Mathematical Modeling and Optimal Control for Malaria Transmission Using Sterile Insect Technique and Insecticide-Treated Nets

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Abstract

We consider a malaria transmission model with SEIR (susceptible-exposed-infected-recovered) classes for the human population, SEI (susceptible-exposed-infected) classes for the wild mosquitoes and an additional class for sterile mosquitoes. We derive the basic reproduction number of infection. We formulate an optimal control problem in which the goal is to minimize both the infected human populations and the cost to implement two control strategies: the release of sterile mosquitoes and the usage of insecticide-treated nets to reduce the malaria transmission. Adjoint equations are derived and the characterization of the optimal controls are established. Finally, we quantify the effectiveness of the two interventions aimed at limiting the spread of Malaria. A combination of both strategies leads to a more rapid elimination of the wild mosquito population that can suppress Malaria transmission. Numerical simulations are provided to illustrate the results.

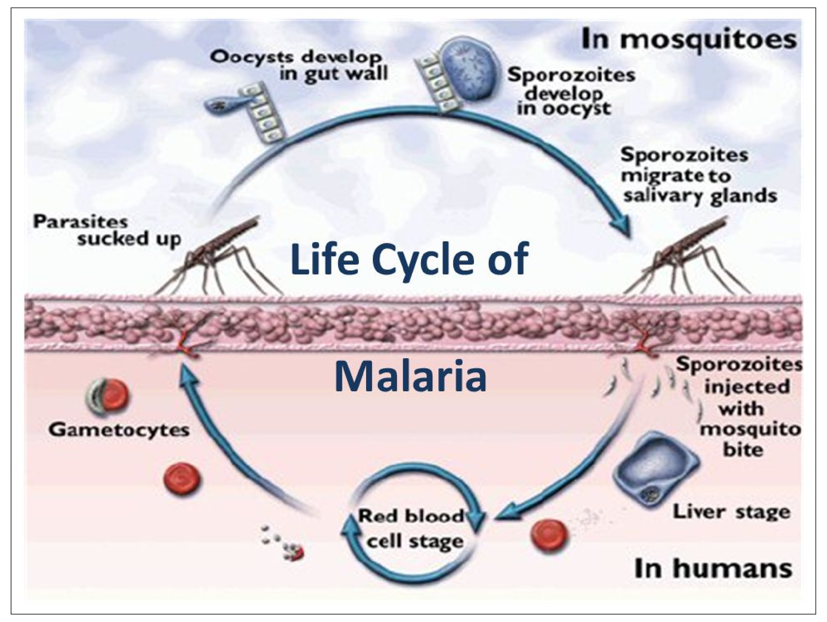


Figure 1:Malaria life cycle.

Malaria Model

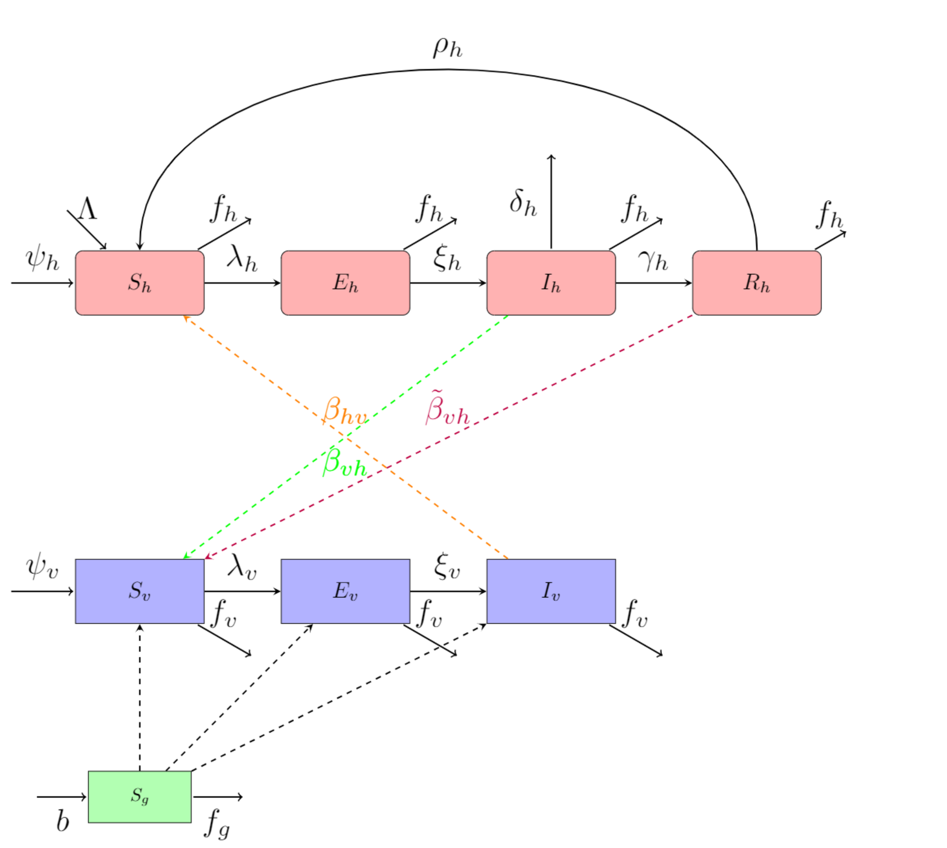


Figure 2:Model Diagram.

Table 1: Description of model parameters

Parameter	Description
Λ_h	Immigrate rate of humans. Humans $\times$ Times^{-1}.
ψ_h	Per capita birth rate of humans.
ψ_v	Per capita birth rate of mosquitoes.
σ_v	Number of times one mosquito would bite humans per unit time.
σ_h	The maximum number of mosquito bites a human can have per unit time.
β_{hv}	Transmission probability from an infectious mosquito to a susceptible human if contact (bite) occurs.
β_{vh}	Transmission probability from an infectious human to a susceptible mosquito if contact (bite) occurs.
$\widetilde{\beta}_{vh}$	Transmission probability from a recovered human to a susceptible mosquito if contact (bite) occurs.
$\xi_h,\ \xi_v$	Per capita rate of progression of human/mosquitoes from the exposed state to the infectious state.
δ_h	Per capita disease-induced death rate for humans.
μ_{1h}, μ_{2h}	Density-independent and density dependent death rate for humans,respectively.
μ_{1v}, μ_{2v}	Density-independent and density dependent death rate for wild mosquitoes, respectively.
μ_{1g}, μ_{2g}	Density-independent and density dependent death rate for sterile mosquitoes, respectively.
ρ_h	Per capita rate of loss of immunity for human.
γ_h	Per capita recovery rate for humans from the infectious state to the recovered state.
$\hat{b}$	release rate of sterile mosquitoes

Our **SEIR**, **SEI**, and Sterile Mosquitoes model:

$$
\begin{aligned}
S_h' &= \Lambda_h + \psi_h N_h + \rho_h R_h - \lambda_h(t) S_h - f_h(N_h) S_h \\
E_h' &= \lambda_h(t) S_h - \xi_h E_h - f_h(N_h) E_h \\
I_h' &= \xi_h E_h - \gamma_h I_h - f_h(N_h) I_h - \delta_h I_h \\
R_h' &= \gamma_h I_h - \rho_h R_h - f_h(N_h) R_h \\
S_v' &= \psi_v \frac{N_v}{N_v + S_g} N_v - \lambda_v(t) S_v - f_v(N_v + S_g) S_v \\
E_v' &= \lambda_v(t) S_v - \xi_v E_v - f_v(N_v + S_g) E_v \\
I_v' &= \xi_v E_v - f_v(N_v + S_g) I_v \\
S_g' &= \hat{b} - f_g(N_v + S_g) S_g
\end{aligned}
\tag{1}
$$

$\mathcal{R}_0$ Calculation

Using the next generation matrix, we obtain the basic reproduction number

$$
\mathcal{R}_0 = \sqrt{\frac{\beta_{hv}\xi_v\sigma_v\sigma_h S_{ho}}{k_1 k_2(\sigma_v S_{vo} + \sigma_h S_{ho})}\left(\frac{\beta_{vh}\xi_v\sigma_v\sigma_h S_{vo}}{k_3 k_4(\sigma_v S_{vo} + \sigma_h S_{ho})} + \frac{\tilde{\beta}_{vh}\xi_v\gamma_h\sigma_v\sigma_h S_{vo}}{k_3 k_4 k_5(\sigma_v S_{vo} + \sigma_h S_{ho})}\right)}
$$

To ensure there is a stable positive mosquito population, we give the conditions that guarantee the existence of this pos. equil.

Define the threshold release rate of sterile mosquitoes:

$$
b_0 := \frac{1}{2\psi_v}\left(\mu_{2v}\mu_{2g}\bar{N}_{vg}^2 + (\psi_v - \mu_{1v})\mu_{1g}\right)\bar{N}_{vg}, \text{ where } N_{vg} = S_v + S_g.
$$

Theorem

Theorem 1: *If $b < b_0$, there exists a disease-free equilibrium in system (1). The disease-free equilibrium of the state system is locally asymptotically stable when $R_0 < 1$, and unstable when $R_0 > 1$.*

Optimal Control

Two controls: $u_1(t)$: the efficacy of the bed net usage, and $u_2(t)$: the rate of releasing sterile mosquitoes. Our goal is to determine an optimal control pair $(u_1^*(t), u_2^*(t))$ that minimizes the objective functional:

$$
J = \int_0^T w_1 I_h + \frac{1}{2}(w_2 u_1^2 + w_3 u_2^2)\ dt.
$$

The state system with two controls is given by

$$
\begin{aligned}
\frac{dS_h}{dt} &= \Lambda_h + \psi_h N_h + \rho_h R_h - (1 - u_1(t))\lambda_h(t) S_h - f_h(N_h)\ldots \\
\frac{dE_h}{dt} &= (1 - u_1(t))\lambda_h(t) S_h - \nu_h E_h - f_h(N_h) E_h, \\
\frac{dI_h}{dt} &= \nu_h E_h - \gamma_h I_h - f_h(N_h) I_h - \delta_h I_h, \\
\frac{dR_h}{dt} &= \gamma_h I_h - \rho_h R_h - f_h(N_h) R_h, \\
\frac{dS_v}{dt} &= \frac{\psi_v N_v^2}{N_v + S_g} - (1 - u_1(t))\lambda_v(t) S_v - f_v(N_v + S_g) S_v, \\
\frac{dE_v}{dt} &= (1 - u_1(t))\lambda_v(t) S_v - \nu_v E_v - f_v(N_v + S_g) E_v, \\
\frac{dI_v}{dt} &= \nu_v E_v - f_v(N_v + S_g) I_v, \\
\frac{dS_g}{dt} &= b u_2(t) - f_g(N_v + S_g) S_g.
\end{aligned}
$$

Characterization of the optimal controls:

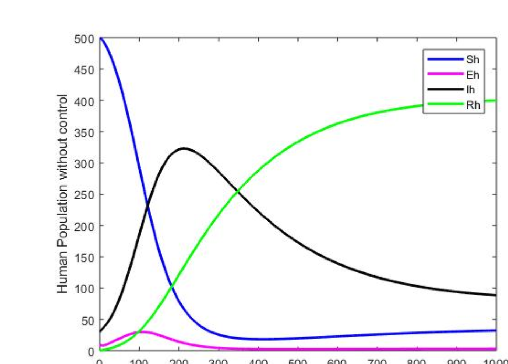
$$
u_1^* = \min\Big(\max(0, m), M\Big)
$$

$$
u_2^* = \min\Big(\max(0, -\hat{b}\lambda_8 / w_3), M\Big)
$$

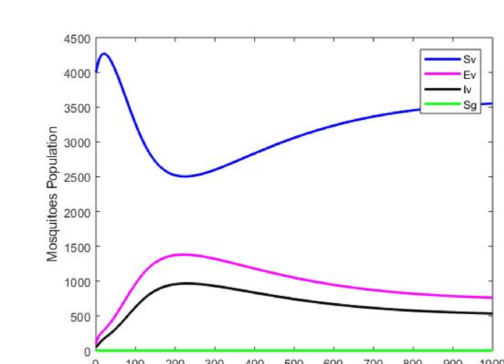
where

$$
m = \left((\lambda_2 - \lambda_1)\frac{\beta_{hv}\sigma_h\sigma_v I_v S_h}{\sigma_h N_h + \sigma_v N_v} + (\lambda_6 - \lambda_5)\frac{\sigma_h\sigma_v(\beta_{vh} I_h + \tilde{\beta}_{vh} R_h)}{\sigma_h N_h + \sigma_v N_v}\right)/w_2.
$$

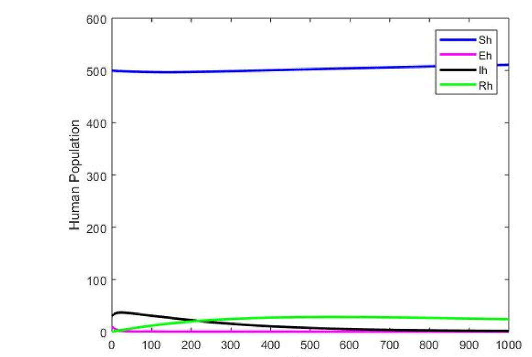
Numerical Simulations



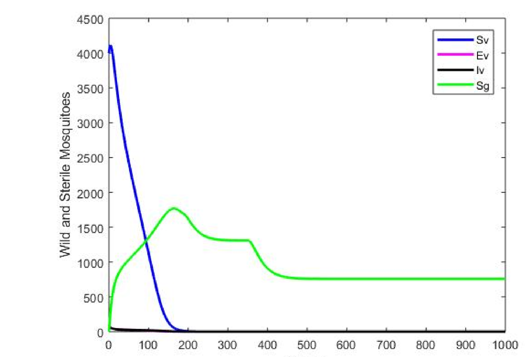
(a) Human Population w/o Control



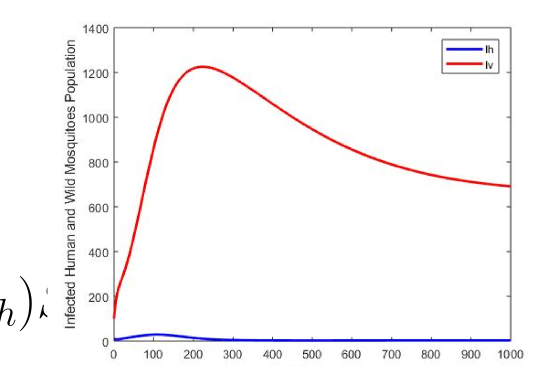
(b) Wild Mosquitoes Population w/o Control



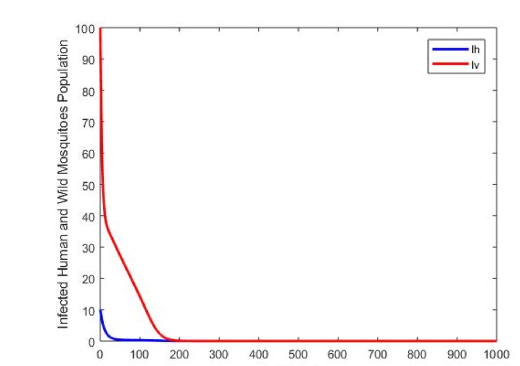
(c) Human Populations with 2 Controls



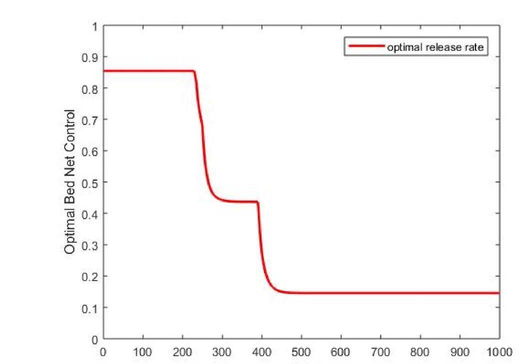
(d) Mosquitoes Populations with 2 Controls



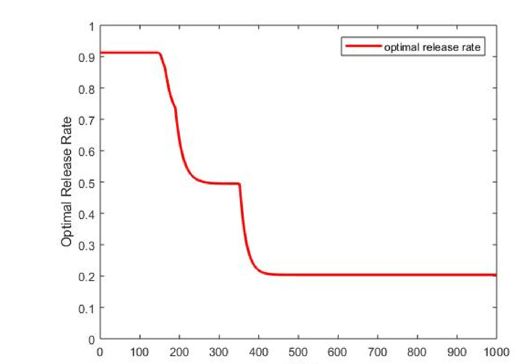
(e) Infected Human and Wild Mosquitoes: No control



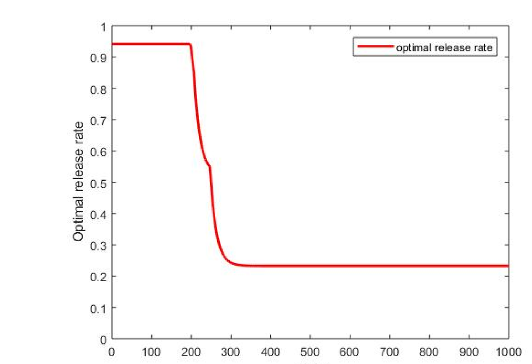
(f) Infected Human and Wild Mosquitoes: with 2 optimal controls



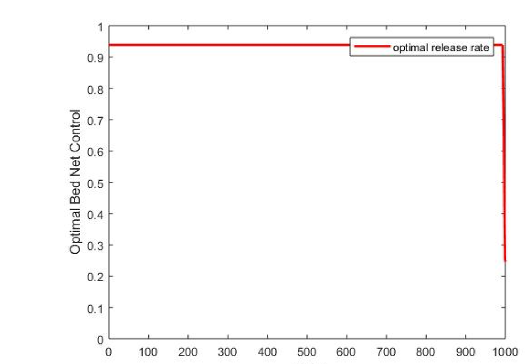
(g) Optimal Two Controls: (1) ITN



(h) Optimal Two Controls: (2) SIT



(i) Optimal Single Control: only SIT



(j) Optimal Single Control: only ITN

References

[1] L. Cai, S. Ai, J. Li, Dynamics of mosquitoes populations with different strategies of releasing sterile mosquitoes, SIAM, J. Appl. Math., 75 (2014), 1223-1237.

[2] N. Chitnis, J. M. Cushing, J. M. Hyman, Bifurcation analysis of a mathematical model for malaria transmission. SIAM, J. Appl. Math., 67 (2006), 24-45.

Acknowledgements

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