

# The Development and the Idea of Calculus

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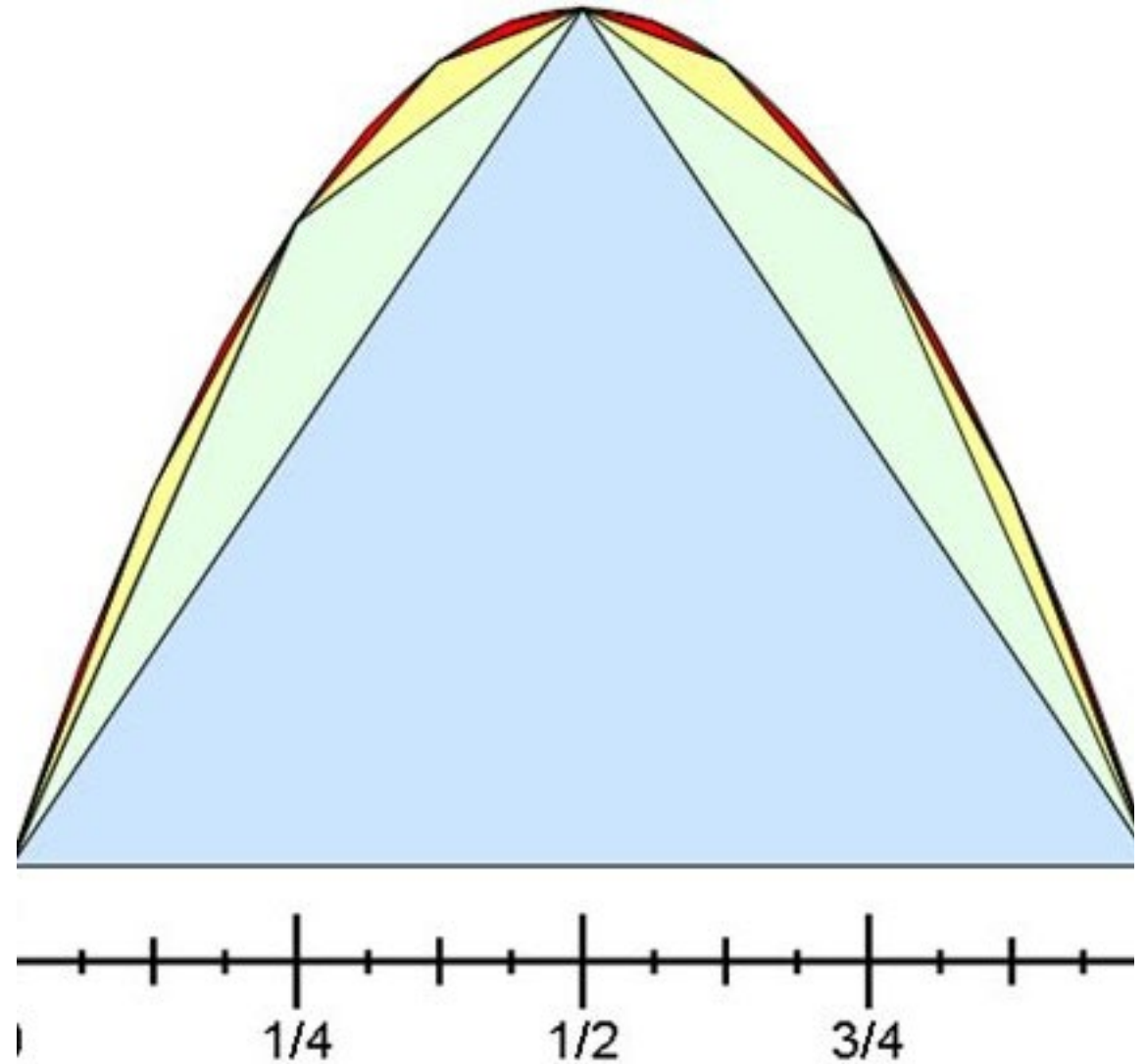
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Theory of Calculus

# The Area of a Parabola

There exist formulas for finding the area of specific shapes such as circles, squares, rectangles, triangles, etc. However, if someone wants to find the area of a randomly drawn curve how can this be done?

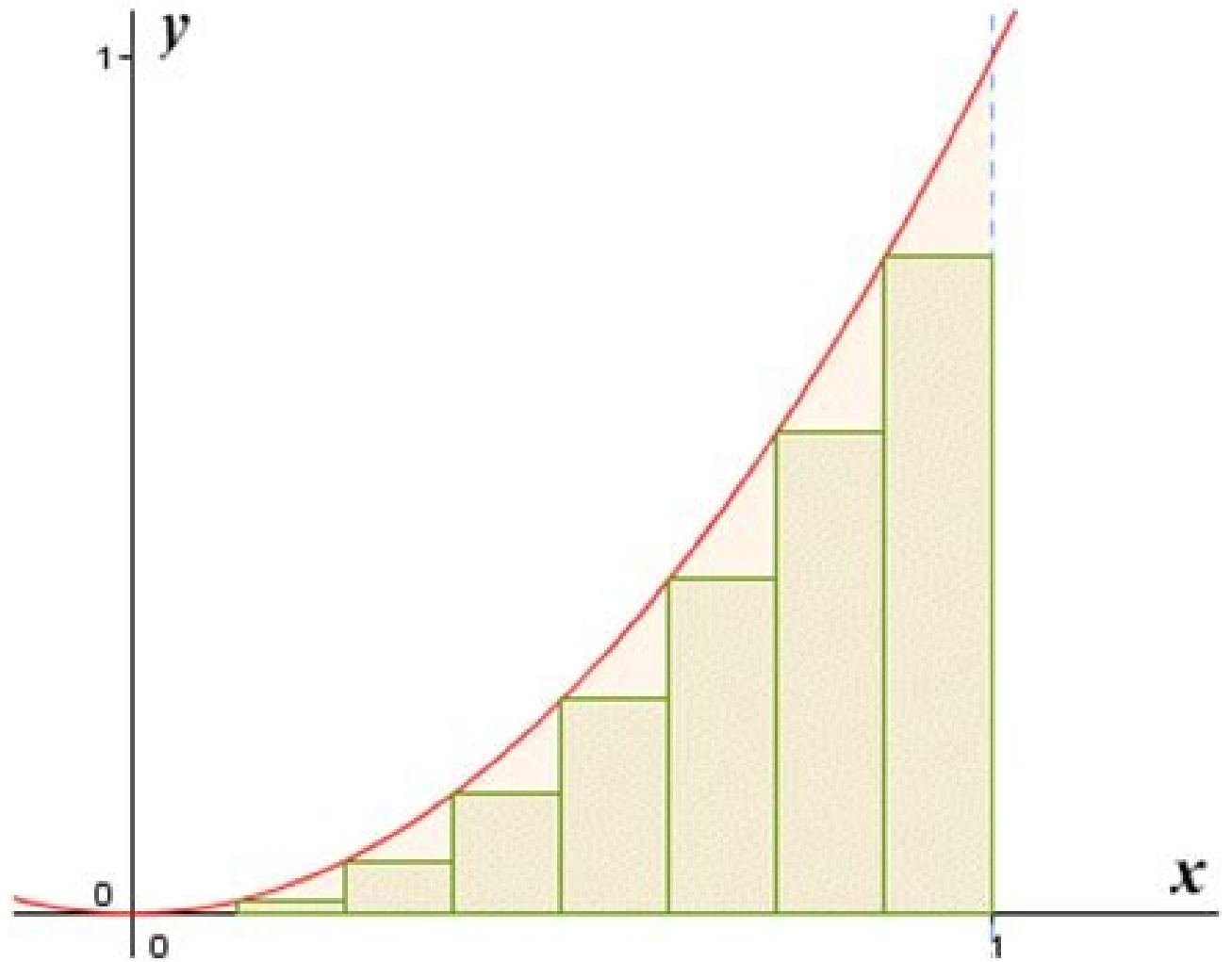
Many early mathematicians, including Archimedes, used known shapes such as triangles to fill the area under the curve and added their areas to approximate the area under the curve. Archimedes eventually found that the more triangles he added, the more the sum of the areas of the triangles approached  $\frac{4}{3}$  of the original triangle. Eventually leading Archimedes to officially discover what we call the “exhaustion method”.

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- Here we have an example of the area of a parabola being approximated by the exhaustion method.



# Rectangles for Approximation

- Approximately 1800 years after Archimedes' invention of the exhaustion method in which he used triangles; mathematicians started using rectangles in the exhaustion method to find the area under curves in the seventeenth century.



## Sum of an Area of Rectangles

- The use of rectangles in the exhaustion method allowed mathematicians to develop a formula for the sum of the area of  $n$  rectangles which allowed them to better approximate the area under a curve.

$$S = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

# Strictness and Rigor vs. Practicality

While the ancient Greeks were aware of the existing formulas for calculation under a curve, they chose to use the more logically rigorous and strict method of exhaustion for their calculations.

Ancient China, however, chose to use the existing formulas for their calculations of area due to the more practical purpose of their mathematics.

# Beginning of Integral Calculus

- The modern notation of an integral in which the differential “dx” is placed on the right-hand side of the expression can be interpreted as the increasingly small or “infinitesimal” base of each rectangle. For this reason, before calculus was completed, the logic behind integration was attacked for being non-rigorous.

$$\int x^2 dx$$

# Beginning of Differential Calculus

- While integral calculus can be understood by taking an integral to simply mean “the area under a curve”, differential calculus hinges on the idea of the “derivative”.
- A derivative can most easily be understood as the slope of the tangent line to a curve at a specific point.

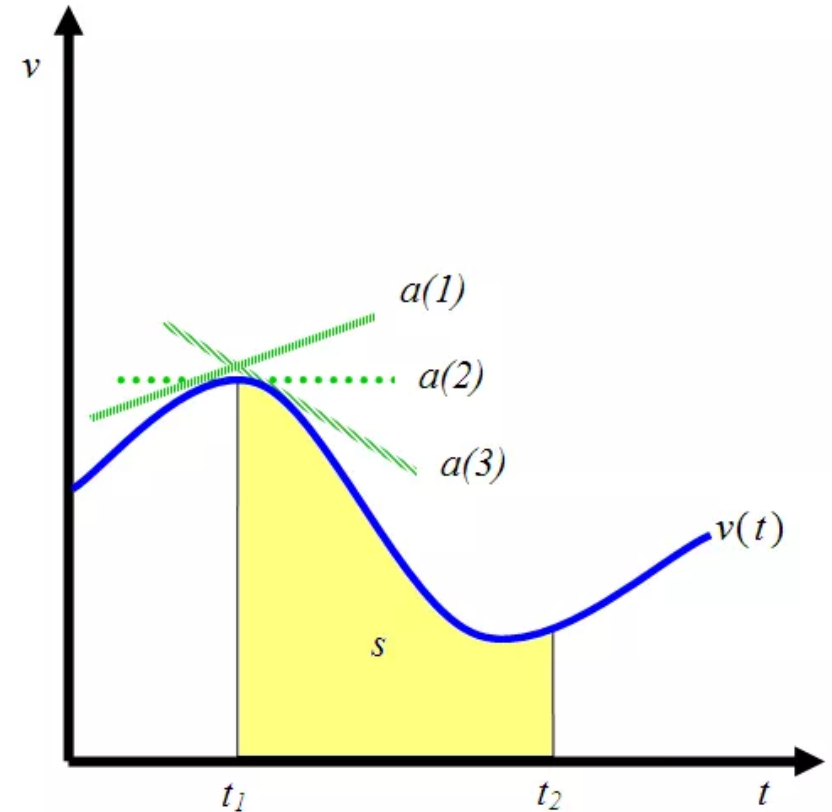


# Reciprocal Operations

- Newton and Leibniz are known as the independent inventors of Calculus.
- One thing of significance about the work of Newton and Leibniz is the fact that they discovered that Differentiation and Integration are reciprocal operations. Therefore, if we integrate a function, we can differentiate its result to get back the original function and vice-versa.

# XY to VT- Calculus in Physics

- While Calculus is known as a topic in mathematics, which it is, Calculus has many very useful applications in physics.
- The v-t image is a type of graph which uses velocity( $v$ ) and time( $t$ ), rather than  $x$  and  $y$  to analyze the movement of an object.
- Integral Calculus can be used to find the area  $S$  which is the magnitude of the displacement of the object.



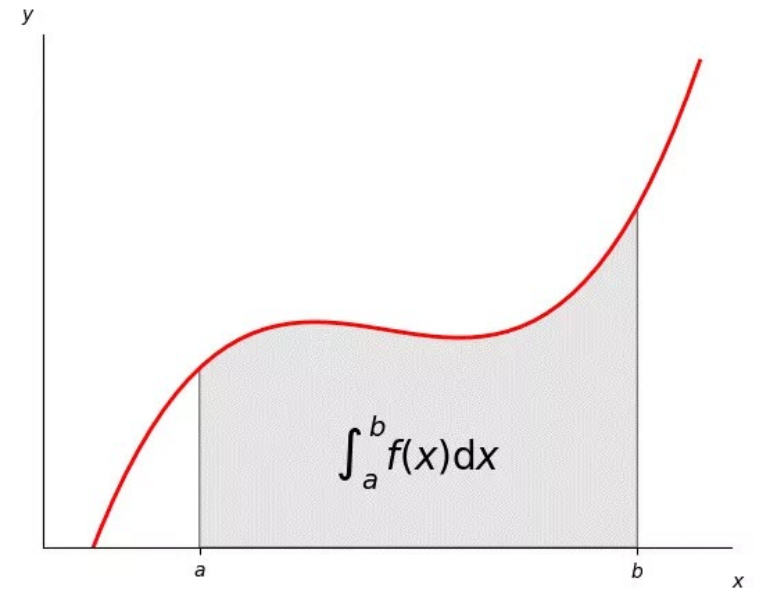
# Anti-differentiation in Place of Integration

- When Integration of a particular function to find the area under the curve proves too challenging, one may use anti-differentiation in its place to more easily achieve the same result. Resulting in a + c on the end of the result, what we know as the “constant of integration”.
- This method takes its place in the advancement of mathematics, as well in applications to physics and other sciences which require the use of Calculus.

# The Fundamental Theorem of Calculus

- Simply put, if the function  $f(x)$  is continuous, then you can calculate  $F(x)$  by the equation on the right.
- This is the Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$





# Objections to Calculus

- One objection to the early foundation of Calculus is Leibniz's use of infinitesimals. If we think of an infinitesimal as something which is close to zero, yet not equal to zero then why can it sometimes be treated as a nonzero entity, and sometimes be discarded as an entity equivalent to zero?
  - These kinds of questions were often ignored by many of the great mathematicians throughout history such as Euler, Lagrange, Laplace, and the Bernoulli brothers.
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# Cauchy

- Cauchy created the following definition of a limit: “when the successive values of a variable approach a certain fixed value infinitely, if the difference between it and this fixed value can be arbitrarily small, then this fixed value is called its limit”.
- This definition allowed for infinitesimal quantities to be considered as simply quantities with a limit of zero. Hence, the objection to the logic behind calculus concerning infinitesimal quantities is successfully repudiated.

# Weierstrass-the Father of Modern Analysis

- Weierstrass improved on Cauchy's definition of a limit with his  $\epsilon$ - $\delta$  limit definition.

$$\left( \lim_{x \rightarrow a} f(x) = L \right) \Leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in D) \\ (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

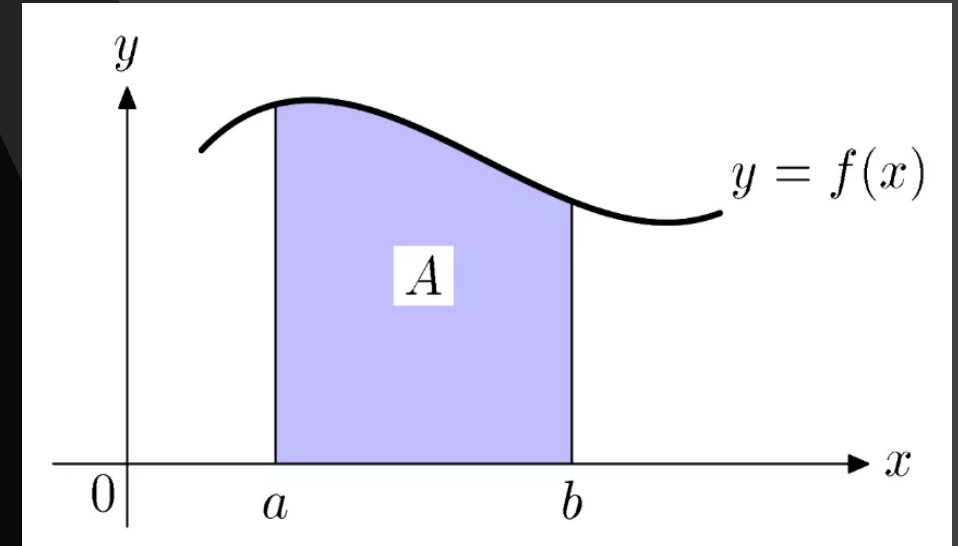
# Weierstrass Limit Continued

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- Weierstrass's definition of a limit solved any remaining problems concerning the ambiguity of infinitesimal quantities.
  - It doesn't matter how  $L$  is approached, so long that it is approached such that the final difference is smaller than epsilon.
  - With this definition of a limit, Calculus's foundation was able to be successfully rebuilt.



# Reconstruction of the Integral

- With this new limit definition, the integral became a limit with a rigorous and logical base. No longer dependent on the infinitesimal quantity which was objected to, ambiguity in integration disappeared. Here  $A$  is a limit which can be understood as the limit of the area of the sum of  $n$  rectangles.
- What is lost here in intuition, is gained in rigor and logic.



$$A = \int_a^b f(x) dx$$

# Reconstruction of the Derivative

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- With the Weierstrass definition of a limit, derivatives are no longer dependent on the ratio of infinitesimal quantities (dy/dx) but can be regarded as a limit.
- This is due to a tangent being regarded as the limit of the secant such that the slope of the tangent line is the limit of the slope of the secant, therefore the derivative becomes a limit itself.

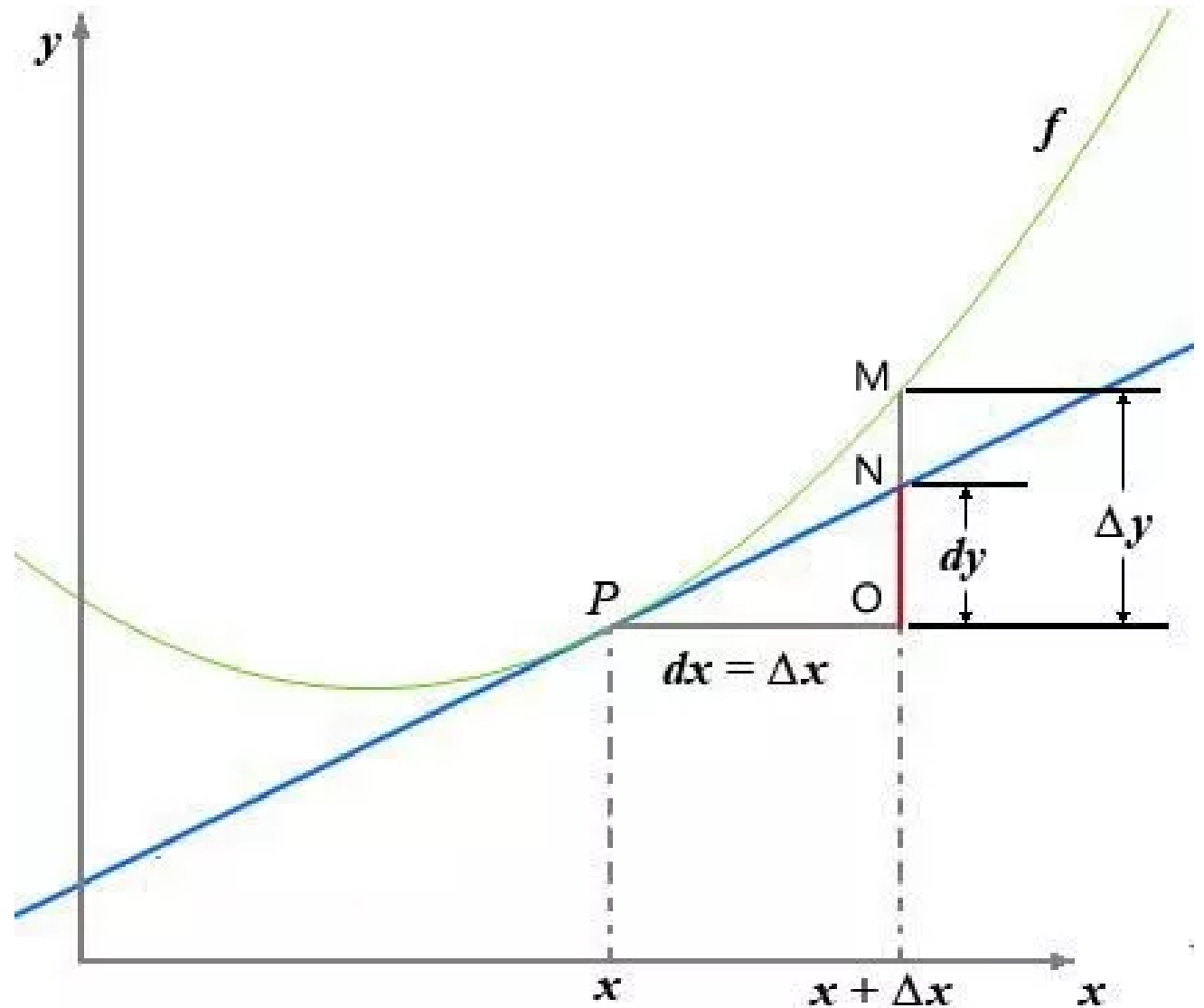
## Reconstruction of Differentials

- The traditional Leibniz notation for differentials ( $dy/dx$ ) is still used, but now we use limits to define the differential rather than simply letting differentials be infinitesimals.

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

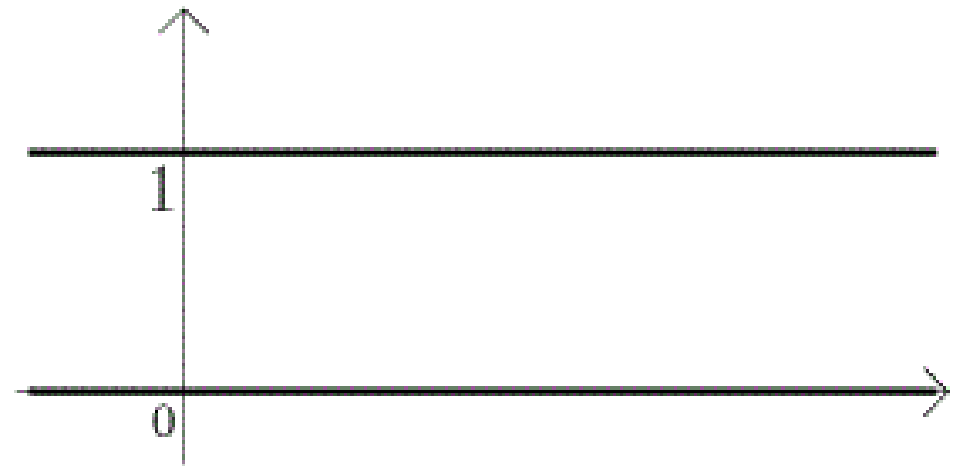
## Reconstruction of Differentials

- Essentially, the differential  $dx$  represents the change on the  $x$ -axis, and the differential  $dy$  represents the amount of change on the  $y$ -axis.



# Lebesgue

- With calculus, while many functions were easily able to be integrated, functions such as the Dirichlet function were not so easily integrable.
- In order to properly determine if a function was integrable, the great mathematician Henri Lebesgue devised the Lebesgue integral which extended the integral to a larger class of functions.



$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

# Sources

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- Mellor, Joseph. “What Is a Limit, Really?” Medium, Cantor's Paradise, 29 Oct. 2021, <https://www.cantorsparadise.com/what-is-a-limit-really-ee1c256c6544>.