How Many Numbers/Infinities Exist?

- Nick Lamay

A Warm Up With Georg Cantor

- Not all infinities are equal, but how can we prove that?
- Compare the Even and Odd natural numbers

  Do they have the same size?

  What is their cardinality?

A Warm Up With Georg Cantor

- Not all infinities are equal, but how can we prove that?
- Compare the Even and Odd natural numbers

  Do they have the same size?

  What is their cardinality?
- Now try the natural numbers with all Real numbers

  Hint: Compare {1,2,3,…} to {1.00001,1.00002,….}

  These cannot be paired off in the same way!

  The second infinity is "bigger"

The Continuum Hypothesis

- In 1873, Georg Cantor first asserts real numbers “outnumber” the natural numbers
- Cantor discovers that the power set of a set has larger cardinality than its original set does
- He then tries to take a power set of a power set, and discovers its cardinality is even larger.
- This poses a very imposing problem onto Cantor

The Continuum Hypothesis

- Cantor focused on the first few power sets and was able to prove that different ways of ordering the natural numbers all lead to a set of the exact same size. No matter the ordering method.
- He then asserts that this cardinality is, $\aleph_1$ (Aleph 1), and that this is precisely the size of the real numbers as well.
- Sadly, he was never able to prove that this was true.

The Zermelo–Fraenkel axioms with the axiom of choice

- In 1931, Austrian logician Kurt Gödel discovered that any set of axioms that you might posit as a foundation for mathematics will inevitably be incomplete.
- In other words, there will always be problems outside of your scope of rules, that your rules simply can't answer.
- Gödel quickly applied this to the Continuum Hypothesis, and found that it was independent of the ZFC

The Zermelo–Fraenkel axioms with the axiom of choice

- What is the ZFC?
- The ZFC are the ten axioms that underpin all of modern mathematics.
- They describe basic set properties and properties of being an object in a set.
- They are almost always sufficient for any proofs in mathematics since most things can be described using set notation.

The Problem

- In 1940, Gödel posited that you can not use the ZFC to disprove his continuum hypothesis.
- Then in 1963, an American mathematician, Paul Cohen, showed you can not use the ZFC to prove it either.
- So, where do we go from here?

We Need a New Axiom!

- Since Cohens discovery, mathematicians have been searching for a new axiom that would solve Cantor's hypothesis.
- Two rival axioms have come out of this search since them, but they seemed to be completely illogical when put together but necessary to solve the hypothesis on their own.

- But to understand this, lets go back to Cohens Work, and a procedure he called 'Forcing"

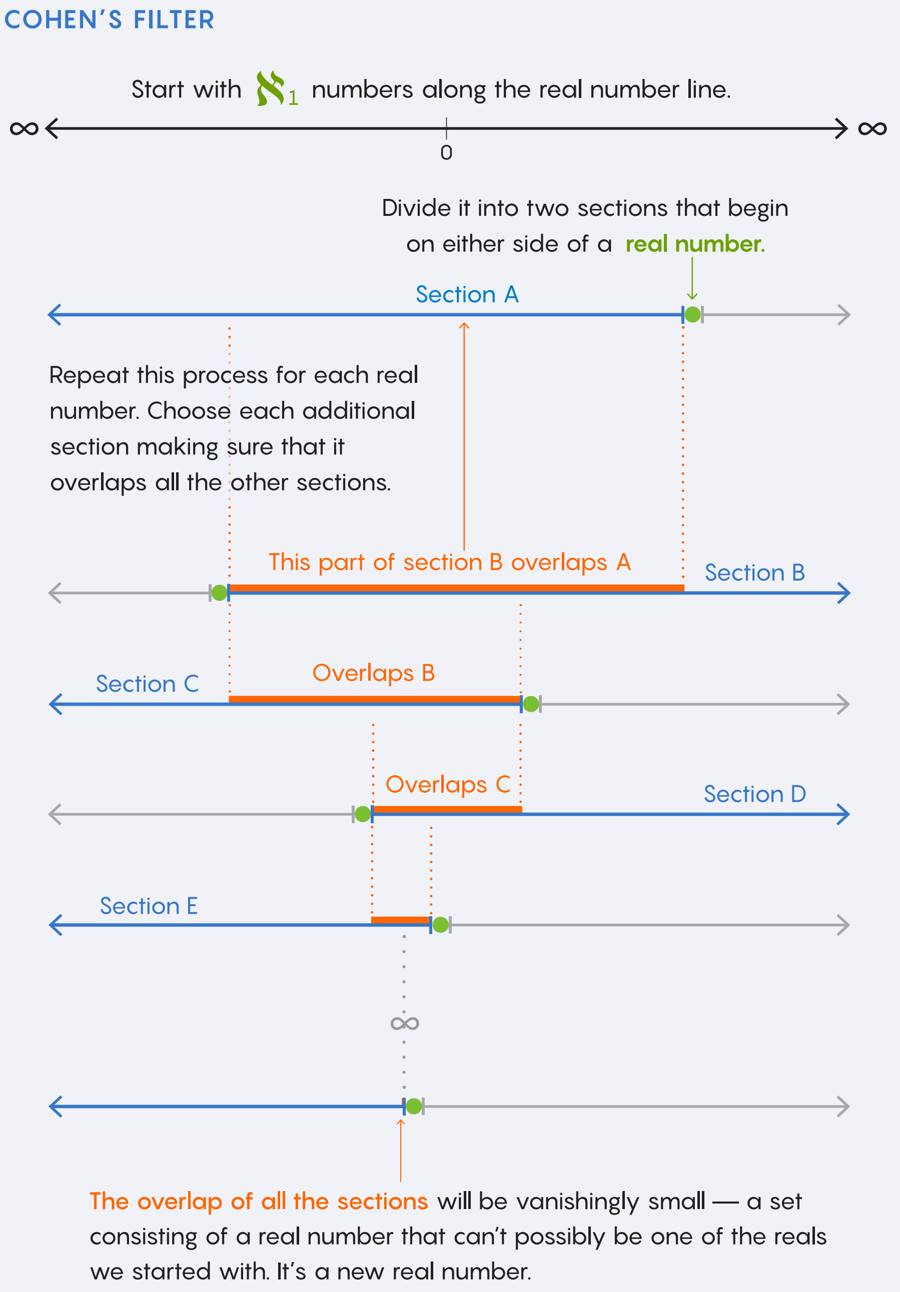
May the Force Be With You!

- Forcing "forces" a new real number to exist from the real number line by the process pictured.

- This process can produce an infinite amount of new real numbers that can provide very exciting properties that may not have been considered before.

Forcing New Reals

To get a sense of how forcing works, imagine $\aleph_1$ real numbers spread out along a number line. You can slice up this line in infinitely many ways. For instance, you can slice the line into two sections that begin on either side of a real number, and then do the same for every other real number. Cohen conceived of a filter that picks one section from each possible slicing of the number line. Each additional section is chosen such that it overlaps all the other sections already in the filter. When this filtering process is complete, the overlap of all the sections is vanishingly small — a set consisting of a single real number. But because each real we started with was one of the points where we sliced the number line in two (and therefore, it's excluded from the section chosen from that slicing), the overlap of all sections in the filter can't possibly contain one of the reals we started with. It's a new real number. Generalizations of this method let you expand the continuum to include however many reals you like.



The Solution….s

- One solution was first introduced in 1988 by Menachem Magidor, Matthew Foreman, and Saharon Shelah. It was called Martin's Maximum.
- Martin's Maximum states that if any mathematical object can be found using forcing, then it must truly exist in some mathematical sense, if the forcing method satisfies a consistency condition. This method utilized the axiom of Determinacy to be used.
- This grand sweeping axiom provides a lot of powerful tools to infinite set exploration and theorem generation.

The Solution….s

- However, in the 1990's, Hugh Woodin proposed and axiom that also solved the Continuum Hypothesis with forcing but in an entirely different way.
- He named the axiom (*), pronounced "Star".
- Woodin devised a model universe of sets that at first only satisfied 9 of the ten axioms of the ZFC, and the axiom of choice.
- Woodin then expanded this model using forcing in such a way that was consistent with the ZFC. So, (*) was also logically sound, because it allowed for statements such as "For all sets of $\aleph_1$, there exists reals not in those sets". This is the pure negation of the Continuum Hypothesis.

The Missing Link

In 2011, David Aspero and Ralf Shindler had been European mathmeticians, reading a paper by Ronald Jensen about a process of forcing called L-Forcing.

Aspero immediately proposed that they may be able to link Martin's Maximum and (*) together by using such a process.

The next year in 2012, they announced they had solved the process. However, Woodin immediately spotted a mistake, and they withdrew their paper in shame.

However, Aspero and Shindler were determined to find an answer

The Witness

Their plan for the proof was to find some process that was like L-forcing but use it to generate something they called a “witness” that verified all statements of the form, (*).

If they could find a procedure that satisfied the consistency principle, Martin’s Maximum guarantees the existence of the witness, and then (*) would follow.

In a “flash experience”, Aspero proposed a recursive method of forcing that requires a set of conditions to function. This was the missing link they needed. Now Martin's Maximum and (*) were connected. At least, it seems so…

Further Questioning

- Now the next steps for mathematicians is to try to tear this new convergent theory apart.
- Mathematicians such as Peter Koellner, and even Woodin himself, doubt the convergent axiom.
- Woodin has been quoted saying “I'm considered a traitor”. He now considers his (*) axiom to be false and believes the Continuum Hypothesis to be true. Such statements spell disaster for the convergent axiom if they could be proved true.
- However, he still claims Aspero and Schindler’s theory to be a fantastic result that should be published and discussed,

The Future

- Even now Woodin, and our European Duo Aspero and Schindler, are working hard to prove their theories right.
- Woodin has stated he is 400 pages into a proof attempt and Aspero and Schindler are anxiously waiting to rip it apart.
- One winner of these two theories could be produced at any time and mathematicians will be waiting for this new foundational axiom to be added to the ZFC and Cantors infamous question finally answered.