the other case can also be obtained easily. Let

$$B_j(t) = I_h^j(0,t), \quad B_{12}(t) = I_h^{12}(0,t), \quad B_{21}(t) = I_h^{21}(0,t).$$

Integrating along the characteristic lines of system (2.2) yields

$$I_h^1(a,t) = \begin{cases} B_1(t-a)\pi_1(a), & a \le t, \\ \psi_1(a-t)\dfrac{\pi_1(a)}{\pi_1(a-t)}, & t < a, \end{cases} \tag{3.1}$$

$$I_h^{21}(a,t) = \begin{cases} B_{21}(t-a)\pi_1(a), & a \le t, \\ \psi_{21}(a-t)\dfrac{\pi_1(a)}{\pi_1(a-t)}, & t < a. \end{cases} \tag{3.2}$$

From the first and the fourth equations of system (2.2), we have

$$\limsup_{t\to\infty} S_v(t) \le \frac{\Lambda_v}{\mu_v}, \quad \limsup_{t\to\infty} S_h(t) \le \frac{\Lambda_h}{\mu_h}. \tag{3.3}$$

Then, from the second equation of system (2.2), when the time t is sufficiently large, we obtain

$$\frac{dI_v^1(t)}{dt} \le \frac{\Lambda_v}{\mu_v}\int_0^\infty \beta_v^1(a)\big(I_h^1(a,t) + \phi I_h^{21}(a,t)\big)da - \mu_v I_v^1(t). \tag{3.4}$$

By the comparison principle, we have

$$\begin{aligned} I_v^1(t) &\le I_v^1(0)e^{-\mu_v t} + \frac{\Lambda_v}{\mu_v}\int_0^t e^{-\mu_v(t-s)}\int_0^\infty \beta_v^1(a)\big(I_h^1(a,s) + \phi I_h^{21}(a,s)\big)dads \\ &= I_v^1(0)e^{-\mu_v t} + \frac{\Lambda_v}{\mu_v}\int_0^t \Bigg(\underbrace{e^{-\mu_v(t-s)}\int_0^s \beta_v^1(a)B_1(s-a)\pi_1(a)da}_{\text{the first term}} \\ &\quad + \underbrace{e^{-\mu_v(t-s)}\int_s^\infty \beta_v^1(a)\psi_1(a-s)\frac{\pi_1(a)}{\pi_1(a-s)}da}_{\text{the second term}}\Bigg)ds \\ &\quad + \frac{\Lambda_v}{\mu_v}\int_0^t \Bigg(\underbrace{e^{-\mu_v(t-s)}\int_0^s \beta_v^1(a)\phi B_{21}(s-a)\pi_1(a)da}_{\text{the third term}} \\ &\quad + \underbrace{e^{-\mu_v(t-s)}\int_s^\infty \beta_v^1(a)\phi\psi_{21}(a-s)\frac{\pi_1(a)}{\pi_1(a-s)}da}_{\text{the fourth term}}\Bigg)ds. \end{aligned} \tag{3.5}$$

Integrating and taking the superior limit of the first term yields

$$\begin{aligned}
&\limsup_{t\to\infty}\int_0^t e^{-\mu_v(t-s)}\int_0^s \beta_v^1(a)B_1(s-a)\pi_1(a)dads\\
&\quad\le \limsup_{t\to\infty}\int_0^t e^{-\mu_v(t-s)}\int_0^\infty \beta_v^1(a)\pi_1(a)da\left(\limsup_{t\to\infty}B_1(t)\right)ds\\
&\quad= \limsup_{t\to\infty}\left(\int_0^t e^{-\mu_v(t-s)}ds\right)\int_0^\infty \beta_v^1(a)\pi_1(a)da\left(\limsup_{t\to\infty}B_1(t)\right)\\
&\quad= \frac{1}{\mu_v}\left(\int_0^\infty \beta_v^1(a)\pi_1(a)da\right)\left(\limsup_{t\to\infty}B_1(t)\right).
\end{aligned}$$

Integrating the second term yields

$$\begin{aligned}
&\int_0^t e^{-\mu_v(t-s)}\int_s^\infty \beta_v^1(a)\psi_1(a-s)\frac{\pi_1(a)}{\pi_1(a-s)}dads\\
&\quad\le \sup_a\{\beta_v^1(a)\}\int_0^t e^{-\mu_v(t-s)}\int_s^\infty \psi_1(a-s)e^{-\mu_h s}e^{-\int_{a-s}^a\gamma_1(\sigma)d\sigma}dads\\
&\quad\le \sup_a\{\beta_v^1(a)\}\int_0^t e^{-\mu_v(t-s)}\int_s^\infty \psi_1(a-s)e^{-\mu_h s}dads\\
&\quad\le \sup_a\{\beta_v^1(a)\}e^{-\mu_v t}\int_0^t\int_s^\infty \psi_1(a-s)e^{(\mu_v-\mu_h)s}dads\\
&\quad\le \sup_a\{\beta_v^1(a)\}e^{-\mu_v t}\int_0^t\int_0^\infty \psi_1(a)e^{(\mu_v-\mu_h)s}dads\\
&\quad\le \sup_a\{\beta_v^1(a)\}e^{-\mu_v t}\int_0^t e^{(\mu_v-\mu_h)s}ds\left(\int_0^\infty \psi_1(a)da\right)\\
&\quad= \sup_a\{\beta_v^1(a)\}e^{-\mu_v t}\frac{e^{(\mu_v-\mu_h)t}-1}{\mu_v-\mu_h}\left(\int_0^\infty \psi_1(a)da\right)\\
&\quad= \sup_a\{\beta_v^1(a)\}\frac{e^{-\mu_h t}-e^{-\mu_v t}}{\mu_v-\mu_h}\left(\int_0^\infty \psi_1(a)da\right).
\end{aligned}$$

Thus,

$$\limsup_{t\to\infty}\int_0^t e^{-\mu_v(t-s)}\int_s^\infty \beta_v^1(a)\psi_1(a-s)\frac{\pi_1(a)}{\pi_1(a-s)}dads = 0.$$

With similar arguments to the third and fourth terms, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\int_0^t e^{-\mu_v(t-s)}\int_0^s \beta_v^1(a)\phi B_{21}(s-a)\pi_1(a)dads\\
&\quad\le \phi\frac{1}{\mu_v}\left(\int_0^\infty \beta_v^1(a)\pi_1(a)da\right)\left(\limsup_{t\to\infty}B_{21}(t)\right)
\end{aligned}$$

and

$$\limsup_{t\to\infty}\int_0^t e^{-\mu_v(t-s)}\int_s^\infty \beta_v^1(a)\phi\psi_{21}(a-s)\frac{\pi_1(a)}{\pi_1(a-s)}dads = 0.$$

Consequently,

$$\begin{aligned}\limsup_{t\to\infty} I_v^1(t) &\le \frac{\Lambda_v}{\mu_v^2}\int_0^\infty \beta_v^1(a)\pi_1(a)da\left(\limsup_{t\to\infty} B_1(t) + \phi\limsup_{t\to\infty} B_{21}(t)\right)\\ &= \frac{\Lambda_v}{\mu_v^2}\int_0^\infty \beta_v^1(a)\pi_1(a)da\left(\limsup_{t\to\infty}\frac{\beta_h^1 S_h(t)I_v^1(t)}{N_h} + \phi\limsup_{t\to\infty}\frac{\sigma_1\beta_h^1 R_h^2(t)I_v^1(t)}{N_h}\right)\\ &\le \frac{\Lambda_v}{\mu_v^2}\int_0^\infty \beta_v^1(a)\pi_1(a)da\left(\limsup_{t\to\infty}\left(\beta_h^1 I_v^1(t)\right) + \phi\limsup_{t\to\infty}\left(\sigma_1\beta_h^1 I_v^1(t)\right)\right) \qquad (3.6)\\ &\le (1+\phi\sigma_1)\frac{\Lambda_v\beta_h^1}{\mu_v^2}\int_0^\infty \beta_v^1(a)\pi_1(a)da\left(\limsup_{t\to\infty}\left(I_v^1(t)\right)\right)\\ &= \mathscr{R}_0^1\left(\limsup_{t\to\infty} I_v^1(t)\right).\end{aligned}$$

Since $\mathscr{R}_0^1 < 1$ and $I_v^1(t)$ is bounded, it follows that

$$\limsup_{t\to\infty} I_v^1(t) = 0.$$

Also notice that

$$0 \le I_h^1(0,t-a) = \frac{\beta_h^1 S_h(t-a)I_v^1(t-a)}{N_h} \le \beta_h^1 I_v^1(t-a) \to 0 \quad (t\to\infty).$$

Hence,

$$\limsup_{t\to\infty} I_h^1(a,t) = \limsup_{t\to\infty} I_h^1(0,t-a)\pi_1(a) = 0.$$

Similarly,

$$\limsup_{t\to\infty} I_h^{21}(a,t) = 0.$$

Therefore, $(I_v^1(t), I_h^1(t), I_h^{21}(t)) \to (0,0,0)$ as $t\to\infty$. This implies strain 1 in system (2.2) will die out. The proof of Theorem 3.1 is completed. □

4. Global Stability of the Disease-Free Equilibrium

In this section, we investigate the global stability of the disease-free equilibrium in system (2.2). System (2.2) always has a unique disease-free equilibrium $\mathscr{E}_0$, which is given by

$$\mathscr{E}_0 = (S_v^0, \mathbf{0}, S_h^0, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

where

$$S_h^0 = \frac{\Lambda_h}{\mu_h}, \quad S_v^0 = \frac{\Lambda_v}{\mu_v},$$

where $\mathbf{0} = (0, 0)$ is a two-dimensional zero vector.

First, we investigate the local stability of the disease-free equilibrium. Linearizing system (2.2) at the disease-free equilibrium $\mathscr{E}_0$, we get the following characteristic equation:

$$\lambda + \mu_v = \frac{\Lambda_v \beta_h^1}{\mu_v} \int_0^\infty \beta_v^1(a) e^{-\lambda a} \pi_1(a) da. \tag{4.1}$$

Similarly, we can obtain another characteristic equation

$$\lambda + \mu_v = \frac{\Lambda_v \beta_h^2}{\mu_v} \int_0^\infty \beta_v^2(a) e^{-\lambda a} \pi_2(a) da. \tag{4.2}$$

Now, we can state the following result.

Theorem 4.1. *If* $\mathscr{R}_0 = \max\{\mathscr{R}_1^1, \mathscr{R}_1^2\} < 1$, *the disease-free equilibrium* $\mathscr{E}_0$ *is locally asymptotically stable, and if* $\mathscr{R}_0 = \max\{\mathscr{R}_1^1, \mathscr{R}_1^2\} > 1$, *it is unstable.*

Proof. Let us assume $\max\{\mathscr{R}_1^1, \mathscr{R}_1^2\} = \mathscr{R}_1^1 > 1$. We rewrite the characteristic equation (4.1) in the form

$$\mathscr{L}_1(\lambda) \triangleq \lambda + \mu_v - \frac{\Lambda_v \beta_h^1}{\mu_v} \int_0^\infty \beta_v^1(a) e^{-\lambda a} \pi_1(a) da = 0. \tag{4.3}$$

Obviously, the function $\mathscr{L}_1(\lambda)$ is strictly increasing in the interval $(0, \infty)$ and

$$\lim_{\lambda \to \infty} \mathscr{L}_1(\lambda) = \infty.$$

Also notice that

$$\begin{aligned} \mathscr{L}_1(0) &= \mu_v - \frac{\Lambda_v \beta_h^1}{\mu_v} \int_0^\infty \beta_v^1(a) \pi_1(a) da \\ &= \mu_v \left(1 - \frac{\Lambda_v \beta_h^1}{\mu_v^2} \int_0^\infty \beta_v^1(a) \pi_1(a) da\right) \\ &= \mu_v \left(1 - \mathscr{R}_1^1\right) < 0. \end{aligned}$$

Therefore, equation $\mathscr{L}_1(\lambda) = 0$ has a real positive root. The disease-free equilibrium is unstable.

Now we consider the case $\mathscr{R}_0 < 1$. Let

$$\begin{aligned} \text{LHS} &\triangleq \lambda + \mu_v, \\ \text{RHS} &\triangleq \mathscr{L}_2(\lambda) = \frac{\Lambda_v \beta_h^1}{\mu_v} \int_0^\infty \beta_v^1(a) e^{-\lambda a} \pi_1(a) da. \end{aligned} \tag{4.4}$$

If λ_0 is any root of Eq. (4.1) with nonnegative real part $Re\lambda_0 \geq 0$, it is easy to show that

$$\begin{aligned} |\text{LHS}| &\geq \mu_v, \\ |\text{RHS}| &\leq \mathscr{L}_2(Re\lambda_0) \leq \mathscr{L}_2(0) = \frac{\Lambda_v \beta_h^1}{\mu_v} \int_0^\infty \beta_v^1(a) \pi_1(a) da \\ &= \mu_v \frac{\Lambda_v \beta_h^1}{\mu_v^2} \int_0^\infty \beta_v^1(a) \pi_1(a) da = \mu_v \mathscr{R}_1^1 < \mu_v \leq |\text{LHS}|. \end{aligned}$$

Obviously, this is a contradiction. Equation (4.1) has no root with nonnegative real parts. Using similar arguments in Eq. (4.1), it is easy to prove that Eq. (4.2) has a real positive root if $\mathscr{R}_1^2 > 1$ and has only the roots with negative real parts if $\mathscr{R}_1^2 < 1$. Therefore, the disease-free equilibrium is still unstable if $\mathscr{R}_1^2 > 1$. This concludes the proof. ☐

Now we show the global behavior of the disease-free equilibrium $\mathscr{E}_0$.

Theorem 4.2. *When* $\mathscr{R}_0^j < 1$, $j = 1, 2$, *the disease-free equilibrium* $\mathscr{E}_0$ *is global asymptotically stable.*

Proof. From Theorem 4.1, we only need to show that the disease-free equilibrium is globally attractive. In fact, from Theorem 3.1, we know that if $\mathscr{R}_0^j < 1$, we have

$$\lim_{t\to\infty} I_v^j(t) = \lim_{t\to\infty} I_h^j(a,t) = \lim_{t\to\infty} I_h^{21}(a,t) = \lim_{t\to\infty} I_h^{12}(a,t) = 0.$$

Also notice that $\lim_{t\to\infty}(S_v(t) + I_v^1(t) + I_v^2(t)) = \frac{\Lambda_v}{\mu_v}$. Then we have $\lim_{t\to\infty} S_v(t) = \frac{\Lambda_v}{\mu_v}$. Solving the fourth equation of system (2.2), we obtain

$$S_h(t) = S_h(0) e^{-\int_0^t \left(\frac{\beta_h^2 I_v^2(\theta)}{N_h} + \frac{\beta_h^1 I_v^1(\theta)}{N_h} + \mu_h\right) d\theta} + \frac{\int_0^t \Lambda_h e^{\int_0^\xi \left(\frac{\beta_h^2 I_v^2(\theta)}{N_h} + \frac{\beta_h^1 I_v^1(\theta)}{N_h} + \mu_h\right) d\theta} d\xi}{e^{\int_0^t \left(\frac{\beta_h^2 I_v^2(\theta)}{N_h} + \frac{\beta_h^1 I_v^1(\theta)}{N_h} + \mu_h\right) d\theta}}.$$

Since

$$0 \leq S_h(0) e^{-\int_0^t \left(\frac{\beta_h^2 I_v^2(\theta)}{N_h} + \frac{\beta_h^1 I_v^1(\theta)}{N_h} + \mu_h\right) d\theta} \leq S_h(0) e^{-\mu_h t} \to 0 \quad (t \to \infty),$$

we have

$$\lim_{t\to\infty} S_h(0)e^{-\int_0^t\left(\frac{\beta_h^2 I_v^2(\theta)}{N_h}+\frac{\beta_h^1 I_v^1(\theta)}{N_h}+\mu_h\right)d\theta} = 0.$$

Using the L'Hospital's rule, we have

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\int_0^t \Lambda_h e^{\int_0^\xi\left(\frac{\beta_h^2 I_v^2(\theta)}{N_h}+\frac{\beta_h^1 I_v^1(\theta)}{N_h}+\mu_h\right)d\theta}d\xi}{e^{\int_0^t\left(\frac{\beta_h^2 I_v^2(\theta)}{N_h}+\frac{\beta_h^1 I_v^1(\theta)}{N_h}+\mu_h\right)d\theta}}\\
&\quad=\lim_{t\to\infty}\frac{\Lambda_h e^{\int_0^t\left(\frac{\beta_h^2 I_v^2(\theta)}{N_h}+\frac{\beta_h^1 I_v^1(\theta)}{N_h}+\mu_h\right)d\theta}}{e^{\int_0^t\left(\frac{\beta_h^2 I_v^2(\theta)}{N_h}+\frac{\beta_h^1 I_v^1(\theta)}{N_h}+\mu_h\right)d\theta}\left(\frac{\beta_h^2 I_v^2(t)}{N_h}+\frac{\beta_h^1 I_v^1(t)}{N_h}+\mu_h\right)}\\
&\quad=\lim_{t\to\infty}\frac{\Lambda_h}{\frac{\beta_h^2 I_v^2(t)}{N_h}+\frac{\beta_h^1 I_v^1(t)}{N_h}+\mu_h}=\frac{\Lambda_h}{\mu_h}=S_h^0.
\end{aligned}$$

Also considering

$$\begin{aligned}
&\lim_{t\to\infty}\left(S_h(t)+\sum_{j=1}^{2}\left(\int_0^\infty I_h^j(a,t)da+R_h^j(t)\right)\right.\\
&\qquad\left.+\int_0^\infty\left(I_h^{12}(a,t)+I_h^{21}(a,t)\right)da+R_h(t)\right)=\frac{\Lambda_h}{\mu_h},
\end{aligned}$$

thus, we obtain

$$\lim_{t\to\infty}\left(\sum_{j=1}^{2}R_h^j(t)+R_h(t)\right)=0.$$

Consequently, by the nonnegativity of the solutions in system (2.2), we have

$$\lim_{t\to\infty}R_h^1(t)=\lim_{t\to\infty}R_h^2(t)=0.$$

These imply the disease-free equilibrium $\mathscr{E}_0$ is globally attractive. This completes the proof. □

Now, we provide a numerical example to confirm the result in Theorem 4.2, where the global stability of the disease-free equilibrium is guaranteed if $\mathscr{R}_0^j < 1$, $j = 1, 2$. We choose constant parameters by $\Lambda_h = 1000, \Lambda_v = 20, \phi = 0.6, \sigma_1 = 0.01, \sigma_2 = 0.1, \beta_h^1 = 0.18, \beta_h^2 = 0.2, \gamma_1 = 0.12, \gamma_2 = 0.1, \mu_v = 0.8, \mu_h = 0.6, \beta_v^1 = 0.08, \beta_v^2 = 0.06$. Then $\mathcal{R}_0 + \varsigma = 0.0044 < 1$. The numerical solutions of $S_h(t), I_h^j(a, t), R_h^j(a), I_h^{ji}, S_v(t)$ and $I_v^j(t)$ are displayed in Figs. 2 and 3. Clearly, the solutions $I_h^j(a, t), I_h^{ij}(a, t), R_h^j(t)$ and $I_v^j(t)$ all approach zero when t is large enough. Besides, the solutions $S_h(t)$ and $S_v(t)$ both approach corresponding components of the disease-free equilibrium.

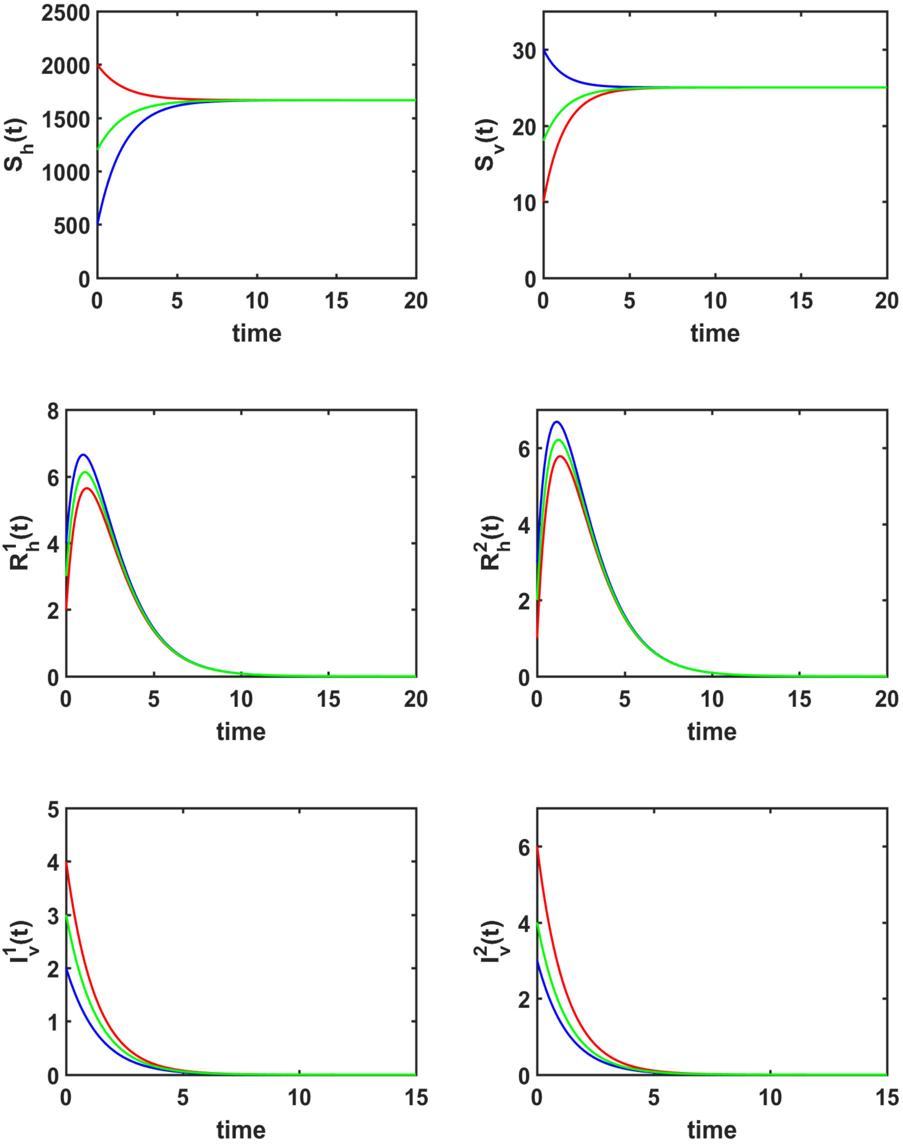

Fig. 2. (Color online) A typical scenario showing the time series of $S_h(t)$, $R_h^j(t)$, $S_v(t)$ and $I_v^j(t)$ when $\mathscr{R}_0^j < 1$, $j = 1, 2$. The difference in curve color only means that the initial values of variables are different.

5. Existence and Stability of Dominant Equilibria

In this section, we mainly investigate the existence and the local stability of the dominant equilibria. For the sake of simplicity, let

$$b_j = \int_0^{\infty} \beta_v^j(a)\pi_j(a)da, \quad b_j(\lambda) = \int_0^{\infty} \beta_v^j(a)e^{-\lambda a}\pi_j(a)da, \quad c_j = \int_0^{\infty} \gamma_j(a)\pi_j(a)da.$$

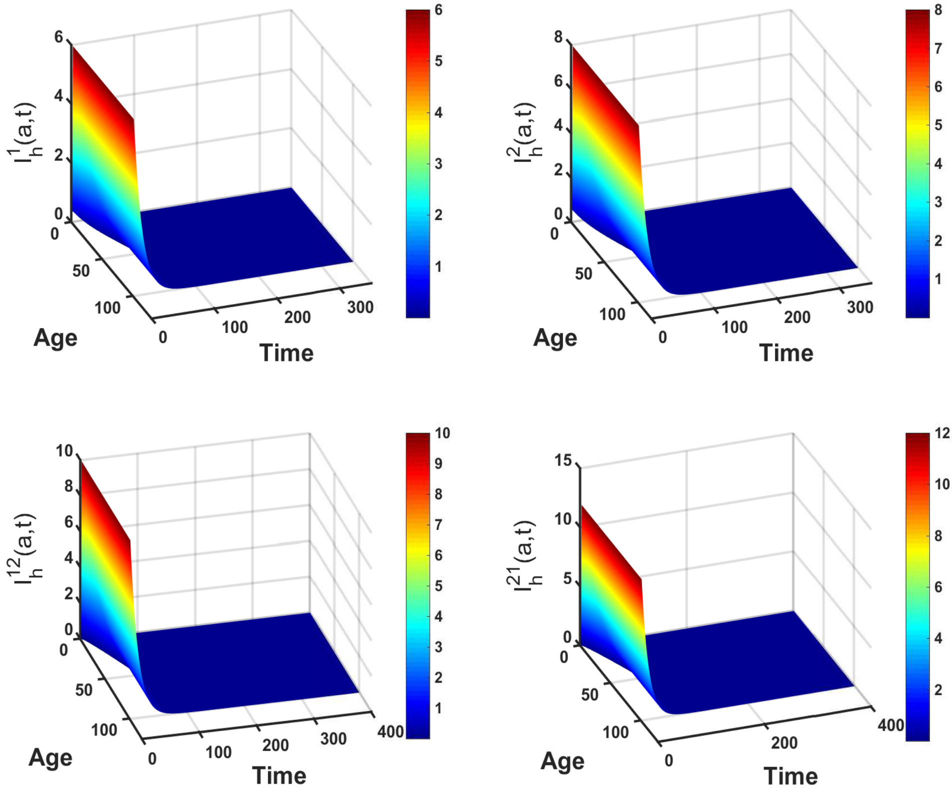

Fig. 3. A typical scenario showing the time and age series of $I_h^j(a,t)$ and $I_h^{ij}(a,t)$ when $\mathscr{R}_0^j < 1, j = 1, 2$.

From Theorem 3.1, we know that strain j will die out when $\mathscr{R}_0^j < 1$. Thus, in the following, it is assumed that $\mathscr{R}_0^j > 1$. We will give the result in the case where $j = 1$, and the conclusion also holds for $j = 2$.

Theorem 5.1. *If $\mathscr{R}_1^1 > 1$, there exists a unique strain 1 dominant equilibrium $\mathscr{E}_1$ in system (2.2), where $\mathscr{E}_1$ is given by*

$$\mathscr{E}_1 = (S_{v1}^*, I_{v1}^*, 0, S_{h1}^*, I_{h1}^*(a), 0, R_{h1}^*, 0, 0, 0).$$

The nonzero components are given as follows:

$$I_{v1}^* = \frac{\mu_v \Lambda_h (\mathscr{R}_1^1 - 1)}{\beta_h^1 (\mu_v + \Lambda_h b_1)}, \quad S_{v1}^* = \frac{\Lambda_v}{\mu_v} - I_{v1}^*, \quad S_{h1}^* = \frac{\Lambda_h}{\mu_h + \frac{\beta_h^1 I_{v1}^*}{N_h}},$$

$$I_{h1}^*(a) = I_{h1}^*(0)\pi_1(a), \quad I_{h1}^*(0) = \frac{\beta_h^1 S_{h1}^* I_{v1}^*}{N_h}, \quad R_{h1}^* = \frac{1}{\mu_h}\int_0^\infty \gamma_1(a) I_{h1}^*(a)da.$$

Next, we first investigate the local stability of the dominant equilibrium $\mathscr{E}_1$. Let $\mathscr{R}_1^1 > 1$,

$$\varsigma = \frac{1}{1 + \frac{\phi\sigma_2\beta_h^1 I_{h1}^* c_1}{\Lambda_h}} = \frac{1}{1 + \frac{\phi\sigma_2 c_1 \mu_v(\mathscr{R}_1^1 - 1)}{\mu_v + \Lambda_h b_1}},$$

obviously, $\varsigma \in (0,1)$.

Theorem 5.2. *Assuming $\mathscr{R}_1^1 > 1$. Then, the strain 1 dominance equilibrium $\mathscr{E}_1$ of system (2.2) is unstable if $\mathscr{R}_1^2/\mathscr{R}_1^1 > 1$ and locally asymptotically stable if $\mathscr{R}_1^2/\mathscr{R}_1^1 < \varsigma\,(< 1)$.*

Proof. Linearizing system (2.2) at $\mathscr{E}_1$, we have

$$\begin{aligned}
\frac{d\bar{S}_v(t)}{dt} &= -\bar{S}_v(t)\int_0^\infty \beta_v^1(a) I_{h1}^*(a)da \\
&\quad - S_{v1}^* \sum_{j=1, j\neq i}^{j=2} \int_0^\infty \beta_v^j(a)\left(\bar{I}_h^j(a,t) + \phi \bar{I}_h^{ij}(a,t)\right)da - \mu_v \bar{S}_v(t), \\
\frac{d\bar{I}_v^1(t)}{dt} &= \bar{S}_v(t)\int_0^\infty \beta_v^1(a) I_{h1}^*(a)da \\
&\quad + S_{v1}^* \int_0^\infty \beta_v^1(a)\left(\bar{I}_h^1(a,t) + \phi\bar{I}_h^{21}(a,t)\right)da - \mu_v \bar{I}_v^1(t), \\
\frac{d\bar{I}_v^2(t)}{dt} &= S_{v1}^* \int_0^\infty \beta_v^2(a)\left(\bar{I}_h^2(a,t) + \phi\bar{I}_h^{12}(a,t)\right)da - \mu_v \bar{I}_v^2(t), \\
\frac{d\bar{S}_h(t)}{dt} &= -\frac{\beta_h^1}{N_h}\left(S_{h1}^*\bar{I}_v^1(t) + I_{v1}^*\bar{S}_h(t)\right) - \frac{\beta_h^2 S_{h1}^* \bar{I}_v^2}{N_h} - \mu_h \bar{S}_h(t), \\
\frac{\partial \bar{I}_h^1(a,t)}{\partial a} &+ \frac{\partial \bar{I}_h^1(a,t)}{\partial t} = -\left(\mu_h + \gamma_1(a)\right)\bar{I}_h^1(a,t), \\
\bar{I}_h^1(0,t) &= \frac{\beta_h^1}{N_h}\left(S_{h1}^*\bar{I}_v^1(t) + I_{v1}^*\bar{S}_h(t)\right), \\
\frac{\partial \bar{I}_h^2(a,t)}{\partial a} &+ \frac{\partial \bar{I}_h^2(a,t)}{\partial t} = -\left(\mu_h + \gamma_2(a)\right)\bar{I}_h^2(a,t), \quad \bar{I}_h^2(0,t) = \frac{\beta_h^2}{N_h} S_{h1}^* \bar{I}_v^2(t), \\
\frac{d\bar{R}_h^1(t)}{dt} &= \int_0^\infty \gamma_1(a)\bar{I}_h^1(a,t)da - \frac{\sigma_2\beta_h^2}{N_h} R_{h1}^* \bar{I}_v^2(t) - \mu_h \bar{R}_h^1(t), \\
\frac{d\bar{R}_h^2(t)}{dt} &= \int_0^\infty \gamma_2(a)\bar{I}_h^2(a,t)da - \frac{\sigma_1\beta_h^1}{N_h} I_{v1}^* \bar{R}_h^2(t) - \mu_h \bar{R}_h^2(t),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \bar{I}_h^{12}(a,t)}{\partial a}+\frac{\partial \bar{I}_h^{12}(a,t)}{\partial t} = -\left(\mu_h+\gamma_2(a)\right)\bar{I}_h^{12}(a,t), \quad \bar{I}_h^{12}(0,t)=\frac{\sigma_2\beta_h^2}{N_h}R_{h1}^*\bar{I}_v^2(t),\\
&\frac{\partial \bar{I}_h^{21}(a,t)}{\partial a}+\frac{\partial \bar{I}_h^{21}(a,t)}{\partial t} = -\left(\mu_h+\gamma_1(a)\right)\bar{I}_h^{21}(a,t),\ \bar{I}_h^{21}(0,t)=\frac{\sigma_1\beta_h^1}{N_h}I_{v1}^*\bar{R}_h^2(t).
\end{aligned} \tag{5.1}$$

To investigate the stability of $\mathscr{E}_1$, we let

$$\begin{aligned}
&\bar{S}_v(t)=\bar{S}_v e^{\lambda t}, \quad \bar{I}_v^j(t)=\bar{I}_v^j e^{\lambda t}, \quad \bar{S}_h(t)=\bar{S}_h e^{\lambda t}, \quad \bar{I}_h^j(a,t)=\bar{I}_h^j(a)e^{\lambda t},\\
&\bar{R}_h^j(t)=\bar{R}_h^j e^{\lambda t}, \quad \bar{I}_h^{ji}(a,t)=\bar{I}_h^{ji}(a)e^{\lambda t}.
\end{aligned}$$

Substituting the expressions above into system (5.1) and omitting the bar, we have

$$\begin{aligned}
\lambda S_v &= -S_v\int_0^\infty \beta_v^1(a)I_{h1}^*(a)da - S_{v1}^*\int_0^\infty \beta_v^1(a)\left(I_h^1(a)+\phi I_h^{21}(a)\right)da\\
&\quad -\ S_{v1}^*\int_0^\infty \beta_v^2(a)\left(I_h^2(a)+\phi I_h^{12}(a)\right)da-\mu_v S_v,\\
\lambda I_v^1 &= S_{v1}^*\int_0^\infty \beta_v^1(a)\left(I_h^1(a)+\phi I_h^{21}(a)\right)da + S_v\int_0^\infty \beta_v^1(a)I_{h1}^*(a)da-\mu_v I_v^1,\\
\lambda I_v^2 &= S_{v1}^*\int_0^\infty \beta_v^2(a)\left(I_h^2(a)+\phi I_h^{12}(a)\right)da-\mu_v I_v^2,\\
\lambda S_h &= -\frac{\beta_h^1}{N_h}\left(S_{h1}^*I_v^1+I_{v1}^*S_h\right)-\frac{\beta_h^2}{N_h}S_{h1}^*I_v^2-\mu_h S_h,\\
&\frac{dI_h^1(a)}{da}+\lambda I_h^1(a) = -\left(\mu_h+\gamma_1(a)\right)I_h^1(a), \quad I_h^1(0)=\frac{\beta_h^1}{N_h}\left(S_{h1}^*I_v^1+I_{v1}^*S_h\right),\\
&\frac{dI_h^2(a)}{da}+\lambda I_h^2(a) = -\left(\mu_h+\gamma_2(a)\right)I_h^2(a), \quad I_h^2(0)=\frac{\beta_h^2}{N_h}S_{h1}^*I_v^2,\\
\lambda R_h^1 &= \int_0^\infty \gamma_1(a)I_h^1(a)da-\frac{\sigma_2\beta_h^2}{N_h}R_{h1}^*I_v^2-\mu_h R_h^1,\\
\lambda R_h^2 &= \int_0^\infty \gamma_2(a)I_h^2(a)da-\frac{\sigma_1\beta_h^1}{N_h}I_{v1}^*R_h^2-\mu_h R_h^2,\\
&\frac{dI_h^{12}(a)}{da}+\lambda I_h^{12}(a) = -\left(\mu_h+\gamma_2(a)\right)I_h^{12}(a), \quad I_h^{12}(0)=\frac{\sigma_2\beta_h^2}{N_h}R_{h1}^*I_v^2,\\
&\frac{dI_h^{21}(a)}{da}+\lambda I_h^{21}(a) = -\left(\mu_h+\gamma_1(a)\right)I_h^{21}(a), \quad I_h^{21}(0)=\frac{\sigma_1\beta_h^1}{N_h}I_{v1}^*R_h^2.
\end{aligned} \tag{5.2}$$

First, solving the sixth and ninth differential equations in system (5.2), we get

$$I_h^2(a)=\frac{\beta_h^2}{N_h}S_{h1}^*I_v^2e^{-\lambda a}\pi_2(a), \quad I_h^{12}(a)=\frac{\sigma_2\beta_h^2}{N_h}R_{h1}^*I_v^2e^{-\lambda a}\pi_2(a).$$

Substituting the expressions for $I_h^2(a)$ and $I_h^{12}(a)$ into the third equation in system (5.2), we get one characteristic equation:

$$\lambda + \mu_v = S_{v1}^* \int_0^\infty \beta_v^2(a) \left(\frac{\beta_h^2}{N_h} S_{h1}^* e^{-\lambda a} \pi_2(a) + \frac{\phi\sigma_2\beta_h^2}{N_h} R_{h1}^* e^{-\lambda a} \pi_2(a) \right) da. \tag{5.3}$$

By some computing, we have

$$S_{v1}^* S_{h1}^* = \frac{\mu_v N_h}{b_1 \beta_h^1} = \frac{\Lambda_v N_h}{\mu_v \mathscr{R}_1^1}.$$

Also noting that $R_{h1}^* = \frac{1}{\mu_h}\int_0^\infty \gamma_1(a) I_{h1}^*(a) da$, by further simplifying Eq. (5.3), we have

$$\lambda + \mu_v = \frac{\beta_h^2 \Lambda_v}{\mu_v \mathscr{R}_1^1} b_2(\lambda) + \frac{\phi\sigma_2 c_1 \beta_h^1 \beta_h^2 \Lambda_v I_{v1}^*}{\mu_v \Lambda_h \mathscr{R}_1^1} b_2(\lambda). \tag{5.4}$$

Let us assume $\mathscr{R}_1^2/\mathscr{R}_1^1 > 1$. We rewrite the characteristic equation (5.4) in the form of

$$\mathscr{L}_3(\lambda) \triangleq \lambda + \mu_v - \frac{\beta_h^2 \Lambda_v}{\mu_v \mathscr{R}_1^1} b_2(\lambda) - \frac{\phi\sigma_2 c_1 \beta_h^1 \beta_h^2 \Lambda_v I_{v1}^*}{\mu_v \Lambda_h \mathscr{R}_1^1} b_2(\lambda) = 0.$$

Then, it is easy to know that $\mathscr{L}_3(\lambda)$ is strictly increasing in the interval $(0, \infty)$ and

$$\lim_{\lambda\to\infty} \mathscr{L}_3(\lambda) = \infty.$$

Considering

$$\begin{aligned}
\mathscr{L}_3(0) &= \mu_v - \frac{\beta_h^2 \Lambda_v}{\mu_v \mathscr{R}_1^1} b_2 - \frac{\phi\sigma_2 c_1 \beta_h^1 \beta_h^2 \Lambda_v I_{v1}^*}{\mu_v \Lambda_h \mathscr{R}_1^1} b_2 \\
&= \mu_v \left(1 - \frac{\beta_h^2 \Lambda_v}{\mu_v^2 \mathscr{R}_1^1} b_2 \right) - \frac{\phi\sigma_2 c_1 \beta_h^1 \beta_h^2 \Lambda_v I_{v1}^*}{\mu_v \Lambda_h \mathscr{R}_1^1} b_2 \\
&= \mu_v \left(1 - \frac{\mathscr{R}_1^2}{\mathscr{R}_1^1} \right) - \frac{\phi\sigma_2 c_1 \beta_h^1 \beta_h^2 \Lambda_v I_{v1}^*}{\mu_v \Lambda_h \mathscr{R}_1^1} b_2.
\end{aligned}$$

As $\mathscr{R}_1^2/\mathscr{R}_1^1 > 1$, we have $\mathscr{L}_3(0) < 0$. By the intermediate value theorem, it can see that the equation $\mathscr{L}_3(\lambda) = 0$ has a real positive root. Consequently, the strain 1 dominance equilibrium $\mathscr{E}_1$ is unstable.

Next, assume $\mathscr{R}_1^2/\mathscr{R}_1^1 < \varsigma\,(< 1)$ and set

$$\mathscr{L}_4(\lambda) = \lambda + \mu_v, \quad \mathscr{L}_5(\lambda) = \frac{\beta_h^2 \Lambda_v}{\mu_v \mathscr{R}_1^1} b_2(\lambda) + \frac{\phi\sigma_2 c_1 \beta_h^1 \beta_h^2 \Lambda_v I_{v1}^*}{\mu_v \Lambda_h \mathscr{R}_1^1} b_2(\lambda).$$

Assuming that the characteristic equation (5.4) has a root λ_0 with nonnegative real part $Re\lambda_0 \geq 0$, then we have

$$\begin{aligned}
|\mathscr{L}_4(\lambda_0)| &\geq \mu_v,\\
|\mathscr{L}_5(\lambda_0)| &\leq \frac{\beta_h^2\Lambda_v}{\mu_v\mathscr{R}_1^1}b_2(Re\lambda_0) + \frac{\phi\sigma_2 c_1\beta_h^1\beta_h^2\Lambda_v I_{v1}^*}{\mu_v\Lambda_h\mathscr{R}_1^1}b_2(Re\lambda_0)\\
&\leq \frac{\beta_h^2\Lambda_v}{\mu_v\mathscr{R}_1^1}b_2 + \frac{\phi\sigma_2 c_1\beta_h^1\beta_h^2\Lambda_v I_{v1}^*}{\mu_v\Lambda_h\mathscr{R}_1^1}b_2\\
&= \mu_v\frac{\mathscr{R}_1^2}{\mathscr{R}_1^1} + \mu_v\frac{\mathscr{R}_1^2}{\mathscr{R}_1^1}\frac{\phi\sigma_2 c_1\mu_v(\mathscr{R}_1^1-1)}{\mu_v+\Lambda_h b_1}\\
&= \mu_v\frac{\mathscr{R}_1^2}{\mathscr{R}_1^1}\left(1+\frac{\phi\sigma_2 c_1\mu_v(\mathscr{R}_1^1-1)}{\mu_v+\Lambda_h b_1}\right)\\
&= \mu_v\frac{\mathscr{R}_1^2}{\varsigma\mathscr{R}_1^1} < \mu_v,
\end{aligned}$$

which is obviously a contradiction. Hence, the characteristic equation (5.4) has no roots with nonnegative real parts. Therefore, the local stability of the strain 1 dominance equilibrium $\mathscr{E}_1$ is determined by the following system:

$$\begin{aligned}
&\lambda S_v = -S_v\int_0^\infty \beta_v^1(a)I_{h1}^*(a)da - S_{v1}^*\int_0^\infty \beta_v^1(a)\left(I_h^1(a)+\phi I_h^{21}(a)\right)da - \mu_v S_v,\\
&\lambda I_v^1 = S_{v1}^*\int_0^\infty \beta_v^1(a)\left(I_h^1(a)+\phi I_h^{21}(a)\right)da + S_v\int_0^\infty \beta_v^1(a)I_{h1}^*(a)da - \mu_v I_v^1,\\
&\lambda S_h = -\frac{\beta_h^1}{N_h}\left(S_{h1}^*I_v^1 + I_{v1}^*S_h\right) - \mu_h S_h,\\
&\frac{dI_h^1(a)}{da} + \lambda I_h^1(a) = -\left(\mu_h+\gamma_1(a)\right)I_h^1(a), \quad I_h^1(0) = \frac{\beta_h^1}{N_h}\left(S_{h1}^*I_v^1 + I_{v1}^*S_h\right), \qquad (5.5)\\
&\lambda R_h^1 = \int_0^\infty \gamma_1(a)I_h^1(a)da - \mu_h R_h^1,\\
&\lambda R_h^2 = -\frac{\sigma_1\beta_h^1}{N_h}I_{v1}^*R_h^2 - \mu_h R_h^2,\\
&\frac{dI_h^{21}(a)}{da} + \lambda I_h^{21}(a) = -(\mu_h+\gamma_1(a))I_h^{21}(a), \quad I_h^{21}(0) = \frac{\sigma_1\beta_h^1}{N_h}I_{v1}^*R_h^2.
\end{aligned}$$

Straightforward computing, we get

$$\mathscr{L}_6(\lambda) = \mathscr{L}_7(\lambda), \tag{5.6}$$

where

$$\mathscr{L}_6(\lambda) = \frac{1}{\lambda+\mu_h}\left(\lambda+\mu_h+\frac{\beta_h^1 I_{v1}^*}{N_h}\right)\left(\lambda+\mu_v+\int_0^\infty \beta_v^1(a) I_{h1}^*(a)da\right),$$

$$\mathscr{L}_7(\lambda) = \frac{1}{N_h}\beta_h^1 S_{v1}^* S_{h1}^* b_1(\lambda) = \frac{1}{N_h}\beta_h^1 S_{v1}^* S_{h1}^* \int_0^\infty \beta_v^1(a) e^{-\lambda a}\pi_1(a)da.$$

Here we assume λ_0 is any root of Eq. (5.6) with $Re\lambda_0 \geq 0$. Then, it is easy to obtain

$$|\mathscr{L}_6(\lambda_0)| \geq |\lambda_0+\mu_v| > \mu_v,$$

$$|\mathscr{L}_7(\lambda_0)| \leq \frac{1}{N_h}\beta_h^1 S_{v1}^* S_{h1}^* b_1(\lambda_0) = \frac{1}{N_h}\beta_h^1 S_{v1}^* S_{h1}^* b_1 = \mu_v < |\mathscr{L}_6(\lambda_0)|.$$

This results in a contradiction, which implies the characteristic equation (5.6) has only roots with negative real parts. Hence, the strain 1 dominance equilibrium $\mathscr{E}_1$ is locally asymptotically stable. This completes the proof. □

6. The Uniform Persistence

In the previous section, we obtained that if both reproduction numbers are less than one, all strains are eliminated and the disease dies out. Now, we shall prove that the competitive exclusion principle holds in system (2.2). In the following, we always assume that $\mathscr{R}_0^1 > 1$, and show that strain 1 persists, while strain 2 dies out if $\mathscr{R}_1^2/\mathscr{R}_1^1 < b_2/b_1 < 1$ and $0 < \phi \leq 1$, where b_j is the force of infection of the primarily infective humans at the equilibria. Under some conditions, strain 1 can eliminate the other strain, and the competitive exclusion principle holds in system (2.2). Mathematically speaking, it means the global asymptotically stability of the strain 1 dominance equilibrium $\mathscr{E}_1$. Noticing that the strain 1 dominance equilibrium $\mathscr{E}_1$ is locally asymptotically stable if $\mathscr{R}_1^2/\mathscr{R}_1^1 < \varsigma\,(< 1)$. Hence, we only need to show that $\mathscr{E}_1$ is a global attractor. Here we apply the Lyapunov functional methods in [2, 5, 23, 26] to prove this result.

Let

$$f(x) = x - 1 - \ln x, \quad x > 0.$$

Obviously, $f(x) \geq 0$ for all $x \geq 0$ and reaches its global minimum value $f(1) = 0$ at $x = 1$. Then, we formulate the following Lyapunov function:

$$\begin{aligned} U(t) &= U_1(t) + U_2^1(t) + U_2^2(t) + U_3(t) + U_4^1(t) + U_4^2(t) \\ &\quad + U_5^1(t) + U_5^2(t) + U_6^1(t) + U_6^2(t) + U_7(t), \end{aligned} \tag{6.1}$$

where

$$U_1(t) = \frac{1}{b_1} f\left(\frac{S_v(t)}{S_{v1}^*}\right), \quad U_2^1(t) = \frac{I_{v1}^*}{b_1 S_{v1}^*} f\left(\frac{I_v^1(t)}{I_{v1}^*}\right), \quad U_2^2(t) = \frac{I_v^2(t)}{b_1 S_{v1}^*},$$

$$U_3(t) = S_{h1}^* f\left(\frac{S_h(t)}{S_{h1}^*}\right),$$

$$U_4^1(t) = \frac{1}{b_1}\int_0^\infty q_1(a) I_{h1}^*(a) f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right) da + \int_0^\infty p_1(a) I_{h1}^*(a) f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right) da,$$

$$U_4^2(t) = \frac{1}{b_2}\int_0^\infty q_2(a) I_h^2(a,t) da + \int_0^\infty p_2(a) I_h^2(a,t) da,$$

$$U_5^1(t) = R_{h1}^* f\left(\frac{R_h^1(t)}{R_{h1}^*}\right), \quad U_5^2(t) = R_h^2(t),$$

$$U_6^1(t) = \frac{1}{b_1}\int_0^\infty q_1(a) I_h^{21}(a,t) da, \quad U_6^2(t) = \frac{1}{b_2}\int_0^\infty q_2(a) I_h^{12}(a,t) da,$$

$$U_7(t) = \int_t^\infty c_1 I_{h1}^*(0) f\left(\frac{I_h^1(0,s)}{I_{h1}^*(0)}\right) ds + \int_t^\infty c_2 I_h^2(0,s) ds + \int_t^\infty \frac{R_{h1}^*}{R_h^1(s)} I_h^{12}(0,s) ds$$

and

$$q_j(a) = \int_a^\infty \beta_v^j(s) e^{-\int_a^s (\mu_h + \gamma_j(\sigma)) d\sigma} ds, \quad p_j(a) = \int_a^\infty \gamma_j(s) e^{-\int_a^s (\mu_h + \gamma_j(\sigma)) d\sigma} ds.$$

Straightforward computation yields

$$q_j'(a) = -\beta_v^j(a) + (\mu_h + \gamma_j(a)) q_j(a), \quad q_j(0) = b_j$$

and

$$p_j'(a) = -\gamma_j(a) + (\mu_h + \gamma_j(a)) p_j(a), \quad p_j(0) = c_j.$$

The difficulty with the Lyapunov function $U(t)$ in (6.1) may be not defined when $S_v = 0$, $I_v^1(t) = 0$, $S_h(t) = 0$, $I_h^1(a,t) = 0$, $R_h^1(t) = 0$. In order to overcome this, we need to show that strain 1 persists for humans and mosquitoes. Let

$$\hat{X} = \left\{\psi_1 \in L_+^1(0,\infty) \,|\, \exists s \geq 0 : \int_0^\infty \beta_v^1(a+s)\psi_1(a) da > 0\right\}$$

and set

$$X_0 = \prod_{j=1}^{3} \mathbb{R}_+ \times \mathbb{R}_+ \times \hat{X} \times L^1(0,\infty) \times \mathbb{R}_+^2 \times \prod_{j=1}^{2} L^1(0,\infty),$$

$$\Omega_0 = \Omega \cap X_0.$$

In the following, we justify Ω_0 is forward invariant. To do so, we first demonstrate $\hat{X}$ is forward invariant. If it is true for the initial condition, the support of $\beta_v^1(a)$ will intersect the support of the initial condition when it is translated s units to the right. Also noticing that the support of the initial condition only moves to the right, the intersection will occur for any other time if it takes place at the initial time. Thus, it is easy to obtain that Ω_0 is a forward invariant set.

To do so, let us give the following important definitions.

Definition 6.1. Strain 1 in system (2.2) is uniformly weakly persistent if there exists some $\gamma > 0$ independent of the initial conditions such that

$$\limsup_{t\to\infty} \int_0^\infty \left(I_h^1(a,t) + I_h^{21}(a,t)\right) da > \gamma \quad \text{whenever} \quad \int_0^\infty \psi_1(a) da > 0$$

and

$$\limsup_{t\to\infty} I_v^1(t) > \gamma \quad \text{whenever} \quad I_{v0}^1 > 0$$

for all solution of system (2.2).

Obviously, the uniform weak persistence of the disease shows that the disease-free equilibrium is unstable.

Definition 6.2. Strain 1 in system (2.2) is uniformly strongly persistent if there exists some $\gamma > 0$ independent of the initial conditions such that

$$\liminf_{t\to\infty} \int_0^\infty \left(I_h^1(a,t) + I_h^{21}(a,t)\right) da > \gamma \quad \text{whenever} \quad \int_0^\infty \psi_1(a) da > 0$$

and

$$\liminf_{t\to\infty} I_v^1(t) > \gamma \text{ whenever } I_{v0}^1 > 0$$

for all solution of system (2.2).

By the above definitions, it is easy to know that strain 1 is also uniformly weakly persistent if it is uniformly strongly persistent. In order to prove uniform strong persistence of strain 1, we need to prove that there exists a global compact attractor for the semiflow in system (2.2).

First, we show the uniform weak persistence of strain 1 providing that strain 2 in system (2.2) goes extinct. We have the following theorem.

Theorem 6.3. *Assume* $\mathscr{R}_1^1 > 1$ *and* $\mathscr{R}_1^2/\mathscr{R}_1^1 < \varsigma\,(< 1)$. *Further, assume that strain* 2 *in system* (2.2) *will go extinct, i.e.*

$$\limsup_{t\to\infty} I_v^2(t) = 0 \quad \textit{and} \quad \limsup_{t\to\infty} \int_0^\infty \left(I_h^2(a,t) + I_h^{12}(a,t)\right) da = 0.$$

Then strain 1 *in system* (2.2) *is uniformly weakly persistent for the initial conditions that belong to* Ω_0, *i.e. there exists* $\varepsilon > 0$ *such that*

$$\limsup_{t\to\infty} \frac{\beta_h^1 I_v^1(t)}{N_h} \geq \varepsilon \quad \textit{and} \quad \limsup_{t\to\infty} \int_0^\infty \beta_v^1(a) \left(I_h^1(a,t) + \phi I_h^{21}(a,t)\right) da \geq \varepsilon.$$

Proof. We argue by contradiction. Assume that strain 1 also goes extinct. For every $\varepsilon > 0$ and any initial conditions in Ω_0, we have

$$\limsup_{t\to\infty} \frac{\beta_h^1 I_v^1(t)}{N_h} < \varepsilon \quad \text{and} \quad \limsup_{t\to\infty} \int_0^\infty \beta_v^1(a)\left(I_h^1(a,t) + \phi I_h^{21}(a,t)\right) da < \varepsilon.$$

Therefore, there exists $T \ge 0$ such that for all $t > T$, we obtain

$$\frac{\beta_h^j I_v^j(t)}{N_h} \le \varepsilon,\ j = 1, 2,\ \int_0^\infty \beta_v^2(a)\left(I_h^2(a,t) + \phi I_h^{12}(a,t)\right) da \le \varepsilon$$

and

$$\int_0^\infty \beta_v^1(a)\left(I_h^1(a,t) + \phi I_h^{21}(a,t)\right) da \le \varepsilon.$$

Without loss of generality, we assume that the inequalities above hold for all $t \ge 0$. From the first and fourth equations in (2.2), we get

$$S_v'(t) \ge \Lambda_v - 2\varepsilon S_v(t) - \mu_v S_v(t), \quad S_h'(t) \ge \Lambda_h - 2\varepsilon S_h(t) - \mu_h S_h(t).$$

By the comparison principle, we get

$$\limsup_{t\to\infty} S_v(t) \ge \liminf_{t\to\infty} S_v(t) \ge \frac{\Lambda_v}{2\varepsilon + \mu_v},$$

$$\limsup_{t\to\infty} S_h(t) \ge \liminf_{t\to\infty} S_h(t) \ge \frac{\Lambda_h}{2\varepsilon + \mu_h}.$$

Recall that $B_1(t) = I_h^1(0,t) = \frac{\beta_h^1 S_h(t) I_v^1(t)}{N_h}$, it then follows from system (2.2) that

$$\begin{aligned} \frac{dI_v^1(t)}{dt} &\ge \frac{\Lambda_v}{2\varepsilon + \mu_v} \int_0^\infty \beta_v^1(a)\left(I_h^1(a,t) + \phi I_h^{21}(a,t)\right) da - \mu_v I_v^1(t) \\ &\ge \frac{\Lambda_v}{2\varepsilon + \mu_v} \int_0^\infty \beta_v^1(a) I_h^1(a,t) da - \mu_v I_v^1(t). \end{aligned} \tag{6.2}$$

By using (3.1), we have

$$B_1(t) \ge \frac{\beta_h^1 \Lambda_h I_v^1(t)}{(2\varepsilon + \mu_h) N_h}, \quad \frac{dI_v^1(t)}{dt} \ge \frac{\Lambda_v}{2\varepsilon + \mu_v} \int_0^t \beta_v^1(a) B_1(t-a) \pi_1(a) da - \mu_v I_v^1(t).$$

Since all the above functions are bounded, their Laplace transforms exist for $\lambda > 0$. We denote by $\hat{B}_1(\lambda)$ the Laplace transform of $B_1(t)$ and by $\hat{I}_v^1(\lambda)$ the Laplace transform of $I_v^1(t)$. Further, set

$$\hat{L}(\lambda) = \int_0^\infty e^{-\lambda a} \beta_v^1(a) \pi_1(a) da.$$

Using the Laplace transform, we have

$$\hat{B}_1(\lambda) \ge \frac{\beta_h^1 \Lambda_h}{(2\varepsilon + \mu_h) N_h} \hat{I}_v^1(\lambda), \quad \lambda \hat{I}_v^1(\lambda) - I_v^1(0) \ge \frac{\Lambda_v}{2\varepsilon + \mu_v} \hat{B}_1(\lambda) \hat{L}(\lambda) - \mu_v \hat{I}_v^1(\lambda).$$

Eliminating $\hat{I}_v^1(\lambda)$, we have

$$\hat{B}_1(\lambda) \ge \frac{\beta_h^1 \Lambda_h \Lambda_v \hat{L}(\lambda)}{(2\varepsilon + \mu_h) N_h (\lambda + \mu_v)(2\varepsilon + \mu_v)} \hat{B}_1(\lambda) + \frac{\beta_h^1 \Lambda_h I_v^1(0)}{(2\varepsilon + \mu_h)(\lambda + \mu_v) N_h}.$$

The inequality should hold for the given $\varepsilon \approx 0$ and any $\lambda > 0$. However, this is impossible since for $\varepsilon \approx 0$ and $\lambda \approx 0$, the coefficient in front of $\hat{B}_1(\lambda)$ on the right-hand side is approximately $\mathscr{R}_1^1 > 1$, namely,

$$\frac{\beta_h^1 \Lambda_h \Lambda_v \hat{L}(\lambda)}{(2\varepsilon + \mu_h) N_h (\lambda + \mu_v)(2\varepsilon + \mu_v)} \approx \mathscr{R}_1^1 > 1.$$

Besides, there is another positive term on the right-hand side of the above inequality. The contradiction means that there exists $\varepsilon > 0$ such that for any initial conditions in Ω_0, we have

$$\limsup_{t\to\infty} \frac{\beta_h^1 I_v^1(t)}{N_h} \geq \varepsilon \quad \text{and} \quad \limsup_{t\to\infty} \int_0^\infty \beta_v^1(a) \left(I_h^1(a,t) + \phi I_h^{21}(a,t) \right) da \geq \varepsilon.$$

This completes the proof. □

Now we show that system (2.2) has a global compact attractor $\mathcal{A}$. As a first step, we define the semiflow Ψ of system (2.2):

$$\begin{aligned} &\Psi\left(t, S_{v0}, I_{v0}^1, I_{v0}^2, S_{h0}, \psi_1(\cdot), \psi_2(\cdot), R_{h0}^1, R_{h0}^2, \psi_{12}(\cdot), \psi_{21}(\cdot)\right) \\ &\quad = \left(S_v(t), I_v^1(t), I_v^2(t), S_h(t), I_h^1(a,t), I_h^2(a,t), R_h^1(t), R_h^2(t), I_h^{12}(a,t), I_h^{21}(a,t)\right). \end{aligned} \tag{6.3}$$

The semiflow actually is a mapping $\Psi : [0,\infty) \times \Omega_0 \to \Omega_0$. Now, let us give the definition of a global compact attractor in [16, 24].

Definition 6.4. A set $\mathcal{A}$ in Ω_0 is called a global compact attractor for Ψ if $\mathcal{A}$ is a maximal compact invariant set and if for all open sets $\mathcal{U}$ containing $\mathcal{A}$ and all bounded sets $\mathcal{B}$ of Ω_0, there exists some constant $T > 0$ such that $\Psi(t, \mathcal{B}) \subseteq \mathcal{U}$ for all $t \geq T$.

The following theorem gives the presence of a global compact attractor.

Theorem 6.5. *Under the hypothesis of Theorem* 6.3, *there exists* $\mathcal{A}$, *a compact subset of* Ω_0 *which is a global attractor for the solution semiflow* Ψ *on* Ω_0. *Moreover,* $\mathcal{A}$ *is invariant under the solution semiflow, that is*

$$\Psi(t, x^0) \subseteq \mathcal{A} \text{ for every } x^0 \in \mathcal{A}, \quad \forall\, t \geq 0.$$

Proof. To establish the result, we will use [16, Lemma 3.2.3 and Theorem 3.4.6]. To show that Ψ satisfies the assumptions of Lemma 3.2.3, we split the semiflow into the two components. For an initial condition $x^0 \in \Omega_0$, let $\Psi(t, x^0) = \hat{\Psi}(t, x^0) + \tilde{\Psi}(t, x^0)$,

where

$$\begin{aligned}
&\hat{\Psi}\left(t, S_{v0}, I_{v0}^1, I_{v0}^2, S_{h0}, \psi_1(\cdot), \psi_2(\cdot), R_{h0}^1, R_{h0}^2, \psi_{12}(\cdot), \psi_{21}(\cdot)\right)\\
&\qquad= \left(0, 0, 0, 0, \hat{I}_h^1(a,t), \hat{I}_h^2(a,t), 0, 0, \hat{I}_h^{12}(a,t), \hat{I}_h^{21}(a,t)\right),\\
&\tilde{\Psi}\left(t, S_{v0}, I_{v0}^1, I_{v0}^2, S_{h0}, \psi_1(\cdot), \psi_2(\cdot), R_{h0}^1, R_{h0}^2, \psi_{12}(\cdot), \psi_{21}(\cdot)\right)\\
&\qquad= \left(S_v(t), I_v^1(t), I_v^2(t), S_h(t), \tilde{I}_h^1(a,t), \tilde{I}_h^2(a,t), R_h^1(t), R_h^2(t), \tilde{I}_h^{12}(a,t), \tilde{I}_h^{21}(a,t)\right)
\end{aligned}$$

and

$$\begin{aligned}
&I_h^j(a,t) = \hat{I}_h^j(a,t) + \tilde{I}_h^j(a,t), \quad I_h^{12}(a,t) = \hat{I}_h^{12}(a,t) + \tilde{I}_h^{12}(a,t),\\
&I_h^{21}(a,t) = \hat{I}_h^{21}(a,t) + \tilde{I}_h^{21}(a,t).
\end{aligned}$$

Furthermore, $\hat{I}_h^j(a,t)$ and $\tilde{I}_h^j(a,t)$ satisfy the following equations:

$$\begin{cases}
\dfrac{\partial \hat{I}_h^j}{\partial a} + \dfrac{\partial \hat{I}_h^j}{\partial t} = -(\mu_h + \gamma_j(a))\,\hat{I}_h^j(a,t),\\
\hat{I}_h^j(0,t) = 0,\\
\hat{I}_h^j(a,0) = \psi_j(a)
\end{cases} \quad \text{and}$$

$$\begin{cases}
\dfrac{\partial \tilde{I}_h^j}{\partial a} + \dfrac{\partial \tilde{I}_h^j}{\partial t} = -(\mu_h + \gamma_j(a))\,\tilde{I}_h^j(a,t),\\
\tilde{I}_h^j(0,t) = \dfrac{\beta_h^j S_h I_v^j}{N_h},\\
\tilde{I}_h^j(a,0) = 0.
\end{cases}$$

$\hat{I}_h^{12}(a,t)$ and $\tilde{I}_h^{12}(a,t)$ satisfy the following equations:

$$\begin{cases}
\dfrac{\partial \hat{I}_h^{12}}{\partial a} + \dfrac{\partial \hat{I}_h^{12}}{\partial t} = -(\mu_h + \gamma_2(a))\,\hat{I}_h^{12}(a,t),\\
\hat{I}_h^j(0,t) = 0,\\
\hat{I}_h^j(a,0) = \psi_{12}(a)
\end{cases} \quad \text{and}$$

$$\begin{cases}
\dfrac{\partial \tilde{I}_h^{12}}{\partial a} + \dfrac{\partial \tilde{I}_h^{12}}{\partial t} = -\,(\mu_h + \gamma_2(a))\,\tilde{I}_h^{12}(a,t),\\
\tilde{I}_h^j(0,t) = \dfrac{\sigma_2 \beta_h^2 R_h^1 I_v^2}{N_h},\\
\tilde{I}_h^j(a,0) = 0.
\end{cases}$$

$\hat{I}_h^{21}(a,t)$ and $\tilde{I}_h^{21}(a,t)$ satisfy the following equations:

$$\begin{cases} \dfrac{\partial \hat{I}_h^{21}}{\partial a} + \dfrac{\partial \hat{I}_h^{21}}{\partial t} = -(\mu_h + \gamma_1(a))\, \hat{I}_h^{21}(a,t), \\ \hat{I}_h^{21}(0,t) = 0, \\ \hat{I}_h^{21}(a,0) = \psi_{21}(a) \end{cases} \quad \text{and}$$

$$\begin{cases} \dfrac{\partial \tilde{I}_h^{21}}{\partial a} + \dfrac{\partial \tilde{I}_h^{21}}{\partial t} = -(\mu_h + \gamma_1(a))\, \tilde{I}_h^{21}(a,t), \\ \tilde{I}_h^{21}(0,t) = \dfrac{\sigma_1 \beta_h^1 R_h^2 I_v^1}{N_h}, \\ \tilde{I}_h^{21}(a,0) = 0. \end{cases}$$

Integrating along the characteristic lines, we get

$$\hat{I}_h^j(a,t) = \begin{cases} 0, & a \le t, \\ \psi_j(a-t)\dfrac{\pi_j(a)}{\pi_j(a-t)}, & t < a. \end{cases} \tag{6.4}$$

Again integrating $\hat{I}_h^j(a,t)$ with regard to a in the interval $(0,\infty)$, we obtain

$$\begin{aligned} \int_0^\infty \hat{I}_h^j(a,t)da &= \int_t^\infty \psi_j(a-t)\frac{\pi_j(a)}{\pi_j(a-t)}da \\ &= \int_0^\infty \psi_j(a)\frac{\pi_j(a+t)}{\pi_j(a)}da \\ &\le \int_0^\infty \psi_j(a)e^{-\mu_h t}da = e^{-\mu_h t}\int_0^\infty \psi_j(a)da \to 0 \quad (t \to \infty). \end{aligned} \tag{6.5}$$

Hence, we have

$$\lim_{t\to\infty} \int_0^\infty \hat{I}_h^j(a,t)da = 0.$$

Using the similar method, we also get

$$\lim_{t\to\infty} \int_0^\infty \hat{I}_h^{12}(a,t)da = 0, \quad \lim_{t\to\infty} \int_0^\infty \hat{I}_h^{21}(a,t)da = 0.$$

Therefore, $\hat{\Psi}(t,x^0) \to 0$ as $t \to \infty$ uniformly for any $x^0 \in \mathscr{B} \subseteq \Omega_0$, where $\mathscr{B}$ is a ball of a given radius.

Now, we show that $\tilde{\Psi}(t,x^0)$ is completely continuous. We fix t and let $x^0 \in \Omega_0$. Considering Ω_0 is bounded, we need to prove the family of functions defined by

$$\begin{aligned} &\tilde{\Psi}(t,x^0) \\ &\quad = \Big(S_v(t), I_v^1(t), I_v^2(t), S_h(t), \tilde{I}_h^1(a,t), \tilde{I}_h^2(a,t), R_h^1(t), R_h^2(t), \tilde{I}_h^{12}(a,t), \tilde{I}_h^{21}(a,t)\Big) \end{aligned}$$

is a compact family of functions for that fixed t, which is obtained by taking different initial conditions in Ω_0. Since

$$\{\tilde{\Psi}(t, x^0) \,|\, x^0 \in \Omega_0, t - \text{fixed}\} \subseteq \Omega_0,$$

it is bounded. Also noticing

$$\tilde{I}_h^j(a,t) = \begin{cases} \tilde{B}_j(t-a)\pi_j(a), & a \leq t, \\ 0, & t < a, \end{cases} \tag{6.6}$$

where

$$\tilde{B}_j(t) = \frac{\beta_h^j S_h(t) I_v^j(t)}{N_h}.$$

Thus, we have

$$\lim_{t\to\infty} \int_t^\infty \tilde{I}_h^j(a,t)da = 0.$$

Using the same argument, we obtain

$$\lim_{t\to\infty} \int_t^\infty \tilde{I}_h^{12}(a,t)da = 0, \quad \lim_{t\to\infty} \int_t^\infty \tilde{I}_h^{21}(a,t)da = 0.$$

Since $S_h(t), R_h^j(t)$ and $I_v^j(t)$ all are bounded, $\tilde{B}_j(t)$ is also bounded. Consequently, there exists a positive constant k_1 such that

$$|\tilde{B}_j(t)| \leq k_1.$$

Then, differentiating (6.6) with respect to a, we have

$$\left|\frac{\partial \tilde{I}_h^j(a,t)}{\partial a}\right| \leq \begin{cases} \left|\dfrac{\partial \tilde{B}_j(t-a)}{\partial a}\right| \pi_j(a) + \tilde{B}_j(t-a)\pi_j'(a), & a \leq t, \\ 0, & t < a. \end{cases} \tag{6.7}$$

Also considering S_h' and $(I_v^j)'$ are bounded, there exists a positive constant k_2 satisfying

$$|(\tilde{B}_j)'(t)| \leq \left|\frac{\beta_h^j}{N_h}\left(S_h' I_v^j + S_h (I_v^j)'\right)\right| \leq k_2.$$

Consequently, we conclude that there exists a positive constant k_3 satisfying

$$\begin{aligned} \left\|\frac{\partial \tilde{I}_h^j(a,t)}{\partial a}\right\| &= \int_0^t \left|\frac{\partial \tilde{I}_h^j(a,t)}{\partial a}\right| da + \int_t^\infty \left|\frac{\partial \tilde{I}_h^j(a,t)}{\partial a}\right| da \\ &\leq \int_0^t (k_2\pi_j(a) + k_1(\mu_h + \gamma_j(a))\pi_j(a))\, da \\ &\leq k_2 \int_0^t \pi_j(a)da + k_1(\mu_h + \bar{\gamma}) \int_0^t \pi_j(a)da \\ &\leq k_2 \int_0^\infty \pi_j(a)da + k_1(\mu_h + \bar{\gamma}) \int_0^\infty \pi_j(a)da \leq k_3, \end{aligned} \tag{6.8}$$

where $\bar{\gamma} = \sup_{a,j}\{\gamma_j(a)\}$. Therefore, we have

$$\int_0^\infty |\tilde{I}_h^j(a+h,t) - \tilde{I}_h^j(a,t)|da \le \left\| \frac{\partial \tilde{I}_h^j(a,t)}{\partial a} \right\| |h| \le k_3|h|.$$

Using same arguments, we can determine that it also holds for $\tilde{I}_h^{12}(a,t)$ and $\tilde{I}_h^{21}(a,t)$. Thus, the integral can be made arbitrary small uniformly in the family of functions. By the Frechet–Kolmogorov theorem [40], we obtain that the $\tilde{\Psi}(t,x^0)$ is precompact. Consequently, the solution semiflow Ψ is asymptotically smooth. Also by the definition of Ω, we know the solution semiflow Ψ is point dissipative and the forward orbit of boundedness sets is bounded in Ω_0. Therefore, the solution semiflow Ψ has a global compact attractor. This completes the proof. □

Now, we establish the uniform strong persistence and give the following theorem.

Theorem 6.6. *Under the hypothesis of Theorem* 6.3, *strain* 1 *in system* (2.2) *is uniformly strongly persistent for all initial conditions in* Ω_0, *that is, there exists* $\gamma > 0$ *satisfying*

$$\liminf_{t\to\infty} \frac{\beta_h^1 I_v^1(t)}{N_h} \ge \gamma \quad \textit{and} \quad \liminf_{t\to\infty} \int_0^\infty \beta_v^1(a)\left(I_h^1(a,t) + \phi I_h^{21}(a,t)\right)da \ge \gamma.$$

Proof. We use [36, Theorem 2.6] to prove this result by considering the solution semiflow Ψ on Ω_0. Formulating two mappings $\rho_j : \Omega_0 \to \mathbb{R}_+$ as follows:

$$\rho_1(\Phi(t,x^0)) = \frac{\beta_h^1 I_v^1(t)}{N_h}, \quad \rho_2(\Phi(t,x^0)) = \int_0^\infty \beta_v^1(a)\left(\tilde{I}_h^1(a,t) + \phi \tilde{I}_h^{21}(a,t)\right)da.$$

From previous arguments, the semiflow of system (2.2) is uniformly weakly persistent and has a global compact attractor. Total orbits are solutions to system (2.2) defined for all times $t \in \mathbb{R}$. As the solution semiflow is nonnegative, we obtain

$$\begin{aligned}
\rho_1(\Phi(t,x^0)) &= \frac{\beta_h^1 I_v^1(t)}{N_h} \ge \frac{\beta_h^1}{N_h} I_v^1(s)e^{-\mu_v(t-s)} > 0,\\
\rho_2(\Phi(t,x^0)) &= \int_0^\infty \beta_v^1(a)\left(\tilde{I}_h^1(a,t) + \phi\tilde{I}_h^{21}(a,t)\right)da\\
&\ge \int_0^\infty \beta_v^1(a)\tilde{I}_h^1(a,t)da \ge \int_0^t \beta_v^1(a)\tilde{B}_1(t-a)\pi_1(a)da\\
&\ge \inf_a\{\beta_v^1(a)\}\pi_1(t)\int_0^t \tilde{B}_1(t-a)da \ge \inf_a\{\beta_v^1(a)\}\pi_1(t)\int_0^t \tilde{B}_1(a)da
\end{aligned}$$

$$\begin{aligned}
&\geq \inf_a\{\beta_v^1(a)\}\pi_1(t)\int_0^t \frac{\beta_h^1 S_h(a) I_v^1(a)}{N_h}da\\
&\geq \inf_a\{\beta_v^1(a)\}\pi_1(t)\int_s^t \frac{\beta_h^1 S_h(a) I_v^1(a)}{N_h}da\\
&= \inf_a\{\beta_v^1(a)\}\pi_1(t)\int_s^t \frac{\beta_h^1 S_h(a)}{N_h} I_v^1(s)e^{-\mu_v(a-s)}da > 0
\end{aligned}$$

for any $t > s$ provided $I_v^1(s) > 0$. By [36, Theorem 2.6], we know that the semiflow is uniformly strongly persistent. So, there exists $\gamma > 0$ satisfying

$$\liminf_{t\to\infty} \frac{\beta_h^1 I_v^1(t)}{N_h} \geq \gamma \quad \text{and} \quad \liminf_{t\to\infty} \int_0^\infty \beta_v^1(a)\left(I_h^1(a,t) + \phi I_h^{21}(a,t)\right)da \geq \gamma. \qquad \square$$

By Theorem 6.6, strain 1 persists for all initial conditions in Ω_0. Also notice that the solutions in (2.2) are in a positively invariant set. Therefore, it follows the following conclusion from Theorem 6.6.

Theorem 6.7. *Under the hypothesis of Theorem* 6.3, *there exist constants* $m > 0$ *and* $M > 0$ *satisfying*

$$m \leq S_v(t) \leq M, \quad m \leq S_h(t) \leq M, \ \forall\, t \in R$$

and

$$m \leq \frac{\beta_h^1 I_v^1(t)}{N_h} \leq M, \quad m \leq \int_0^\infty \beta_v^1(a)(I_h^1(a,t) + \phi I_h^{21}(a,t))da \leq M, \ \forall\, t \in R$$

for each orbit $(S_v(t), I_v^1(t), I_v^2(t), S_h(t), I_h^1(a,t), I_h^2(a,t), R_h^1(t), R_h^2(t), I_h^{12}(a,t), I_h^{21}(a,t))$ *of* Ψ *in* $\mathcal{A}$.

7. Principle of Competitive Exclusion

Now, we demonstrate the important conclusion in this paper. To simplify our analysis, we consider the case for $0 < \phi \leq 1$, which has practical significance, see Sec. 2.

Theorem 7.1. *Assume* $\mathscr{R}_1^1 > 1$, $\mathscr{R}_1^2/\mathscr{R}_1^1 < \min\{b_2/b_1, \varsigma\}$, $b_2/b_1 < 1$. *Then the equilibrium* $\mathscr{E}_1$ *in system* (2.2) *is globally asymptotically stable.*

Proof. By the above arguments, the equilibrium $\mathscr{E}_1$ is locally asymptotically stable and the solution semiflow has a global compact attractor $\mathcal{A}$. Hence, we only need to prove that the equilibrium $\mathscr{E}_1$ is a global attractor. Furthermore, it follows from Theorem 6.7 that there exist $m_1 > 0$ and $M_1 > 0$ satisfying

$$m_1 \leq \frac{I_v^1(t)}{I_{v1}^*} \leq M_1, \quad m_1 \leq \frac{I_h^1(a,t)}{I_{h1}^*(a)} \leq M_1, \quad m_1 \leq \frac{R_h^1(t)}{R_{h1}^*} \leq M_1$$

for all solutions in Ψ. This shows the Lyapunov function well-defined in (6.1).

Differentiating each component of $U(t)$ along the solutions of system (2.2), we have

$$\begin{aligned}
\frac{dU_1(t)}{dt} &= \frac{1}{b_1 S_{v1}^*}\left(1-\frac{S_{v1}^*}{S_v(t)}\right)\Bigg(\Lambda_v - S_v(t)\int_0^\infty \beta_v^1(a)\big(I_h^1(a,t)+\phi I_h^{21}(a,t)\big)da\\
&\quad - S_v(t)\int_0^\infty \beta_v^2(a)\left(I_h^2(a,t)+\phi I_h^{12}(a,t)\right)da - \mu_v S_v(t)\Bigg)\\
&= \frac{-\mu_v(S_v(t)-S_{v1}^*)^2}{b_1 S_v(t) S_{v1}^*} + \frac{1}{b_1}\int_0^\infty \beta_v^1(a) I_{h1}^*(a)\\
&\quad \times\left(1-\frac{S_{v1}^*}{S_v(t)}-\frac{S_v(t)I_h^1(a,t)}{S_{v1}^*I_{h1}^*(a)}+\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)da\\
&\quad - \frac{1}{b_1 S_{v1}^*}\left(S_v(t)\int_0^\infty \beta_v^2(a)I_h^2(a,t)da - S_{v1}^*\int_0^\infty \beta_v^2(a)I_h^2(a,t)da\right)\\
&\quad + \left(\frac{1}{b_1}-\frac{S_v(t)}{b_1 S_{v1}^*}\right)\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da\\
&\quad + \left(\frac{1}{b_1}-\frac{S_v(t)}{b_1 S_{v1}^*}\right)\int_0^\infty \beta_v^2(a)\phi I_h^{12}(a,t)da,\\
\frac{dU_2^1(t)}{dt} &= \frac{1}{b_1 S_{v1}^*}\left(1-\frac{I_{v1}^*}{I_v^1(t)}\right)\left(S_v(t)\int_0^\infty \beta_v^1(a)\big(I_h^1(a,t)+\phi I_h^{21}(a,t)\big)da - \mu_v I_v^1(t)\right)\\
&= \frac{1}{b_1 S_{v1}^*}\left(1-\frac{I_{v1}^*}{I_v^1(t)}\right)\Bigg(S_v(t)\int_0^\infty \beta_v^1(a)I_h^1(a,t)da\\
&\quad + S_v(t)\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da - \frac{I_v^1(t)}{I_{v1}^*}S_{v1}^*\int_0^\infty \beta_v^1(a)I_{h1}^*(a)da\Bigg)\\
&= \frac{1}{b_1 S_{v1}^*}\left(1-\frac{I_{v1}^*}{I_v^1(t)}\right)\Bigg(\int_0^\infty \beta_v^1(a)S_{v1}^*I_{h1}^*(a)\left(\frac{S_v(t)I_h^1(a,t)}{S_{v1}^*I_{h1}^*(a)}-\frac{I_v^1(t)}{I_{v1}^*}\right)da\\
&\quad + S_v(t)\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da\Bigg)\\
&= \frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_{h1}^*(a)\left(\frac{S_v(t)I_h^1(a,t)}{S_{v1}^*I_{h1}^*(a)}-\frac{I_v^1(t)}{I_{v1}^*}-\frac{S_v(t)I_h^1(a,t)I_{v1}^*}{S_{v1}^*I_{h1}^*(a)I_v^1(t)}+1\right)da\\
&\quad + \frac{S_v(t)}{b_1 S_{v1}^*}\left(1-\frac{I_{v1}^*}{I_v^1(t)}\right)\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da,
\end{aligned}$$

$$
\begin{aligned}
\frac{dU_2^2(t)}{dt} &= \frac{1}{b_1 S_{v1}^*}\left(S_v(t)\int_0^\infty \beta_v^2(a)\big(I_h^2(a,t)+\phi I_h^{12}(a,t)\big)da - \mu_v I_v^2(t)\right)\\
&= \frac{S_v(t)}{b_1 S_{v1}^*}\int_0^\infty \beta_v^2(a)\phi I_h^{12}(a,t)da + \frac{S_v(t)}{b_1 S_{v1}^*}\int_0^\infty \beta_v^2(a) I_h^2(a,t)da - \frac{\mu_v I_v^2(t)}{b_1 S_{v1}^*},\\
\frac{dU_3(t)}{dt} &= \left(1-\frac{S_{h1}^*}{S_h(t)}\right)\left(\Lambda_h - \mu_h S_h(t) - \frac{\beta_h^1 S_h(t) I_v^1(t)}{N_h} - \frac{\beta_h^2 S_h(t) I_v^2(t)}{N_h}\right)\\
&= \left(-I_h^1(0,t) + \mu_h S_{h1}^* + \frac{\beta_h^1 S_{h1}^* I_{v1}^*}{N_h} - I_h^2(0,t) - \mu_h S_h(t)\right)\left(1-\frac{S_{h1}^*}{S_h(t)}\right)\\
&= I_{h1}^*(0) - \frac{S_{h1}^*}{S_h(t)} I_{h1}^*(0) - I_h^2(0,t) - I_h^1(0,t) + \frac{S_{h1}^*}{S_h(t)} I_h^1(0,t) + \frac{\beta_h^2 S_{h1}^* I_v^2(t)}{N_h}\\
&\quad - \frac{\mu_h (S_h(t) - S_{h1}^*)^2}{S_h(t)},\\
\frac{dU_4^1(t)}{dt} &= \int_0^\infty p_1(a) I_{h1}^*(a)\frac{1}{I_{h1}^*(a)} f'\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\frac{\partial I_h^1(a,t)}{\partial t}da\\
&\quad + \frac{1}{b_1}\int_0^\infty q_1(a) I_{h1}^*(a) f'\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\frac{1}{I_{h1}^*(a)}\frac{\partial I_h^1(a,t)}{\partial t}da\\
&= \frac{-1}{b_1}\int_0^\infty q_1(a) I_{h1}^*(a) f'\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\frac{1}{I_{h1}^*(a)}\\
&\quad \times\left((\mu_h+\gamma_1(a))\, I_h^1(a,t) + \frac{\partial I_h^1(a,t)}{\partial a}\right)da\\
&\quad - \int_0^\infty p_1(a) I_{h1}^*(a) f'\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\frac{1}{I_{h1}^*(a)}\\
&\quad \times\left((\mu_h+\gamma_1(a))\, I_h^1(a,t) + \frac{\partial I_h^1(a,t)}{\partial a}\right)da\\
&= -\frac{1}{b_1}\int_0^\infty q_1(a) I_{h1}^*(a) df\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right) - \int_0^\infty p_1(a) I_{h1}^*(a) df\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\\
&= \frac{-1}{b_1}\left(f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\Bigg|_0^\infty q_1(a) I_{h1}^*(a) - \int_0^\infty f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right) d\left(q_1(a) I_{h1}^*(a)\right)\right)\\
&\quad - \left(-\int_0^\infty f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right) d\left(p_1(a) I_{h1}^*(a)\right) + p_1(a) I_{h1}^*(a) f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\Bigg|_0^\infty\right)\\
&= -\frac{1}{b_1}\left(-q_1(0) I_{h1}^*(0) f\left(\frac{I_h^1(0,t)}{I_{h1}^*(0)}\right) + \int_0^\infty \beta_v^1(a) I_{h1}^*(a) f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)da\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \left(-p_1(0)I^*_{h1}(0)f\left(\frac{I^1_h(0,t)}{I^*_{h1}(0)}\right) + \int_0^\infty \gamma_1(a)I^*_{h1}(a)f\left(\frac{I^1_h(a,t)}{I^*_{h1}(a)}\right)da\right) \\
&= -\frac{1}{b_1}\int_0^\infty \beta^1_v(a)I^*_{h1}(a)f\left(\frac{I^1_h(a,t)}{I^*_{h1}(a)}\right)da + I^*_{h1}(0)f\left(\frac{I^1_h(0,t)}{I^*_{h1}(0)}\right) \\
&\quad - \int_0^\infty \gamma_1(a)I^*_{h1}(a)f\left(\frac{I^1_h(a,t)}{I^*_{h1}(a)}\right)da + c_1I^*_{h1}(0)f\left(\frac{I^1_h(0,t)}{I^*_{h1}(0)}\right) \\
&= I^1_h(0,t) - I^*_{h1}(0) - I^*_{h1}(0)\ln\frac{I^1_h(0,t)}{I^*_{h1}(0)} - \frac{1}{b_1}\int_0^\infty \beta^1_v(a)I^*_{h1}(a)f\left(\frac{I^1_h(a,t)}{I^*_{h1}(a)}\right)da \\
&\quad + c_1I^*_{h1}(0)f\left(\frac{I^1_h(0,t)}{I^*_{h1}(0)}\right) - \int_0^\infty \gamma_1(a)I^*_{h1}(a)f\left(\frac{I^1_h(a,t)}{I^*_{h1}(a)}\right)da, \\
\frac{dU^2_4(t)}{dt} &= \frac{1}{b_2}\int_0^\infty q_2(a)\frac{\partial I^2_h(a,t)}{\partial t}da + \int_0^\infty p_2(a)\frac{\partial I^2_h(a,t)}{\partial t}da \\
&= \frac{-1}{b_2}\int_0^\infty q_2(a)\left(\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t) + \frac{\partial I^2_h(a,t)}{\partial a}\right)da \\
&\quad - \int_0^\infty p_2(a)\left(\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t) + \frac{\partial I^2_h(a,t)}{\partial a}\right)da \\
&= -\frac{1}{b_2}\int_0^\infty q_2(a)\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t)da - \frac{1}{b_2}\int_0^\infty q_2(a)\frac{\partial I^2_h(a,t)}{\partial a}da \\
&\quad - \int_0^\infty p_2(a)\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t)da - \int_0^\infty p_2(a)\frac{\partial I^2_h(a,t)}{\partial a}da \\
&= \frac{-1}{b_2}\int_0^\infty q_2(a)\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t)da \\
&\quad - \frac{1}{b_2}\left(q_2(a)I^2_h(a,t)|_0^\infty - \int_0^\infty I^2_h(a,t)dq_2(a)\right) \\
&\quad - \int_0^\infty p_2(a)\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t)da \\
&\quad - \left(p_2(a)I^2_h(a,t)|_0^\infty - \int_0^\infty I^2_h(a,t)dp_2(a)\right) \\
&= -\frac{1}{b_2}\int_0^\infty q_2(a)\big(\mu_h + \gamma_2(a)\big)I^2_h(a,t)da + I^2_h(0,t) - \frac{1}{b_2}\int_0^\infty \beta^2_v(a)I^2_h(a,t)da \\
&\quad + \frac{1}{b_2}\int_0^\infty \big(\mu_h + \gamma_2(a)\big)q_2(a)I^2_h(a,t)da - \int_0^\infty p_2(a)\left(\mu_h + \gamma_2(a)\right)I^2_h(a,t)da \\
&\quad + c_2I^2_h(0,t) - \int_0^\infty \gamma_2(a)I^2_h(a,t)da + \int_0^\infty p_2(a)\left(\mu_h + \gamma_2(a)\right)I^2_h(a,t)da
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{b_2}\int_0^\infty \beta_v^2(a) I_h^2(a,t)da + c_2 I_h^2(0,t)\\
&\quad - \int_0^\infty \gamma_2(a) I_h^2(a,t)da + I_h^2(0,t),\\
\frac{dU_5^1(t)}{dt} &= \left(1 - \frac{R_{h1}^*}{R_h^1(t)}\right)\left(\int_0^\infty \gamma_1(a) I_h^1(a,t)da - I_h^{12}(0,t) - \mu_h R_h^1(t)\right)\\
&= \left(1 - \frac{R_{h1}^*}{R_h^1(t)}\right)\left(\int_0^\infty \gamma_1(a) I_h^1(a,t)da - I_h^{12}(0,t) - \frac{R_h^1(t)}{R_{h1}^*}\int_0^\infty \gamma_1(a) I_{h1}^*(a)da\right)\\
&= \left(1 - \frac{R_{h1}^*}{R_h^1(t)}\right)\left(\int_0^\infty \gamma_1(a)\left(I_h^1(a,t) - \frac{I_{h1}^*(a)R_h^1(t)}{R_{h1}^*}\right)da - I_h^{12}(0,t)\right)\\
&= -\left(1 - \frac{R_{h1}^*}{R_h^1(t)}\right)I_h^{12}(0,t) + \int_0^\infty \gamma_1(a) I_{h1}^*(a)\Bigg(- \frac{R_h^1(t)}{R_{h1}^*} + \frac{I_h^1(a,t)}{I_{h1}^*(a)}\\
&\quad - \frac{I_h^1(a,t)R_{h1}^*}{I_{h1}^*(a)R_h^1(t)} + 1\Bigg)da,\\
\frac{dU_5^2(t)}{dt} &= \int_0^\infty \gamma_2(a) I_h^2(a,t)da - I_h^{21}(0,t) - \mu_h R_h^2(t),\\
\frac{dU_6^1(t)}{dt} &= \frac{1}{b_1}\int_0^\infty \frac{\partial I_h^{21}(a,t)}{\partial t} q_1(a)da\\
&= \frac{-1}{b_1}\int_0^\infty q_1(a)\left(\big(\mu_h + \gamma_1(a)\big)I_h^{21}(a,t) + \frac{\partial I_h^{21}(a,t)}{\partial a}\right)da\\
&= -\frac{1}{b_1}\int_0^\infty q_1(a)\big(\mu_h + \gamma_1(a)\big)I_h^{21}(a,t)da - \frac{1}{b_1}\int_0^\infty q_1(a)\frac{\partial I_h^{21}(a,t)}{\partial a}da\\
&= \frac{-1}{b_1}\int_0^\infty q_1(a)\big(\mu_h + \gamma_1(a)\big)I_h^{21}(a,t)da\\
&\quad - \frac{1}{b_1}\left(q_1(a)I_h^{21}(a,t)\big|_0^\infty - \int_0^\infty I_h^{21}(a,t)dq_1(a)\right)\\
&= \frac{-1}{b_1}\int_0^\infty q_1(a)\big(\mu_h + \gamma_1(a)\big)I_h^{21}(a,t)da + I_h^{21}(0,t)\\
&\quad - \frac{1}{b_1}\int_0^\infty \beta_v^1(a) I_h^{21}(a,t)da + \frac{1}{b_1}\int_0^\infty \big(\mu_h + \gamma_1(a)\big)q_1(a)I_h^{21}(a,t)da\\
&= -\frac{1}{b_1}\int_0^\infty \beta_v^1(a) I_h^{21}(a,t)da + I_h^{21}(0,t),
\end{aligned}
$$

$$\begin{aligned}\frac{dU_6^2(t)}{dt} &= \int_0^\infty \frac{\partial I_h^{12}(a,t)}{\partial t}\frac{q_2(a)}{b_2}da\\
&= \frac{-1}{b_2}\int_0^\infty q_2(a)\left((\mu_h+\gamma_2(a))\,I_h^{12}(a,t)+\frac{\partial I_h^{12}(a,t)}{\partial a}\right)da\\
&= \frac{-1}{b_2}\int_0^\infty q_2(a)\big(\mu_h+\gamma_2(a)\big)I_h^{12}(a,t)da-\frac{1}{b_2}\Bigg(q_2(a)I_h^{12}(a,t)|_0^\infty\\
&\quad -\int_0^\infty I_h^{12}(a,t)dq_2(a)\Bigg)\\
&= \frac{-1}{b_2}\int_0^\infty q_2(a)\big(\mu_h+\gamma_2(a)\big)I_h^{12}(a,t)da+I_h^{12}(0,t)\\
&\quad -\frac{1}{b_2}\int_0^\infty \beta_v^2(a)I_h^{12}(a,t)da+\frac{1}{b_2}\int_0^\infty q_2(a)\big(\mu_h+\gamma_2(a)\big)I_h^{12}(a,t)da\\
&= -\frac{1}{b_2}\int_0^\infty \beta_v^2(a)I_h^{12}(a,t)da+I_h^{12}(0,t),\\
\frac{dU_7(t)}{dt} &= -c_2I_h^2(0,t)-I_h^{12}(0,t)\frac{R_{h1}^*}{R_h^1(t)}-c_1I_{h1}^*(0)f\left(\frac{I_h^1(0,t)}{I_{h1}^*(0)}\right).\end{aligned}$$

Adding the eleven components of the Lyapunov function, we have

$$U' = U^1(t)+U^2(t)+U^3(t),$$

where

$$\begin{aligned}U^1(t) &= -\frac{\mu_v\left(S_v(t)-S_{v1}^*\right)^2}{b_1S_v(t)S_{v1}^*}-\frac{\mu_h(S_h(t)-S_{h1}^*)^2}{S_h(t)}\\
&\quad +\frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_{h1}^*(a)\left(1-\frac{S_v(t)I_h^1(a,t)}{S_{v1}^*I_{h1}^*(a)}+\frac{I_h^1(a,t)}{I_{h1}^*(a)}-\frac{S_{v1}^*}{S_v(t)}\right)da\\
&\quad +\frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_{h1}^*(a)\left(\frac{S_v(t)I_h^1(a,t)}{S_{v1}^*I_{h1}^*(a)}-\frac{S_v(t)I_h^1(a,t)I_{v1}^*}{S_{v1}^*I_{h1}^*(a)I_v^1(t)}+1-\frac{I_v^1(t)}{I_{v1}^*}\right)da\\
&\quad +I_{h1}^*(0)-I_h^1(0,t)-\frac{S_{h1}^*}{S_h(t)}I_{h1}^*(0)+\frac{S_{h1}^*}{S_h(t)}I_h^1(0,t)+I_h^1(0,t)\\
&\quad -I_{h1}^*(0)-I_{h1}^*(0)\ln\left(\frac{I_h^1(0,t)}{I_{h1}^*(0)}\right)\\
&= -\frac{\mu_v\left(S_v(t)-S_{v1}^*\right)^2}{b_1S_v(t)S_{v1}^*}-\frac{\mu_h(S_h(t)-S_{h1}^*)^2}{S_h(t)}+\frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_{h1}^*(a)\\
&\quad \times\left(3-\frac{S_{v1}^*}{S_v(t)}-\frac{I_v^1(t)}{I_{v1}^*}-\frac{S_v(t)I_h^1(a,t)I_{v1}^*}{S_{v1}^*I_{h1}^*(a)I_v^1(t)}+\ln\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)\right)da\\
&\quad +I_{h1}^*(0)\Bigg(-\frac{S_{h1}^*}{S_h(t)}+\frac{S_{h1}^*I_h^1(0,t)}{S_h(t)I_{h1}^*(0)}-\ln\left(\frac{I_h^1(0,t)}{I_{h1}^*(0)}\right)\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mu_v \left(S_v(t) - S^*_{v1}\right)^2}{b_1 S_v(t) S^*_{v1}} - \frac{\mu_h (S_h(t) - S^*_{h1})^2}{S_h(t)} + \frac{1}{b_1}\int_0^\infty \beta_v^1(a) I^*_{h1}(a) \\
&\quad \times \left(-f\left(\frac{S^*_{v1}}{S_v(t)}\right) - f\left(\frac{I_v^1(t)}{I^*_{v1}}\right) - f\left(\frac{S_v(t) I_h^1(a,t) I^*_{v1}}{S^*_{v1} I^*_{h1}(a) I_v^1(t)}\right)\right) da \\
&\quad + I^*_{h1}(0)\left(-f\left(\frac{S_h(t)}{S^*_{h1}}\right) + f\left(\frac{S^*_{h1} I_h^1(0,t)}{S_h(t) I^*_{h1}(0)}\right)\right) \\
&= -\frac{\mu_v \left(S_v(t) - S^*_{v1}\right)^2}{b_1 S_v(t) S^*_{v1}} - \frac{\mu_h (S_h(t) - S^*_{h1})^2}{S_h(t)} - I^*_{h1}(0) f\left(\frac{S_h(t)}{S^*_{h1}}\right) \\
&\quad - \frac{1}{b_1}\int_0^\infty \beta_v^1(a) I^*_{h1}(a) \times \left(f\left(\frac{S^*_{v1}}{S_v(t)}\right) + f\left(\frac{S_v(t) I_h^1(a,t) I^*_{v1}}{S^*_{v1} I^*_{h1}(a) I_v^1(t)}\right)\right) da, \\
U^2(t) &= -\frac{1}{b_1 S^*_{v1}}\left(S_v(t)\int_0^\infty \beta_v^2(a) I_h^2(a,t)da - S^*_{v1}\int_0^\infty \beta_v^2(a) I_h^2(a,t)da\right) - \frac{\mu_v I_v^2(t)}{b_1 S^*_{v1}} \\
&\quad - I_h^2(0,t) + \frac{S_v(t)}{b_1 S^*_{v1}}\int_0^\infty \beta_v^2(a) I_h^2(a,t)da + \frac{\beta_h^2 S^*_{h1} I_v^2(t)}{N_h} + I_h^2(0,t) \\
&\quad - \frac{1}{b_2}\int_0^\infty \beta_v^2(a) I_h^2(a,t)da \\
&= \left(\frac{1}{b_1} - \frac{1}{b_2}\right)\int_0^\infty \beta_v^2(a) I_h^2(a,t)da + \frac{\beta_h^2 S^*_{h1} I_v^2(t)}{N_h} - \frac{\mu_v I_v^2(t)}{b_1 S^*_{v1}} \\
&= \left(\frac{1}{b_1} - \frac{1}{b_2}\right)\int_0^\infty \beta_v^2(a) I_h^2(a,t)da + \frac{\beta_h^2 S^*_{h1} I_v^2(t)}{N_h}\left(1 - \frac{\mu_v N_h}{b_1 \beta_h^2 S^*_{v1} S^*_{h1}}\right) \\
&= \left(\frac{1}{b_1} - \frac{1}{b_2}\right)\int_0^\infty \beta_v^2(a) I_h^2(a,t)da + \frac{\beta_h^2 S^*_{h1} I_v^2(t)}{N_h}\left(1 - \frac{\mathscr{R}_0^1 b_2}{\mathscr{R}_0^2 b_1}\right), \\
U^3(t) &= \left(\frac{1}{b_1} - \frac{S_v(t)}{b_1 S^*_{v1}}\right)\int_0^\infty \beta_v^1(a) \phi I_h^{21}(a,t)da \\
&\quad + \left(\frac{1}{b_1} - \frac{S_v(t)}{b_1 S^*_{v1}}\right)\int_0^\infty \beta_v^2(a) \phi I_h^{12}(a,t)da + \frac{S_v(t)}{b_1 S^*_{v1}}\left(1 - \frac{I^*_{v1}}{I_v^1(t)}\right) \\
&\quad \times \int_0^\infty \beta_v^1(a) \phi I_h^{21}(a,t)da + \frac{S_v(t)}{b_1 S^*_{v1}}\int_0^\infty \beta_v^2(a) \phi I_h^{12}(a,t)da \\
&\quad + c_1 I^*_{h1}(0) f\left(\frac{I_h^1(0,t)}{I^*_{h1}(0)}\right) - \int_0^\infty \gamma_1(a) I^*_{h1}(a) f\left(\frac{I_h^1(a,t)}{I^*_{h1}(a)}\right) da + c_2 I_h^2(0,t) \\
&\quad - \int_0^\infty \gamma_2(a) I_h^2(a,t)da - \left(1 - \frac{R^*_{h1}}{R_h^1(t)}\right) I_h^{12}(0,t) - I_h^{21}(0,t) - \mu_h R_h^2(t) \\
&\quad + \int_0^\infty \gamma_1(a) I^*_{h1}(a)\left(\frac{I_h^1(a,t)}{I^*_{h1}(a)} - \frac{R_h^1(t)}{R^*_{h1}} - \frac{I_h^1(a,t) R^*_{h1}}{I^*_{h1}(a) R_h^1(t)} + 1\right) da
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_h^{21}(a,t)da + I_h^{21}(0,t) + I_h^{12}(0,t) - \frac{1}{b_2}\int_0^\infty \beta_v^2(a)I_h^{12}(a,t)da\\
&-c_1 I_{h1}^*(0) f\left(\frac{I_h^1(0,t)}{I_{h1}^*(0)}\right) - c_2 I_h^2(0,t) - \frac{R_{h1}^*}{R_h^1(t)} I_h^{12}(0,t) + \int_0^\infty \gamma_2(a) I_h^2(a,t)da\\
={}& \frac{1}{b_1}\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da + \frac{1}{b_1}\int_0^\infty \beta_v^2(a)\phi I_h^{12}(a,t)da\\
&-\frac{1}{b_2}\int_0^\infty \beta_v^2(a)I_h^{12}(a,t)da - \frac{S_v(t)I_{v1}^*}{b_1 S_{v1}^* I_v^1(t)}\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da\\
&-\int_0^\infty \gamma_1(a)I_{h1}^*(a) f\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)}\right)da - \mu_h R_h^2(t)\\
&+\int_0^\infty \gamma_1(a)I_{h1}^*(a)\left(\frac{I_h^1(a,t)}{I_{h1}^*(a)} - \frac{R_h^1(t)}{R_{h1}^*} - \frac{I_h^1(a,t)R_{h1}^*}{I_{h1}^*(a)R_h^1} + 1\right)da\\
&-\frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_h^{21}(a,t)da\\
={}& -\frac{S_v(t)I_{v1}^*}{b_1 S_{v1}^* I_v^1(t)}\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da - \mu_h R_h^2(t)\\
&+\frac{1}{b_1}\left(\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da - \int_0^\infty \beta_v^1(a)I_h^{21}(a,t)da\right)\\
&+\frac{1}{b_1}\int_0^\infty \beta_v^2(a)\phi I_h^{12}(a,t)da - \frac{1}{b_2}\int_0^\infty \beta_v^2(a)I_h^{12}(a,t)da\\
&-\int_0^\infty \gamma_1(a)I_{h1}^*(a)\left(f\left(\frac{R_h^1(t)}{R_{h1}^*}\right) + f\left(\frac{I_h^1(a,t)R_{h1}^*}{I_{h1}^*(a)R_h^1(t)}\right)\right)da.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
U'(t) ={}& -\frac{\mu_v\left(S_v(t)-S_{v1}^*\right)^2}{b_1 S_v(t)S_{v1}^*} - \frac{\mu_h(S_h(t)-S_{h1}^*)^2}{S_h(t)} - I_{h1}^*(0) f\left(\frac{S_{h1}^*}{S_h(t)}\right) - \mu_h R_h^2(t)\\
&-\frac{1}{b_1}\int_0^\infty \beta_v^1(a)I_{h1}^*(a)\left(f\left(\frac{S_{v1}^*}{S_v(t)}\right) + f\left(\frac{S_v(t)I_h^1(a,t)I_{v1}^*}{S_{v1}^* I_{h1}^*(a)I_v^1(t)}\right)\right)da\\
&-\frac{S_v(t)I_{v1}^*}{b_1 S_{v1}^* I_v^1(t)}\int_0^\infty \beta_v^1(a)\phi I_h^{21}(a,t)da + \frac{\beta_h^2 S_{h1}^* I_v^2(t)}{N_h}\left(1 - \frac{b_2\mathscr{R}_0^1}{b_1\mathscr{R}_0^2}\right)\\
&+\frac{\phi-1}{b_1}\int_0^\infty \beta_v^1(a)I_h^{21}(a,t)da + \left(\frac{\phi}{b_1} - \frac{1}{b_2}\right)\int_0^\infty \beta_v^2(a)I_h^{12}(a,t)da\\
&-\int_0^\infty \gamma_1(a)I_{h1}^*(a)\left(f\left(\frac{R_h^1(t)}{R_{h1}^*}\right) + f\left(\frac{I_h^1(a,t)R_{h1}^*}{I_{h1}^*(a)R_h^1(t)}\right)\right)da\\
&+\left(\frac{1}{b_1} - \frac{1}{b_2}\right)\int_0^\infty \beta_v^2(a)I_h^2(a,t)da. \qquad (7.1)
\end{aligned}
$$

As $f(x) \geq 0$ for $x > 0$, $\phi \leq 1$ and $\mathscr{R}_1^2/\mathscr{R}_1^1 < b_2/b_1 < 1$, we have $U'(t) \leq 0$. Set

$$\Theta = \{(S_v, I_v^1, I_v^2, S_h, I_h^1, I_h^2, R_h^1, R_h^2, I_h^{12}, I_h^{21}) \in \Omega_0 \,|\, U'(t) = 0\}.$$

Obviously, the equality in (7.1) holds if and only if

$$S_v = S_{v1}^*, \quad I_v^1 = I_{v1}^*, \quad I_v^2 = 0, \quad S_h = S_{h1}^*, \quad I_h^1 = I_{h1}^*, \quad I_h^2 = 0,$$
$$R_h^1 = R_{h1}^*, \quad R_h^2 = 0, \quad I_h^{12} = 0, \quad I_h^{21} = 0.$$

It is easy to see that $\mathscr{E}_1$ is the largest invariant set in Θ. By applying the LaSalle invariance principle, $\mathscr{E}_1$ is globally attractive. This completes the proof. □

Now, we provide a numerical example to confirm the result in Theorem 7.1, where the global stability of the strain 1 dominant equilibrium $\mathscr{E}_1$ is guaranteed if the conditions of Theorem 7.1 hold. We choose constant parameters by $\Lambda_h = 800$, $\Lambda_v = 80$, $\phi = 1.5$, $\sigma_1 = 0.5$, $\sigma_2 = 0.01$, $\beta_h^1 = 0.5$, $\beta_h^2 = 0.2$, $\gamma_1 = 0.5$, $\gamma_2 = 0.01$, $\mu_v = 0.002$, $\mu_h = 0.0001$, $\beta_v^1 = 15$, $\beta_v^2 = 0.0001$.The numerical solutions of $S_h(t)$, $I_h^j(a,\, t)$, $R_h^j(a)$, I_h^{ji}, $S_v(t)$ and $I_v^j(t)$ are displayed in Figs. 4 and 5. Clearly, the solutions $I_h^j(a,\, t)$, $I_h^{ij}(a,\, t)$, $R_h^j(t)$, $I_v^j(t)$, $S_h(t)$ and $S_v(t)$ all approach corresponding components of $\mathscr{E}_1$ when t is large enough.

8. Coexistence Equilibrium and Backward Bifurcation

In this section, we investigate the coexistence equilibrium where both strains are present in humans and mosquitoes. For simplicity, it is assumed that

$$\sigma_1 = \sigma_2 = \sigma, \quad \beta_v^1(a) = \beta_v^2(a) = \beta_v(a), \quad \gamma_1(a) = \gamma_2(a) = \gamma(a).$$

In this case, we have

$$\mathscr{R}_0 = \mathscr{R}_1^1 = \mathscr{R}_1^2, \quad \pi_1(a) = \pi_2(a), \quad b_1 = b_2, \quad c_1 = c_2.$$

We can establish the following result.

Theorem 8.1. *If $\mathscr{R}_0 > 1$, then the coexistence equilibrium is unique, and if $\mathscr{R}_0 < 1$ and $\Delta = h_1^2 - 4h_0h_2 > 0$, $h_1 < 0$, there are two coexistence equilibria. The expressions of equilibrium satisfy the following:*

$$S_h = \frac{\Lambda_h}{2x + \mu_h}, \quad I_v^1 = I_v^2 = \frac{N_h}{\beta_h}x, \quad I_h^1(a) = I_h^2(a) = \frac{\Lambda_h \pi(a) x}{2x + \mu_h},$$
$$R_h^1 = R_h^2 = \frac{c\Lambda_h x}{(\mu_h + \sigma x)(2x + \mu_h)}, \quad I_h^{12}(a) = I_h^{21}(a) = \frac{c\sigma\Lambda_h \pi(a) x^2}{(\mu_h + \sigma x)(2x + \mu_h)},$$
$$S_v = \frac{\Lambda_v}{\mu_v} - 2x,$$

where x is the positive roots of

$$H(x) \triangleq h_2 x^2 + h_1 x + h_0 = 0$$

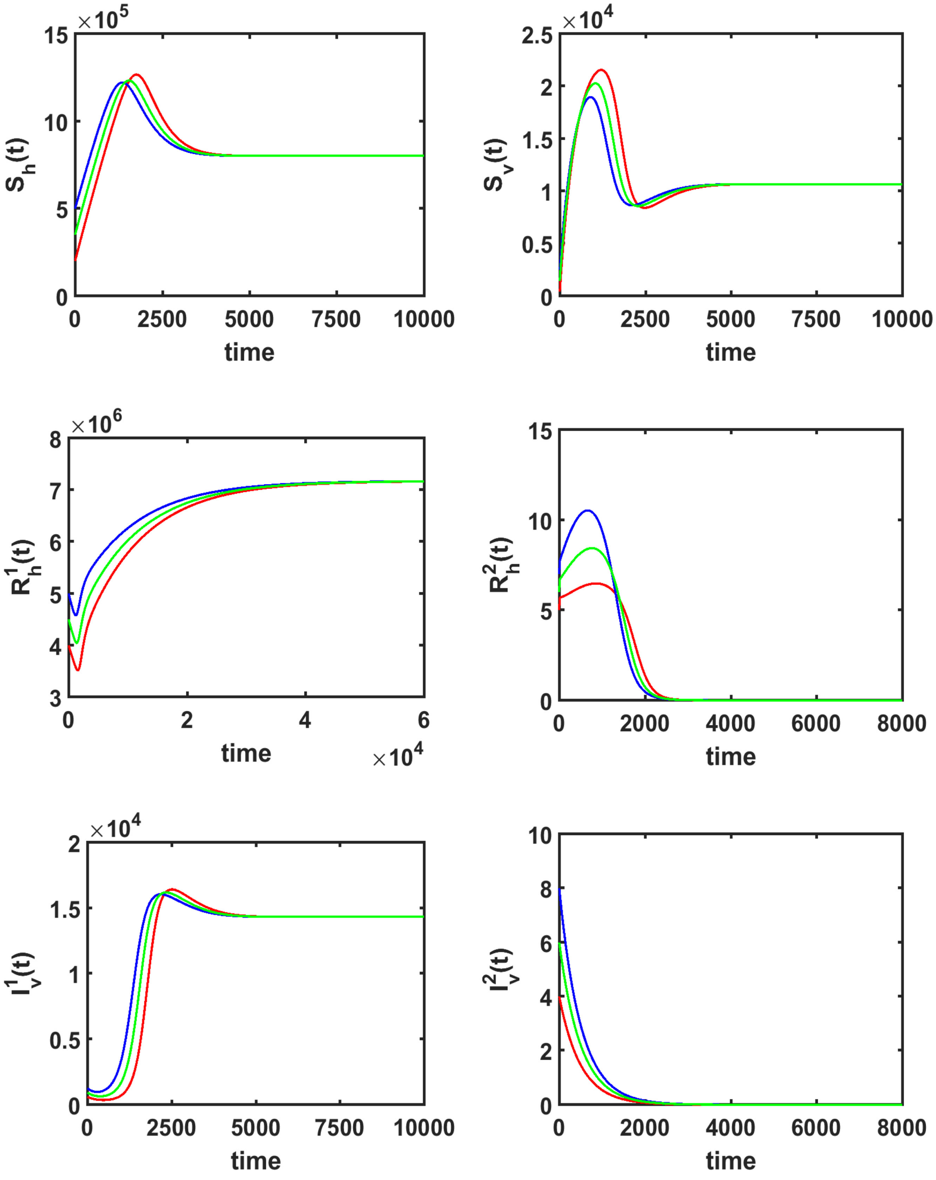

Fig. 4. (Color online) A typical scenario showing the time series of $S_h(t)$, $R_h^j(t)$, $S_v(t)$ and $I_v^j(t)$ when $\mathscr{R}_0^j < 1$, $j = 1, 2$. The difference in curve color only means that the initial values of variables are different.

with the coefficients given by

$$h_2 = 2\sigma\mu_v N_h + 2b\sigma\Lambda_h N_h(1 + \phi c) > 0,$$

$$h_1 = \mu_v\Lambda_h(2 + \sigma) - \left(b\sigma\beta_h\Lambda_h N_v(1 + \phi c) - 2b\Lambda_h^2\right),$$

$$h_0 = \mu_v\mu_h\Lambda_h(1 - \mathscr{R}_1).$$

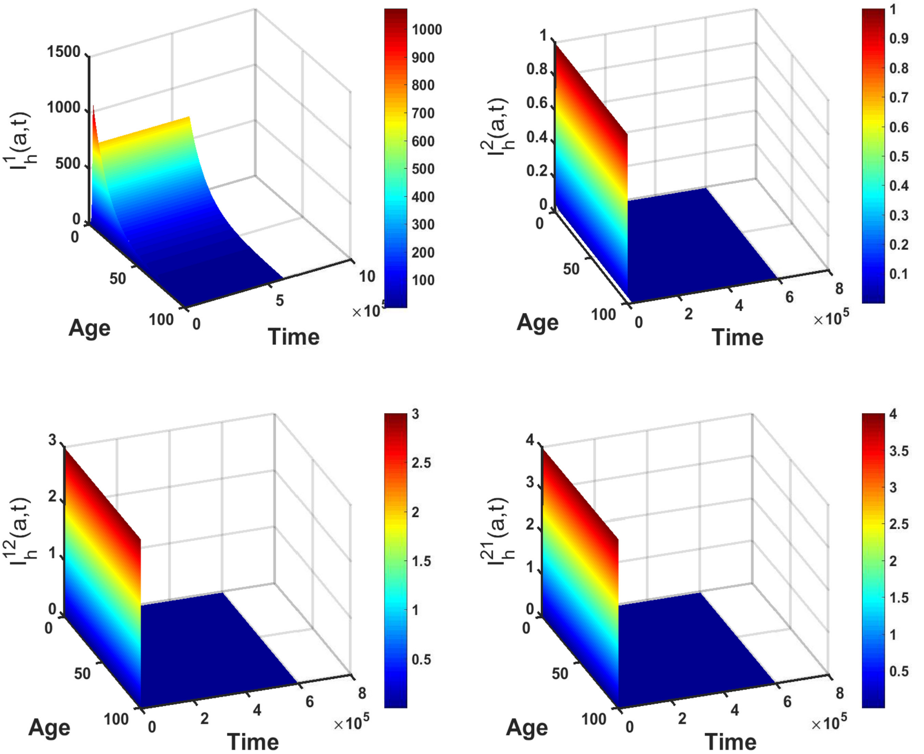

Fig. 5. A typical scenario showing the time and age series of $I_h^j(a,t)$ and $I_h^{ij}(a,t)$ when $\mathscr{R}_0^j < 1$, $j = 1, 2$.

Proof. Set $x_j = \frac{\beta_h}{N_h} I_v^j$. By direct computing, we have

$$S_h = \frac{\Lambda_h}{x_1 + x_2 + \mu_h}, \quad I_h^1(a) = \frac{\Lambda_h \pi(a) x_1}{x_1 + x_2 + \mu_h}, \quad I_h^2(a) = \frac{\Lambda_h \pi(a) x_2}{x_1 + x_2 + \mu_h},$$

$$R_h^1 = \frac{c\Lambda_h x_1}{(\mu_h + \sigma x_2)(x_1 + x_2 + \mu_h)}, \quad R_h^2 = \frac{c\Lambda_h x_2}{(\mu_h + \sigma x_1)(x_1 + x_2 + \mu_h)},$$

$$I_h^{12}(a) = \frac{c\sigma\Lambda_h \pi(a) x_1 x_2}{(\mu_h + \sigma x_2)(x_1 + x_2 + \mu_h)}, \quad I_h^{21}(a) = \frac{c\sigma\Lambda_h \pi(a) x_1 x_2}{(\mu_h + \sigma x_1)(x_1 + x_2 + \mu_h)},$$

$$S_v = \Lambda_v \Big/ \left\{ \int_0^\infty \beta_v(a) \left(I_h^1(a) + \phi I_h^{21}(a) \right) da + \int_0^\infty \beta_v(a) \left(I_h^2(a) + \phi I_h^{12}(a) \right) da + \mu_v \right\}.$$

Obviously, $S_v + I_v^1 + I_v^2 = N_v$. Substituting the equilibrium for the infected classes into the second and third equilibrium equations in (2.2), we have

$$\begin{aligned} \mu_v &= \left(\frac{\beta_h N_v}{N_h} - x_1 - x_2 \right) \left(\frac{b\Lambda_h}{x_1 + x_2 + \mu_h} + \frac{b\phi c\sigma\Lambda_h x_2}{(\mu_h + \sigma x_1)(x_1 + x_2 + \mu_h)} \right), \\ \mu_v &= \left(\frac{\beta_h N_v}{N_h} - x_1 - x_2 \right) \left(\frac{b\Lambda_h}{x_1 + x_2 + \mu_h} + \frac{b\phi c\sigma\Lambda_h x_1}{(\mu_h + \sigma x_2)(x_1 + x_2 + \mu_h)} \right). \end{aligned} \tag{8.1}$$

If there exists the coexistence equilibrium, namely, the positive solution (x_1, x_2) must satisfy the following equation:

$$\frac{b\phi c\sigma\Lambda_h x_2}{(\mu_h + \sigma x_1)(x_1 + x_2 + \mu_h)} = \frac{b\phi c\sigma\Lambda_h x_1}{(\mu_h + \sigma x_2)(x_1 + x_2 + \mu_h)}.$$

From the above, we have $x_1 = x_2$. Set $x_1 = x_2 = x$. From (8.1), we have

$$H(x) \triangleq h_2 x^2 + h_1 x + h_0 = 0, \tag{8.2}$$

where

$$\begin{aligned} h_2 &= 2\sigma\mu_v N_h + 2b\sigma\Lambda_h N_h(1 + \phi c) > 0, \\ h_1 &= \mu_v\Lambda_h(2 + \sigma) - \left(b\sigma\beta_h\Lambda_h N_v(1 + \phi c) - 2b\Lambda_h^2\right), \\ h_0 &= \mu_v\mu_h\Lambda_h(1 - \mathscr{R}_1). \end{aligned}$$

Now, we identify the number of positive roots of (8.2). Obviously, if $\mathscr{R}_1 > 1$, there exists a unique positive root, that is, a unique coexistence equilibrium. However, if

$$\mathscr{R}_1 < 1, \quad \Delta = h_1^2 - 4h_0h_2 > 0, \quad h_1 < 0,$$

there are two coexistence equilibria. □

9. Concluding Remarks

In this paper, we propose a novel two-strain dengue epidemic model coupling PDEs, which can characterize the transmission dynamics of DENV with the infection age and possible secondary infection for humans. For each strain j, the explicit formulas for the reproduction numbers $\mathscr{R}_1^j$ and $\mathscr{R}_2^j$ of strain are derived, respectively. Threshold reproduction numbers of two strain in the proposed model are defined, respectively: $\mathscr{R}_0 = \max\{\mathscr{R}_1^1, \mathscr{R}_1^2\}$ and $\mathscr{R}_c = \max\{\mathscr{R}_2^1, \mathscr{R}_2^2\}$. We show that if $\mathscr{R}_0 < 1$, the disease-free equilibrium $\mathcal{E}_0$ is locally asymptotically stable. It is globally asymptotically stable if $\mathscr{R}_0 + \mathscr{R}_c < 1$, which implies that the DENV goes extinct. If $\mathscr{R}_0 > 1$, without loss of generality, assuming $\mathscr{R}_0 = \mathscr{R}_1^1 > 1$, we show that the dominant equilibrium $\mathscr{E}_1$ exists. The dominant equilibrium $\mathscr{E}_1$ is locally asymptotically stable if $\mathscr{R}_1^1 > 1$ and $\mathscr{R}_1^2 < \varsigma\mathscr{R}_1^1$.

By constructing Lyapunov functions, we show the global stability of the dominant equilibrium $\mathscr{E}_1$ under some conditions that

$$\mathscr{R}_1^2/\mathscr{R}_1^1 < \min\{b_2/b_1, \varsigma\}, \quad b_2/b_1 < 1. \tag{9.1}$$

Condition (9.1) suggests the following inequalities hold simultaneously:

$$\mathscr{R}_1^2 < \mathscr{R}_1^1,\ b_2 < b_1,\ \mathscr{R}_{v1}^2 < \mathscr{R}_{v1}^1.$$

This implies that if two strains circulate in the population only the strain with the lager reproduction number persists, the strain with suboptimal reproduction numbers is eliminated. The competitive exclusion principle holds.

Recall that b_j represents the probability of a susceptible mosquito being transmitted by a primarily infected human with strain j. $0 \leq \phi \leq 1$ means humans with a secondary infection have on average a lower contribution to the overall infectivity of mosquito upon humans than humans with a primary infection. Thus, condition (9.1) for the occurrence of competitive exclusion of strain 1 implies that the reproduction number, the transmission rate of an infectious mosquito during its infectious period, and a susceptible mosquito being transmitted primarily are bigger for strain 1 comparing with the corresponding quantities of strain 2. Besides, it also requires the primary infection has stronger infection force than the secondary infection. At last, under additional conditions, we investigate the existence of the backward bifurcation. This implies that $\mathscr{R}_0 < 1$ is not sufficient to control and eradicate DENV.

Our results do not include the case where $\phi > 1$, which means humans with a secondary infection have on average a higher contribution to the overall infectivity of mosquito upon humans than humans with a primary infection. In this case, phenomena of antibody-dependent enhancement during DENV transmission may occur. It is difficult to deal with it by applying the present methods. Some new techniques must be developed. We would like to investigate it in our future work.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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