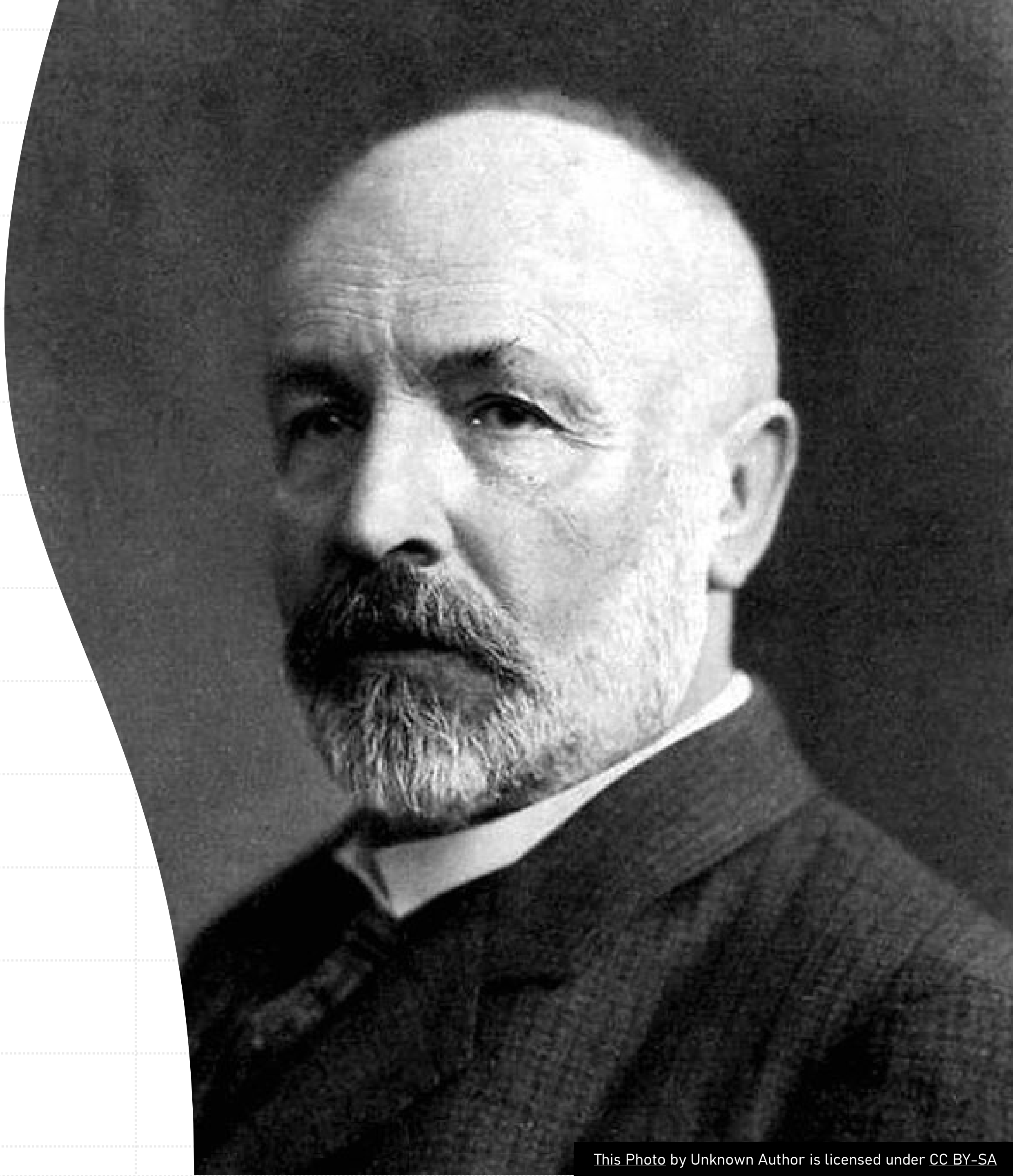
How Many Numbers Exist? Infinity Proof Moves Math Closer to an Answer.

Kevin Lay

Georg Cantor

- In 1873, Georg Cantor, a German mathematician, discovered the "real" numbers filling the number line
- He realizes that if you take two different sets, it can be put one-to-one forming pairs of elements of each set
- In this case, these infinite sets have the same size; Cantor called this "cardinality"
- He designated the size with cardinal number $\aleph_0$ (aleph-zero)



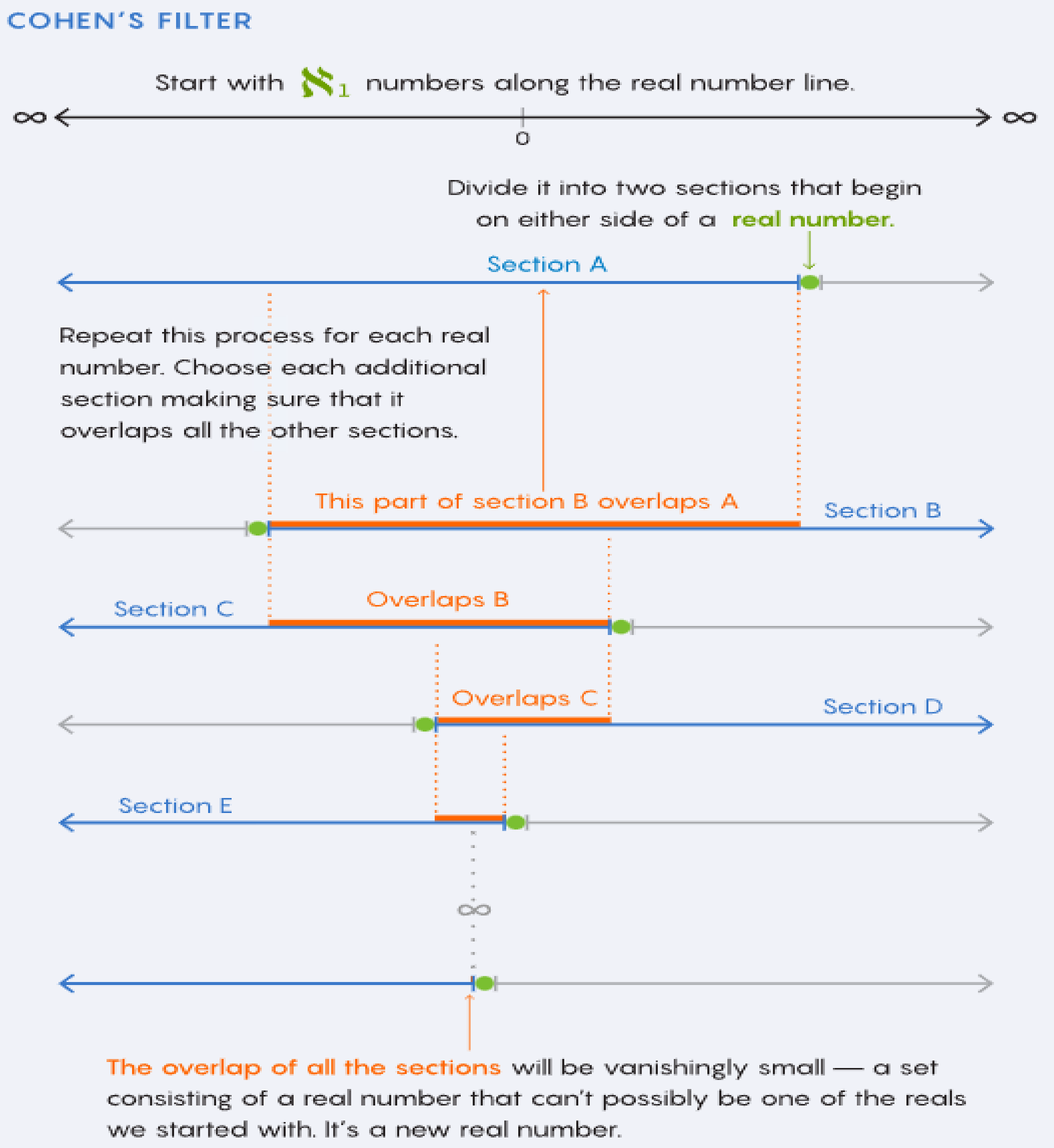
- However, Cantor discovered that natural numbers can't be one-to-one with the continuum of real numbers (ex. 1 paired with 1.00000...). Their cardinality is greater than natural numbers.
- He also discovered that any infinite set's power set has larger cardinality than it does such that every power set has a power set so that cardinal numbers form indefinitely.
- He proved that the set formed from different ways of ordering natural numbers has cardinality $\aleph_1$
- His continuum hypothesis describes that the size of a continuum is precisely $\aleph_1$ real numbers, but he couldn't prove it. (this hypothesis would be called Cantor's paradise by David Hilbert)

ZFC (Zermelo–Fraenkel axioms with the axiom of choice)

- In 1931, Kurt Gödel discovered that any set of axioms will be incomplete in which the continuum hypothesis is an example of it.
- The axioms are described as basic properties of collections of objects or sets.
- In 1940, Gödel shown that the continuum hypothesis can’t use ZFC to disprove it. Then, in 1963, Paul Cohen showed that you can’t use ZFC to prove it. Therefore, the continuum hypothesis is independent from the ZFC axioms.
- This also applies to any infinite sets to be independent from ZFC axioms

Forcing New Reals

To get a sense of how forcing works, imagine $\aleph_1$ real numbers spread out along a number line. You can slice up this line in infinitely many ways. For instance, you can slice the line into two sections that begin on either side of a real number, and then do the same for every other real number. Cohen conceived of a filter that picks one section from each possible slicing of the number line. Each additional section is chosen such that it overlaps all the other sections already in the filter. When this filtering process is complete, the overlap of all the sections is vanishingly small — a set consisting of a single real number. But because each real we started with was one of the points where we sliced the number line in two (and therefore, it's excluded from the section chosen from that slicing), the overlap of all sections in the filter can't possibly contain one of the reals we started with. It's a new real number. Generalizations of this method let you expand the continuum to include however many reals you like.



Samuel Velasco/Quanta Magazine

Martin’s Maximum

- Martin’s maximum says that anything that forces the procedure will be a true mathematical entity if the procedure satisfies a certain condition.
- It has proven that Martin’s maximum is a powerful tool for exploring properties of infinite sets. It was a popular extension to ZFC.

* (Star)

- In the 1990s, Hugh Woodin proposed another axiom (*), which concerns a model universe of sets that satisfies 9 ZF axioms plus the axiom of determinacy.
- The statement, “For all X, there exists Y, such that Z” derived from (*)
- (*) is the contradiction from Martin’s maximum

Schindler and Asperó

- Asperó shows that Martin's maximum++ implies (*)
- Ronald Jensen created L-forcing which Schindler read. He showed this to Asperó in which he said that they can use this technique to derive (*) from Martin's maximum++

Ralf Schindler(left) and David Asperó (right)

- Their plan was to develop a technique like L–forcing to generate a type of object called witness. This verifies all statements in the form of (*). So, if the forcing procedure satisfies a certain condition, then, Martin's maximum++ will force the witness to exist.
- For the forcing procedure to meet the key requirement of Martin's maximum, break up the forcing into recursive sequence of forcing in which each of them satisfies the necessary condition.
- As such, with Martin's maximum++ and (*) tells us that the continuum is $\aleph_2$. However, Paul Koellner questioned that is it true. He implies that knowing the strongest forcing axiom upon (*) can be evidence either for or against it.

Other Stars

- Woodin proposed stronger variants of (*) such as (*)+ and (*)++ which applies to full power set of the reals.
- The variants, (*)+ and (*)++, contradicts Martin's maximum.
- With the existence of these variants, it allowed mathematician to create statements in the form of "There exists a set of reals…" which allows them to describe and analyze those said sets.
- The "ultimate L" conjecture: "the existence of an all-encompassing generalization of Gödel's model universe sets."
- Gödel and Cohen established the independence of continuum hypothesis from ZFC; infinite math became "choose-your-own-adventure-story" that set theorists can force number of reals to any level

Source

- https://www.quantamagazine.org/how-many-numbers-exist-infinity-proof-moves-math-closer-to-an-answer-20210715/