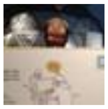
How can you best explain divergence and curl?

How can you best explain the divergence and curl? What is their significance? What are their real world applications and examples?

6 Answers



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tl;dr

You and three friends float down a river, each marking a corner of a square. If your square is getting bigger, the river has positive divergence. If it's shrinking, negative divergence.

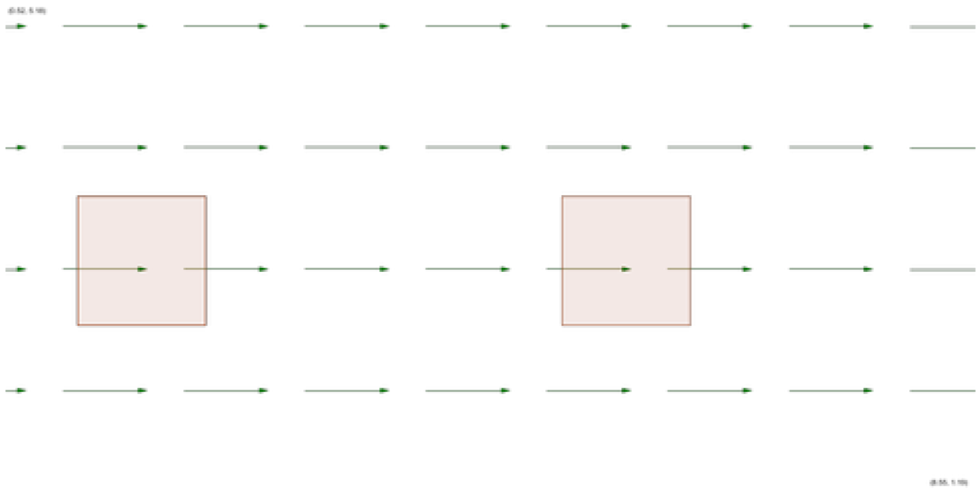
Next, you and your friends are rigidly connected so your square can't change shape. If the square starts rotating like a frisbee as it goes along, the river has curl. Positive curl is counterclockwise rotation. Negative curl is clockwise.

This answer assumes a good knowledge of calculus, including partial derivatives, vectors, and the way we talk about these things in an introductory calculus-based physics course. I'll also assume you know the kinematics of rotations and how to approximate multi-variable functions about an arbitrary point with the first-order terms of its Taylor series.

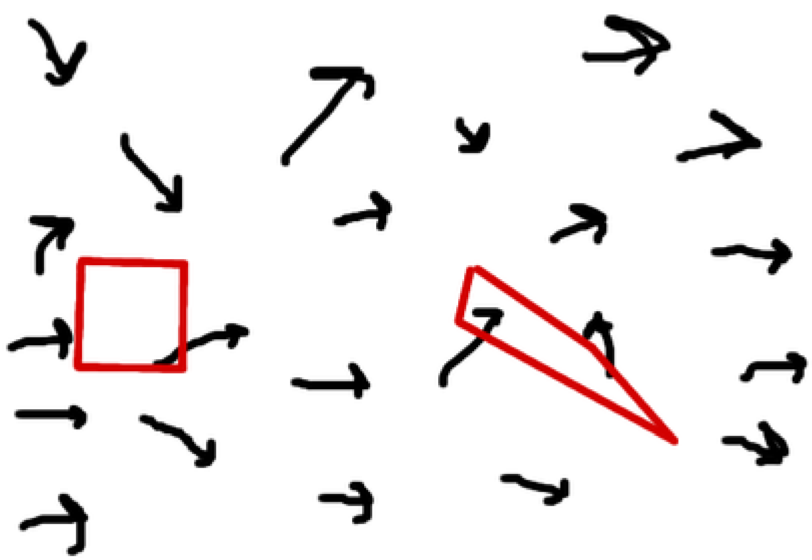
Tubing Down a River

Most students learn the divergence and curl because they're important in Maxwell's equations of electrodynamics. These concepts apply to any vector field, though. Here, let's just visualize you and some friends floating down a river on inner tubes. The vector field is the field giving the velocity of the river's flow. The divergence and curl describe what happens to you and your friends as you float down the river together.

Suppose you are tubing down a river with three friends. You position yourselves into a square formation. Assuming the river flows perfectly-evenly, you'll all float along together and stay in that perfect square.



However, if the river moves faster in some places than others, some of you might get ahead or fall behind. Maybe the flow picks up sideways components and you drift apart. Your square gets messed up.



The black arrows show some sort of messy, complicated flow. The red square that is your initial positions distorts into the pointed shape.

Mathematically, we will treat a person in an inner tube as a point moving in two dimensions. The flow of the river is a vector field - a vector for the velocity of the flow at each point. We imagine that people flow perfectly down the river; their velocity always matches the vector field exactly. (In real life, this is not exactly how it happens; the water can move underneath you.) Also, the people themselves are just test particles - bits of dust floating on the surface that don't affect the river's flow in any way. Because we always know their velocity, we can integrate their motion forward and find their trajectories.

Finally, to avoid complications, imagine that the vector field describing the river's flow is constant in time. Different parts can flow at different speeds (so you could speed up or slow down as you tube along), but the water passing by some particular fixed point always has the same speed and direction.

We'll denote the flow's velocity vector as a function of position as

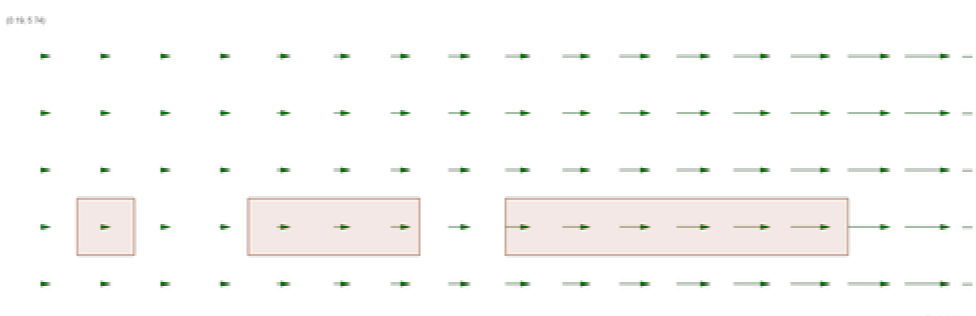
$$\mathbf{V}(x,y)=V_x(x,y)x+V_y(x,y)y$$

V_x and V_y are real-valued functions of two variables. x and y are unit vectors in the x and y directions. To save space, I'll sometimes just write $\mathbf{V}$ instead of $\mathbf{V}(x,y)$, so just keep in mind that it's a function of position.

Area and Divergence

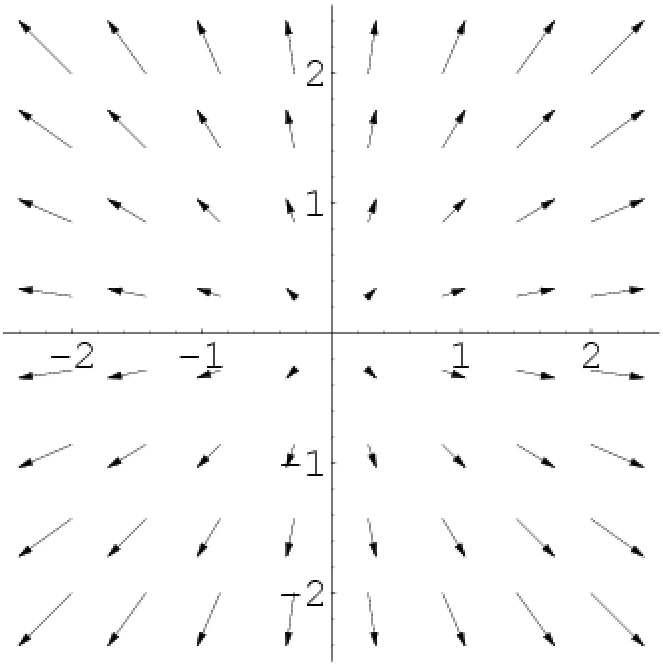
Let's take the shape made by you and your three friends and call that your span. Originally, your span is a square, but as you go along it may distort. Divergence describes how fast the area of your span is changing.

For example, imagine that the river gets faster and faster the further you go downstream. Then your friends in front of you will keep getting further and further ahead, and your span stretches out. This is an example of a positive divergence. This is illustrated below. The shape of the span is accurately calculated for this particular vector field.



On the other hand, if places further up move more slowly so you scrunch in closer to your friends, that's negative divergence; the area of your span goes down.

If you are all floating apart from each other, that's positive divergence.

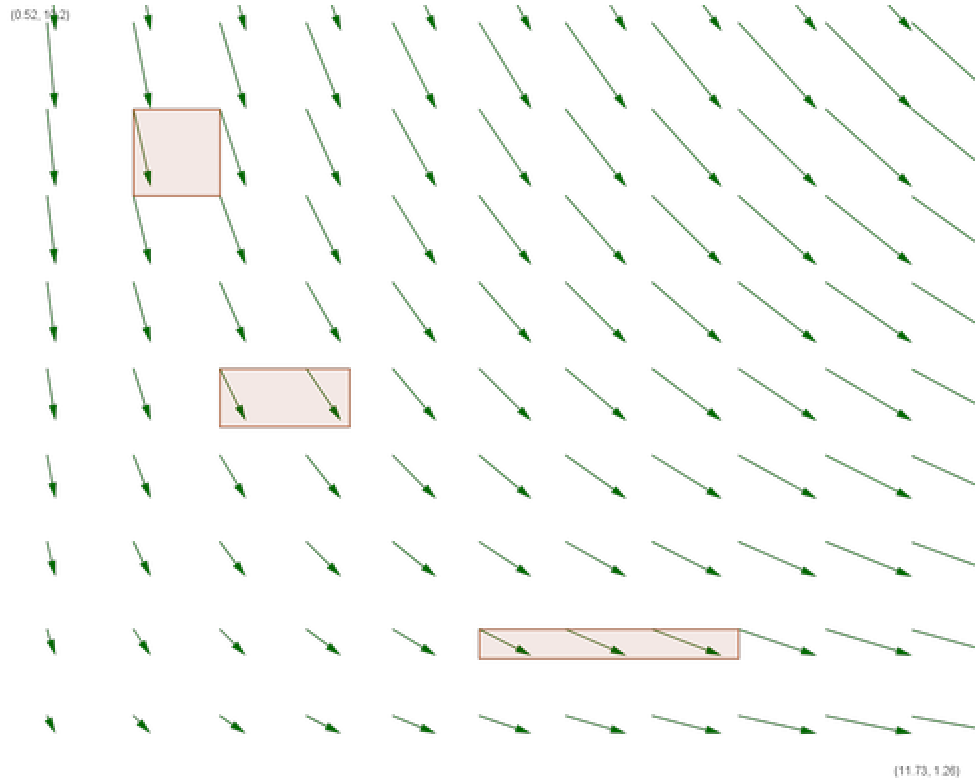


source: The idea of the divergence of a vector field

In this vector field, your square would get bigger in all directions at once.

The opposite, where all the arrows come pointing in, would be negative divergence.

If the river is simultaneously getting faster and narrower, your square gets longer and skinnier. If it does it just right, the "getting longer" part could cancel the "getting skinnier" part so that the area stays the same. This would be zero divergence. Here's an example I constructed:



Preliminary Definition of Divergence

We would like to start calculating the divergence. It seems like we should get something like "such-and-such many square meters per second" for our answer, because the divergence has to do with the rate that an area is changing.

This has a problem, though, because the number will be different if we start with a tiny span than if we start with a large span. We want the divergence to refer to a property of the mathematical vector field. It captures the "spreading apart" nature of the vectors themselves. Its value shouldn't depend on some particular span size we choose.

The solution is to use the percentage area gained per second, which is the same for large and small spans. This could also be called the fractional rate of change. The fractional rate of change of the area is the rate of change of the area in absolute terms divided by the area itself. This our definition of the divergence, which we will denote by Div

$$\mathrm{Div}(\mathbf{V}(\mathbf{x},\mathbf{y})) = \frac{\text{rate of change of area of span}}{\text{area of span}}$$

The units of the area cancel out, and we're left with units of inverse seconds, as is normal for rates.

If the divergence is $2\mathrm{s}^{-1}$, then the span is adding area at a rate of two times its own area once per second. So it would take about 0.005 seconds to increase its area by 1%.

(Side note: if you understand calculus well, you can see that this means that after one second, the span would be e^2 times as large as it originally was.)

We can now calculate a divergence. Let's imagine the river's flow is described by

$$V(x,y)=0.04x\mathrm{s}^{-1}x$$

This is saying that the flow is all in the x-direction and that it gets faster and faster as you go further along in that direction. If you went out 100 meters, the flow rate would be $0.04*100\mathrm{ms}^{-1}=4\mathrm{m/s}$.

You're 100m out, going 4m/s. Let's say you have a span that's a square 1m on a side. Your two friends in front of you are 1m ahead, so they're going 4.04m/s. That means they're pulling away from you at 0.04m/s. That means the square is growing at a rate of $0.04\mathrm{m}^2/\mathrm{s}$.

If you didn't follow where all the numbers came from in the previous paragraph, you should go back and work them out. Putting it together, the divergence is

$$\mathrm{Div}(\mathbf{V})=\frac{0.04\mathrm{m}^2/\mathrm{s}}{1\mathrm{m}^2}=0.04\mathrm{s}^{-1}$$

You can repeat this calculation putting yourself at any position and using any size span (and rotating it if you want). You'll get the same answer.

If you want to test yourself, suppose the flow is given by

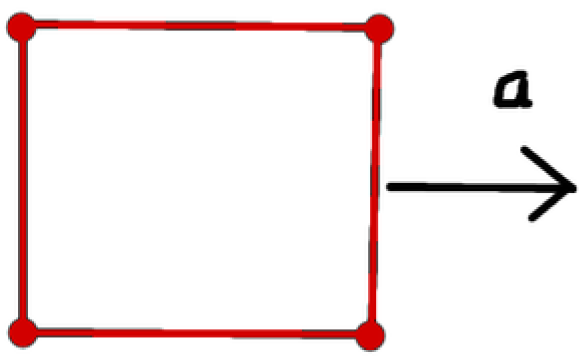
$$V(x,y)=(ax+by+e)x+(cx+dy+f)y$$

Give yourself a starting span somewhere with a certain size. Figure out the rate that its area is changing, and find the divergence. You should find that the divergence is

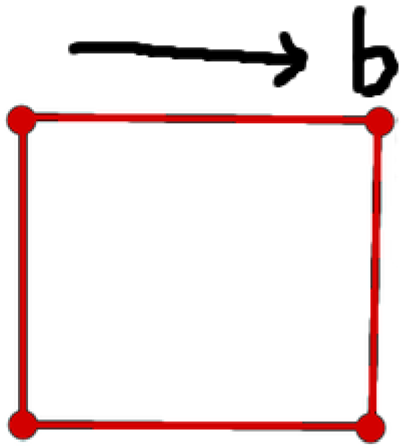
$\text{Div}(\mathbf{V}) = a + d$

This result actually makes intuitive sense. e and f are constants added to the velocity. They just move the entire square along without changing its shape, so they don't cause any divergence. We're only concerned with the relative motion between the corners. Therefore, we can simply take the bottom-left corner and consider it to be fixed.

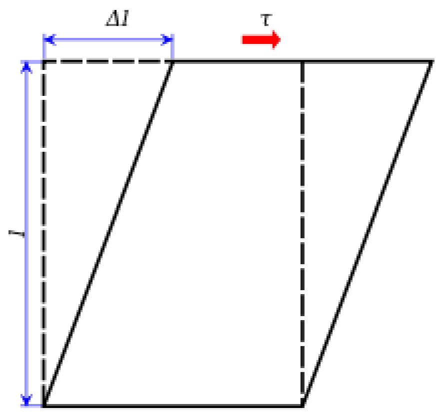
a represents how fast the right hand side of the box is moving out. This adds to the box's area, and so is part of the divergence.



b represents how fast the top part moves to the right.



This causes your square to shear, turning it into a parallelogram, like this:



source: Shear stress

It still has the same base and height, so the area doesn't change. That's why *b*doesn't come into the formula.

The same reasoning applies to *c* (the right hand side sliding up and down, more shear) and *d* (moving the top upward, adding to the area, and affecting the divergence).

Calculus

For the simple example I gave, the divergence is the same no matter where you put your span. However, in a complicated flow, the flow will be doing different things at different places. It may be coming together (negative divergence) in some places and moving apart (positive divergence) in others. Evidently, the divergence needs to be a function of x and y.

This presents a problem, because now the size of the span is going to make a difference. If the divergence is different from spot to spot, then it's different at different spots inside your span, but we're just trying to get a single correct answer.

This is very similar to the problem of finding the slope of a line in calculus. The slope of the line changes from point to point, so if you calculate $\frac{rise}{run}$ between two points, you keep getting different answers depending on how far apart the points are. Slope shouldn't depend on how far apart the points you choose are, and divergence shouldn't depend on how large an original span you choose.

The solution to both problems is the same - we take a limit. In calculus, you take the limit as the distance between your two points on the function goes to zero. In vector calculus, we take the limit as the size of our starting span goes to zero.

In single-variable calculus you'll get the right answer for the slope using a finite-size interval as long as you're measuring the slope of a straight line. That's essentially why the examples I gave in the previous section work; the vector field I gave is the equivalent of a straight line.

Expression for the Divergence

When you zoom in close to a function, it looks locally like a straight line. Likewise, when you zoom in close to a vector field, it looks locally like our equivalent to a straight line, for which we already calculated the divergence.

To be specific, a function of one variable looks locally like

$$f(x) \approx f(x_0) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}(x - x_0)$$

remember that our vector field is

$$\mathbf{V}(x,y) = V_x(x,y)x + V_y(x,y)y$$

We can expand the functions V_x and V_y the same way we expand single-variable functions, so long as we use two partial derivatives.

$$V_x \approx V_x(x_0,y_0) + \frac{\partial V_x}{\partial x}\bigg|_{x_0,y_0}(x-x_0) + \frac{\partial V_x}{\partial y}\bigg|_{x_0,y_0}(y-y_0)$$

and likewise for V_y. Plugging these zoomed-in functions for V_x and V_y into $\mathbf{V}$gives

$$\mathbf{V}(x,y) = \mathbf{V}(x_0,y_0) +$$

$$\left(\frac{\partial V_x}{\partial x}\bigg|_{x_0,y_0}(x-x_0) + \frac{\partial V_x}{\partial y}\bigg|_{x_0,y_0}(y-y_0)\right)\hat{x} +$$

$$\left(\frac{\partial V_y}{\partial x}\bigg|_{x_0,y_0}(x-x_0) + \frac{\partial V_y}{\partial y}\bigg|_{x_0,y_0}(y-y_0)\right)\hat{y}$$

This is the same as our earlier vector field, with partial derivatives standing in for a,b,c,d, and the initial value $\mathbf{V}(x_0,y_0)$ standing in for e and f, so by analogy we can say that the divergence is

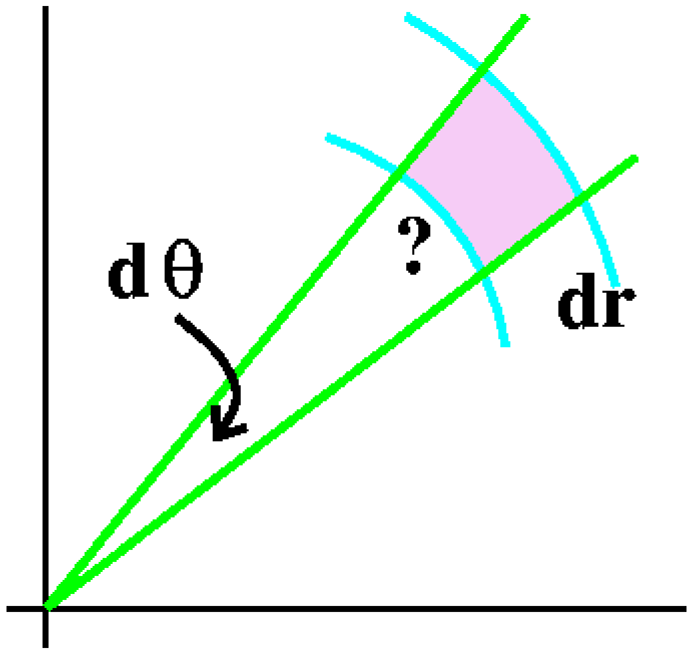
$$\mathrm{Div}(V) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}$$

This is the divergence in two dimensions. Remember that it's a function of $x$and y.

A good test of whether you understand the divergence is to find its formula in polar coordinates when the vector field is

$$\mathbf{V}(r,\theta) = V_r(r,\theta)\hat{r} + V_\theta(r,\theta)\hat{\theta}$$

You can do this in exactly the same way - draw a span, find how fast its area changes, and divide by the area. The square is a little awkward here; use a span that looks like this:



source: Math 251 diary, fall 2010

The answer is

$$\text{Div}(\mathbf{V}) = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_r}{\partial r} + \frac{V_r}{r}$$

Continuity Equation

Suppose that the flow of water in the river is two dimensional, meaning no water moves upwards or downwards. If there were positive divergence, that would mean the water was taking up more and more space. But that in turn means that it would be thinning out, getting less dense. In turn, if the were negative divergence, it would mean all the water in some area was getting smashed together to become more dense.

For water, this is essentially impossible. The pressure it takes to compress water appreciably is too high to reach in a river. You similarly can't thin it out; the water would just collapse. This means that for flowing water the divergence is zero!

More generally, if the water is diverging at 1% per second, that means the same mass is getting spread over 1% more volume, so the density is going down by 1% per second. This relationship between the divergence and density is called the continuity equation. If the density is ρ, then

$$\frac{d\rho}{dt} = -\text{Div}(\rho\mathbf{V})$$

This equation works because mass is conserved. You'll see a similar equation for any conservation law. For example, starting from Maxwell's equations, you can derive a continuity equation like this for electric charge to show that charge is conserved. Starting from Schrodinger's equation in quantum mechanics you can derive a continuity equation to show that probability is conserved - the total probability to find a particle anywhere always adds up to 100%.

Three Dimensions

In three dimensions, everything works the same way. You can think of eight dust particles forming a cube that drifts through the air in your room. The fractional rate of change of the volume of the cube is the divergence in three dimensions. It is

$$\text{Div}(\mathbf{V}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

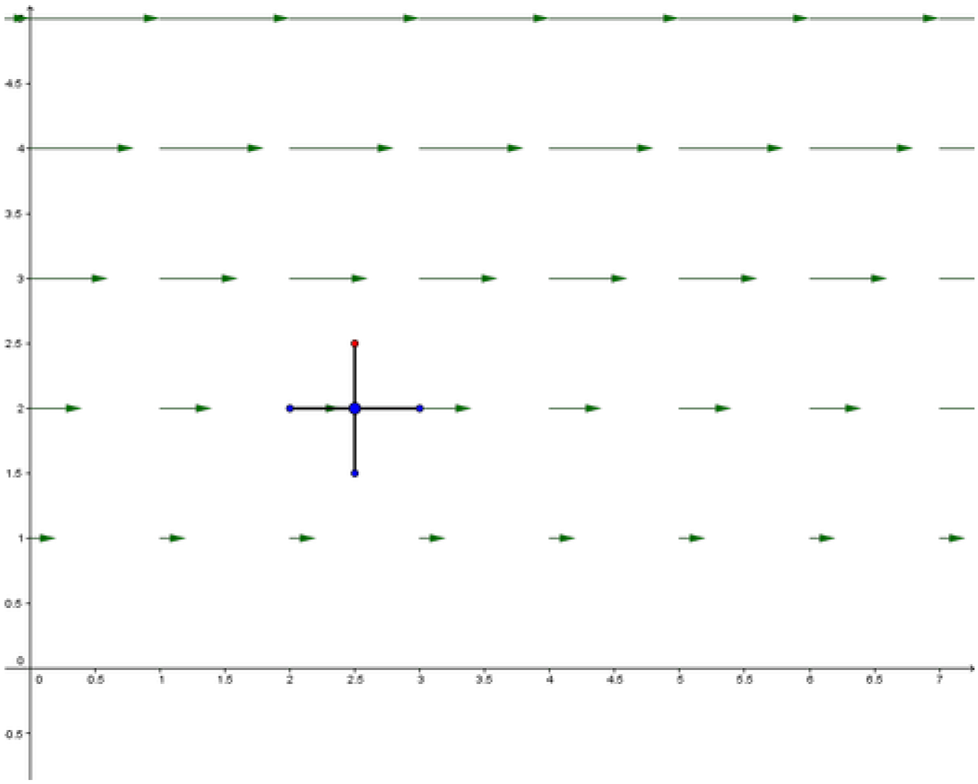
Eulerian vs Lagrangian Pictures

What I've described above might be called the "Lagrangian picture" of the divergence. We imagine following along as you float down the river and talk about what happens to you.

For one reason or another, it is more popular to describe the divergence in the "Eulerian picture", in which we imagine staying fixed in one spot and looking at properties of the flow right there. Ultimately, there is no difference between the two, but you might hear people describe the divergence in a way that sounds superficially different. It's actually the same. You can learn that picture if you want, or stick with this one.

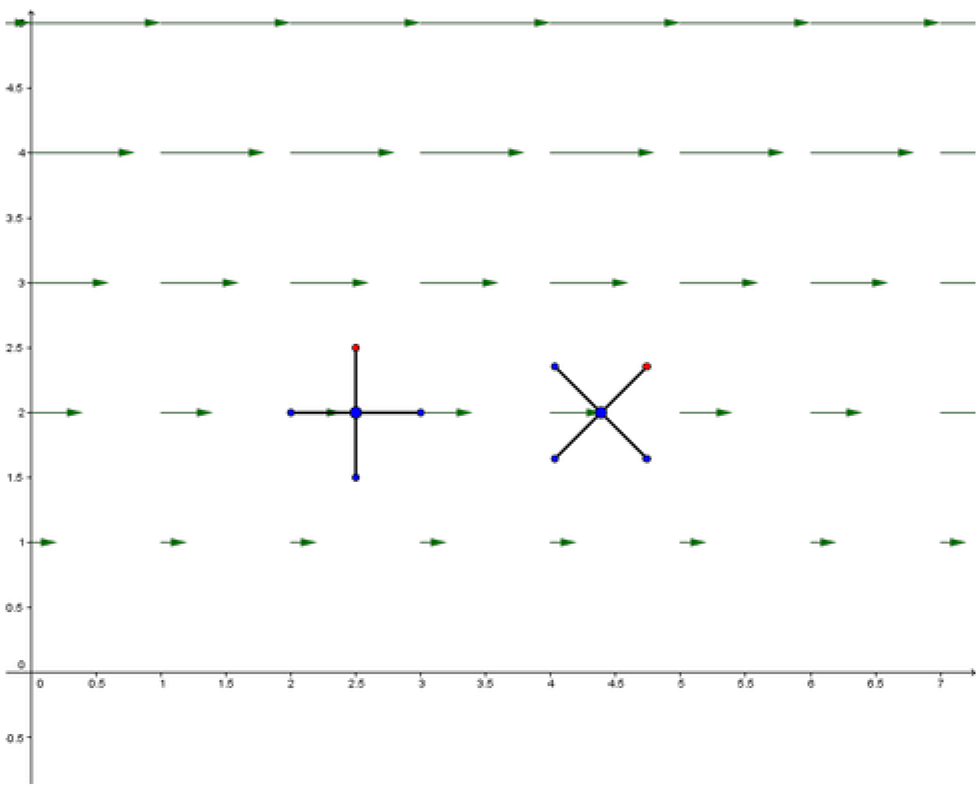
Curl

We now imagine a device made of rods and floaters like this one floating down the river:



It is basically the same thing as before, except rigid. You and your three friends have been replaced by floaters at the end of the rods, and there's a central hub.

If the device spins, the flow has curl. For example, in this flow, the top part of the device is in faster water and so gets ahead. The bottom part lags behind, and all together this causes spin.



If the flow is perfectly uniform, the device won't spin and there's no curl.

In detail, the way we imagine this device working is that for the top/bottom floaters, what matters is the horizontal component of the flow velocity. The vertical component of flow velocity can only push or pull against the vertical rods, and so has no effect. For the left/right floaters, what matters is the vertical components of the flow rate (zero in the above picture).

Let's say the rods have length l. The point (x_0,y_0) is at the center of the device. The velocity in the x-direction at the top floater is

$$V_x \approx V_x(x_0,y_0) + \left.\frac{\partial V_x}{\partial y}\right|_{x_0,y_0} l$$

On the other hand, if the device is moving along as a whole at $\mathbf{V}(x_0,y_0)$ and simultaneously rotating at angular rate ω, the velocity of the top point would be

$$V_x \approx V_x(x_0,y_0) + \omega l$$

So it looks like the thing wants to rotate at a rate

$$\omega = \frac{\partial V_x}{\partial y}$$

We get the same answer if we look at the bottom floater.

If we look at the floater on the right and do the same analysis, it wants the device to rotate at

$$\omega = -\frac{\partial V_y}{\partial x}$$

(You should draw this out to picture it.)

The device can only have one rotation speed, which we will say is the average of the speed the top/bottom floaters want to go and speed the left/right floaters want to go. The curl is then defined to be

$$\text{Curl}(\mathbf{V}) = \frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}$$

It's twice the rotation rate. There's no special reason to make it twice as fast. It just works out better that way when you think of some other possible ways to define the curl. Those other ways are equivalent; you can read about them anywhere. This is just the picture I have.

Three Dimensions

What we found above is the z-component of the curl. That's because the thing is spinning around the z-axis. (The z-axis comes up out of your monitor.)

Imagine putting a toy jack-shaped device with six rods and floaters in the water, one in each direction in 3D. It would spin just the same, but the two extra floaters would contribute some extra torque. The device picks up some spin around the x-axis and y-axis in addition to the familiar z-axis spin.

Adding up the spins around each axis (which you can do because angular velocities are vectors) you find that the entire jack is spinning around some particular axis in 3D. This axis is the direction of the curl in three dimensions. So the direction of the curl is just the direction of the axis around which the device spins. The magnitude of the curl is still how fast it spins.

Further Reading

Div, Grad, Curl, and All That, Schey (An Informal Text on Vector Calculus (Fourth Edition): H. M. Schey: 9780393925166: Amazon.com: Books)

If you have some time to answer a few questions about the effectiveness of this answer, please check out Dan and I Diverge by Mark Eichenlaub on Painting the Cathedral

Related Questions

- What is the actual meaning of divergence and curl in Maxwell's equations?
- What is an intuitive, not heavily technical way, to explain the meaning of divergence, curls, Green's theorem, Stoke's theorem, and Maxwell's ...
- What is the physical meaning of divergence, curl and gradient of a vector field?
- What kind of logic is used in Green's, Stokes and Divergence theorems? Can someone provide the best analogy to understand the logic?
- What is the difference between a curl, divergence and a gradient of a function? Along with their physical significance.