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OPTIMAL HARVESTING OF A SPATIALLY EXPLICIT FISHERY MODEL

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ABSTRACT. We consider an optimal fishery harvesting problem using a spatially explicit model with a semilinear elliptic PDE, Dirichlet boundary conditions, and logistic population growth. We consider two objective functionals: maximizing the yield and minimizing the cost or the variation in the fishing effort (control). Existence, necessary conditions, and uniqueness for the optimal harvesting control for both cases are established. Results for maximizing the yield with Neumann (no-flux) boundary conditions are also given. The optimal control when minimizing the variation is characterized by a variational inequality instead of the usual algebraic characterization, which involves the solutions of an optimality system of nonlinear elliptic partial differential equations. Numerical examples are given to illustrate the results.

KEY WORDS: Optimal fishery harvesting, fisheries management, elliptic partial differential equations, variational inequality.

1. Introduction. There is an ongoingdebate on the benefits of marine reserves (regions of no-harvest) in the management of fisheries. There are several recent issues of *Natural Resource Modeling Journal* devoted to fishery management, and we call attention to the survey paper of Quinn II [2003]. He traced the development of fisheries models from 1900 to the 21st century. He pointed out future fishery models will need to deal better with habitat and spatial concerns and to understand the effects of harvesting on the ecosystem. Managers have begun to use spatial management instruments, in the form of both permanent and temporary closures. Optimal control theory can be used to design

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optimal harvesting strategies including temporal and spatial features. Our work shows that under certain conditions, marine reserves are part of the optimal strategy.

Bioeconomic features and optimization of yield in harvesting models were originally treated using ordinary differential equations (ODEs) with time as the underlying variable. Clark's [1985, 1990] books provide a foundation to use optimal control theory as a useful tool for fisheries management. See also Walters and Martell [2004] for an intensive discussion of fisheries-stock assessment and management.

Spatial features were introduced with space as a discrete variable, in metapopulation models with ODEs. Gordon [1954] recognized the misallocation of harvest effort in space in a simple model of two spatially isolated fishing grounds. One of his conclusions was that the specification of the spatial distribution of harvest effort may yield significant benefits. Tuck and Possingham [2000] used coupled spatially explicit difference equations to model the populations of a single-species 2-patch metapopulation. They considered the problem of optimally exploiting the single-species local population that is connected to an unharvested second local population through the dispersal of larvae. They applied dynamic optimization techniques to determine policies for harvesting the exploited patch by deriving an equation that implicitly defines the optimal equilibrium escapement for the harvested stock. They also considered how a reserve affects yield and spawning stock abundance when compared to policies that have not recognized the spatial structure of the metapopulation. Comparisons of harvest strategies between an exploited metapopulation with and without a harvest refuge were also made.

Sanchirico and Wilen [1999, 2001] studied a collection of discrete substocks connected by dispersal and explored the relationship between efficient spatial exploitation and dynamics of the resource stock. Under different access scenarios, they explored the impacts of a reserve on stock abundance and effort distribution (Sanchirico and Wilen [2001, 2002]). Recently, results have begun to show that the use of reserves may be a part of the optimal harvesting strategy. Reserves may increase efficiency for a variety of reasons (Brown and Roughgarden [1997], Sanchirico and Wilen [2005], Herrera [2007]) because they may react to the spatial dynamics of the resource, and different patterns of "no harvest zones" may result in lower costs of enforcing closures relative to positive harvest quotas.

In contrast to these discrete space models, Neubert [2003] considered a resource existing in a continuous, finite one-dimensional spatial domain with logistic growth and continuous diffusion. He assumed Dirichlet boundary conditions (zero stock at the ends of the habitat, depicting a situation in which the habitat everywhere outside the spatial domain in question cannot support the resource) and solved for the spatial distribution of fishing effort that maximizes the yield at the steady state in which no reserves are imposed *a priori.* After rescaling the variables, he used the model:

$$
\begin{aligned}
&-\frac{d^2u}{dx^2} = u(1-u) - h(x)u, \ \ 0 < x < l, \\
&u(0) = u(l) = 0,
\end{aligned}
$$

where l is the dimensionless length parameter. The assumption of equilibrium changed the problem of optimally controlling a PDE system into one of controlling a coupled ODE system. Using Pontryagin's maximum principle, (Pontryagin et al. [1962]) with x as the underlying variable, he showed that no-take marine reserves are always part of an optimal harvest designed to maximize yield. Also, he found that the sizes and locations of the optimal reserves depend on the length parameter. For small values of this parameter, the maximum yield is obtained by placing a large reserve in the center of the habitat. For large values of this parameter, the optimal harvesting strategy was a spatial "chattering control" with infinite sequences of reserves alternating with areas of intense fishing. Such a chattering strategy would be impossible to actually implement due to the difficulty of monitoring the reserves. In this paper, we extend Neubert's work to a multidimensional spatial domain and consider different types of objective functionals. Note that in the multidimensional PDE case, one cannot use Pontryagin's maximum principle, thus some further analysis is needed to justify the necessary conditions.

There also has been some related work done on harvesting problems from a mathematical viewpoint using partial differential equations (PDEs). Leung and Stojanovic [1993] studied the optimal harvesting control of a biological species, whose growth is governed by the diffusive Volterra–Lotka equation. The species concentration satisfied a steady-state equation with no-flux (Neumann) boundary condition. The optimal control criteria was to maximize profit, which is the difference between economic revenue and cost. They proved existence of an optimal control with a positive lower bound under certain conditions and characterized it in terms of the solution of an elliptic optimality system. They constructed monotone sequences converging from above and below to give upper and lower estimates for the solutions of the optimality system. In the case where the limits of the upper and lower iterates agree, the optimal control was uniquely determined.

Leung [1995] also studied the corresponding optimal control problem for steady-state, prey–predator diffusive Volterra–Lotka systems and obtained similar results to the single equation case (Leung and Stojanovic [1993]). The techniques for *a priori* estimates and the use of the sensitivity and the adjoint system are similar to some of those used here, even though the model is not a fishery application.

Cañada et al. [1998] and Montero [2000] studied an optimal control problem for a nonlinear elliptic equation of the Lotka–Volterra type with Dirichlet boundary condition. The conditions for the optimality system and uniqueness of the optimal control depended on the eigenvalues of the Laplacian operator. Our problem is a bit simpler but with different objective functionals, and we use some of their techniques.

Kurata and Shi [2008] studied a reaction-diffusion model with logistic growth and constant effort harvesting. By minimizing an intrinsic biological energy function that is different from the yield, they obtained an optimal spatial harvesting strategy that would benefit the population the most. They found out a nonharvesting zone should be designed. On the other hand, in the zone that allows harvesting, the effort should be put at the maximum value.

Here, we analyze a harvesting problem, which is an extension of the work of Neubert [2003], to a general multidimensional spatial domain with different objective functionals. We are seeking an optimal fishery harvesting strategy modeled by a semilinear elliptic PDE with Dirichlet boundary condition. The solution $u(x)$ represents fish density, the region Ω is surrounded by a completely hostile habitat, and the control $h(x)$ represents the fishing effort (harvesting). The Laplacian operator Δu describes the movement of the fish by diffusion; the rate of change in the population is described by logistic growth. Instead of a constant harvesting, we let the harvesting depend on spatial location.

Our fishery model is

$$(1)\qquad \begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here, we have used dimensionless variables to simplify the equation, see Neubert [2003] for details. The domain Ω is bounded in $\mathbb{R}^n$ with $\partial\Omega$ being C^1, and the constant $r > 0$ is the growth rate. This is a spatially explicit model for harvesting and is the steady state of the corresponding semilinear evolution problem.

We consider two fishery management problems, which require different control sets and objective functionals. We define two control sets, U_1 and U_2:

$$U_1 = \left\{ h(x) \in L^2(\Omega) \,\middle|\, 0 \le h(x) \le h_{\max} \text{ a.e.} \right\};$$
$$U_2 = \left\{ h(x) \in H_0^1(\Omega) \,\middle|\, 0 \le h(x) \le h_{\max} \text{ a.e.} \right\},$$

where $h_{\max} > 0$ is a constant.

In Sections 2–6, we want to maximize the following objective functional:

$$(2)\quad J_1(h) = \int_\Omega h(x)u(x)\,dx - \int_\Omega (B_1 + B_2 h)h\,dx, \text{ where } h \in U_1,$$

which represents the difference between the yield and cost, where B_1, B_2 are nonnegative constants. The B_1 term is the cost per unit of effort when the level of effort is small, the B_2 term represents the rate at which the wages paid rises as more labor is employed (due to scarcity of labor), and $u = u(h)$ is the solution of (1) with control $h \in U_1$. We explicitly show the dependence of u on h with the understanding that space is the underlying variable. We want to find $h^* \in U_1$, such that

$$(3)\qquad J_1(h^*) = \max_{h \in U_1} J_1(h).$$

The first objective functional generalizes Neubert's work (Neubert [2003]) to treat two and three spatial dimensions. We can take $B_1 = B_2 = 0$ in our analysis to get an optimal control characterization for his objective functional. But we can include the cost of the harvesting effort by considering nonzero B_1, B_2. This first functional is similar to Montero's (Cañada et al. [1998]) when $B_1 = 0$.

In Section 7, we maximize the yield while minimizing the variation of fishing effort, that is, we seek to find $h^* \in U_2$, such that

$$J_2(h^*) = \max_{h \in U_2} J_2(h), \tag{4}$$

and

$$J_2(h) = \int_\Omega h(x)u(x)\,dx - A \int_\Omega |\nabla h|^2\,dx,\ \ h \in U_2, \tag{5}$$

where $A > 0$ is a constant, and $u = u(h)$ is the solution of (1) with control $h \in U_2$. Notice for a Dirichlet boundary condition, by Poincarë inequality, $||\nabla h||_{L^2}$ is equivalent to H_0^1 norm. The analysis needed for the necessary conditions in this case is different from the first case because the derivatives of the control change the calculations in the differentiation of the objective functional. This second objective functional seeks to minimize the variation in the control to eliminate the possibility of "chattering" harvest strategies like in Neubert's work (Neubert [2003]).

In Section 2, we give conditions for unique positive solutions to the state problem. In Section 3, we prove the existence of an optimal control. In Section 4, necessary conditions for an optimal control are obtained, and in particular the optimality system is deduced. We also give the results for maximizing the yield with Neumann boundary conditions. In Section 5, we prove the uniqueness of the optimality system. Note that in Leung [1995], Cañada et al. [1998], and Montero [2000], there were no numerical examples to illustrate the results. In Section 6, numerical examples are given to solve the nonlinear optimality system for dimensions 1 and 2. In Section 7, we give the corresponding theoretical results for $J_2(h)$, and it turns out the optimal control is characterized by a variational inequality instead of the usual algebraic characterization. In Section 7.4, we explain the numerical methods to solve the variational inequality and then give some numerical examples in this case. Finally, we give some relevant conclusions.

2. Existence and uniqueness of a positive solution to the state equation. We say a function $u(x) \in H_0^1(\Omega)$ is a weak solution to (1) if

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega (ru(1-u) - h(x)u)v \, dx, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

Using the extension of the maximum principle to weak solutions (Krylov [1985]), we can show $0 \le u(x) \le 1$.

Clearly $u = 0$ is a solution for (1), but for a fishery problem we need to have a unique positive solution under certain conditions.

We present some results from Cañada et al. [1998] to guarantee the existence and uniqueness of the positive state solution. For a function $q \in L^\infty(\Omega)$, we define $\sigma_1(q)$ to be the principal eigenvalue of the eigenvalue problem

$$\begin{aligned} &-\Delta u(x) + q(x)u(x) = \sigma u(x), && x \in \Omega; \\ &u(x) = 0, && x \in \partial\Omega. \end{aligned} \tag{7}$$

This principal eigenvalue can be expressed as

$$\sigma_1(q) = \inf_{\substack{\phi \in H_0^1(\Omega) \\ \phi \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla \phi|^2 \, dx + \int_\Omega q\phi^2 \, dx}{\displaystyle\int_\Omega \phi^2 \, dx}. \tag{8}$$

It is known that the algebraic multiplicity of $\sigma_1(q)$ is equal to one and the associated eigenfunction is positive.

We obtain the following three properties for $\sigma_1(q)$, and we will refer to them in the following proofs.

Property 1. $\sigma_1(q)$ is increasing with respect to q, that is, if $q_1 < q_2$, then $\sigma_1(q_1) < \sigma_1(q_2)$;
Property 2. $\sigma_1(q)$ is continuous with respect to $q \in L^\infty(\Omega)$;
Property 3. if $\sigma_1(q) > 0$, then there exists $c > 0$, such that

$$c \int_\Omega |\nabla \phi|^2 \, dx \le \int_\Omega |\nabla \phi|^2 \, dx + \int_\Omega q\phi^2 \, dx.$$

Properties 1 and 2 follow from the definition of $\sigma_1(q)$. We present a proof of Property 3: If $0 < c \leq \frac{\sigma_1(q)}{\sigma_1(q)+||q||_\infty}$, that is, $c||q||_\infty \leq (1-c)\sigma_1(q)$, then by the definition of $\sigma_1(q)$, we have

$$(1-c)\int_\Omega (|\nabla\phi|^2 + q\phi^2)\,dx \geq (1-c)\sigma_1(q)\int_\Omega \phi^2\,dx \geq c||q||_\infty \int_\Omega \phi^2\,dx \geq -c\int_\Omega q\phi^2\,dx.$$

By rearranging the above inequality, we have Property 3.

Note if there exist two positive constants M and μ, such that

$$||q||_\infty \leq M,\ \ \sigma_1(q) \geq \mu,$$

then the constant c may be chosen independent of q. We can take $c = \frac{\mu}{\mu+M}$, because

$$\frac{\mu}{\mu+M} \leq \frac{\sigma_1(q)}{\sigma_1(q)+||q||_\infty}.$$

By Berestycki and Lions [1980] and Montero [2000], the nontrivial solution u to equation (1) is unique and strictly positive if and only if $\sigma_1(-r+h) < 0$ and $u \equiv 0$ if and only if $\sigma_1(-r+h) \geq 0$.

We take $u = u(h)$ to be the maximum nonnegative solution of (1). If for some $h \in U_1(\text{or}\,U_2)$, $\sigma_1(-r+h) < 0$, then $u = u(h) > 0$ in Ω. If $\sigma_1(-r+h) \geq 0$ for some $h \in U_1(\text{or}\,U_2)$, then $u = u(h) \equiv 0$. Here, we explicitly show the dependence of the states on the control, and we note that space is still the underlying variable.

Note that Cantrell and Cosner [2003] and Shi and Shivaji [2005] studied a general class of semilinear equations and found that the existence and uniqueness of the positive solutions depend on the principal eigenvalue of the corresponding linear operator.

3. Existence of an optimal control for J_1.

First, we prove the existence of an optimal control for our first objective functional.

Theorem 3.1. *There exists an optimal control $h^* \in U_1$ maximizing the objective functional $J_1(h)$.*

Proof. Because $0 \leq h(x) \leq h_{\max}$ and $0 \leq u(x) \leq 1$, we have $J_1(h) \leq (h_{\max})(meas(\Omega))$, and we can choose a maximizing sequence $\{h^n\} \subset U_1$, s.t.

$$\lim_{n\to\infty} J_1(h^n) = \sup_{h\in U_1} J_1(h). \tag{9}$$

First, we get an *a priori* estimate for u. Let $u^n = u(h^n)$, and take $v = u^n$ as the test function in (6), we have

$$\begin{aligned} \int_\Omega \nabla u^n \cdot \nabla u^n \, dx &= \int_\Omega (ru^n(1-u^n) - h^n u^n)u^n \, dx \\ &\leq \int_\Omega r(u^n)^2 \, dx \leq C_0, \end{aligned} \tag{10}$$

because $0 < u^n \leq 1$ and $h^n \in U_1$, which gives

$$||u^n||_{H_0^1(\Omega)} \leq C_1. \tag{11}$$

Then there exists u^* in $H_0^1(\Omega)$ such that on a subsequence, $u^n \rightharpoonup u^*$ weakly in $H_0^1(\Omega)$. Because $H_0^1(\Omega) \subset\subset L^2(\Omega)$, we obtain

$$u^n \longrightarrow u^* \text{ strongly in } L^2(\Omega),$$

and there is a subsequence $\{u_{n_k}\}$, s.t. $\{u_{n_k}\}$ converges to u^* almost uniformly (Friedman [1982]), so $0 \leq u^* \leq 1$ a.e.

Notice the sequence $\{h^n\}$ in U_1 is uniformly bounded in $L^2(\Omega)$, so on an appropriate subsequence,

$$h^n \rightharpoonup h^* \text{ weakly in } L^2(\Omega).$$

Next we need to prove $u^* = u(h^*)$. The weak solution formulation (6) for u^n gives

$$\int_\Omega \nabla u^n \cdot \nabla v \, dx = \int_\Omega (ru^n(1-u^n) - h^n u^n)v \, dx, \ \forall v \in H_0^1(\Omega). \tag{12}$$

Because $u^n \to$ u* strongly in $L^2(\Omega)$, which implies

$$\int_\Omega |(u^n - u^*)v|\,dx \le \left(\int_\Omega (u^n - u^*)^2\,dx \int_\Omega v^2\,dx\right)^{\frac{1}{2}} \longrightarrow 0, \tag{13}$$

and then

$$\begin{aligned}\left|\int_\Omega (u^n)^2 v - (u^*)^2 v\,dx\right| &\le \int_\Omega |\, u^n + u^*\,||\,(u^n - u^*)v\,|\;dx \\ &\le 2\int_\Omega |\,(u^n - u^*)v\,|\;dx \longrightarrow 0,\end{aligned} \tag{14}$$

because $0 \le u^n,\, u^* \le 1$ and by (13). Then the control are estimated:

$$\begin{aligned}&\left|\int_\Omega (h^n u^n - h^* u^*)v\,dx\right| \\ &\quad\le \left|\int_\Omega h^n(u^n - u^*)v\,dx\right| + \left|\int_\Omega (h^n - h^*)u^* v\,dx\right| \\ &\le \int_\Omega h_{\max}\,|\,(u^n - u^*)v\,dx\,| + \left|\int_\Omega (h^n - h^*)u^* v\,dx\right| \longrightarrow 0,\end{aligned} \tag{15}$$

because $h^n \in U_1$, and using (13) and weak convergence of h^n in $L^2(\Omega)$ with $u^*\,v \in L^2(\Omega)$.

Finally, we obtain

$$\int_\Omega (\nabla u^n \cdot \nabla v - \nabla u^* \cdot \nabla v)dx = \int_\Omega \nabla(u^n - u^*) \cdot \nabla v\;dx \longrightarrow 0, \tag{16}$$

because $u^n \rightharpoonup u^*$ weakly in $H_0^1(\Omega)$ implies $\nabla u^n \rightharpoonup \nabla u^*$ weakly in $L^2(\Omega)$. Then passing to the limit in (12), we have $u^* = u(h^*)$.

Also we need to verify h^* is an optimal control, that is,

$$J_1(h^*) \ge \sup_{h \in U_1} J_1(h). \tag{17}$$

Because

(18)
$$\sup_{h\in U_1} J_1(h) = \lim_{n\to\infty} J_1(h^n) = \lim_{n\to\infty}\int_\Omega h^n u^n - (B_1 + B_2h^n)h^n\,dx$$
$$\leq \int_\Omega h^*u^*\,dx - \underline{\lim}_{n\to\infty}\int_\Omega (B_1 + B_2h^n)h^n\,dx$$
$$\leq \int_\Omega h^*u^*\;dx - \int_\Omega (B_1 + B_2h^*)h^*\,dx = J_1(h^*),$$

where we used (15) and lower semicontinuity of the cost functional with respect to weak L^2 convergence, we have verified (17).

4. Derivation of the optimality system for J_1. In order to characterize the optimal control, we need to differentiate the objective functional with respect to the control h. Because $u = u(h)$ is involved in $J_1(h)$ (and $J_2(h)$), we first must prove appropriate differentiability of the mapping $h \longrightarrow u(h)$, whose derivative is called the *sensitivity*.

Lemma 4.1 (Sensitivity). *Assume for $h_0 \in U_1, \sigma_1(-r + h_0) <$ 0, the mapping $h \in U_1 \longrightarrow u(h)$ is differentiable at h_0 in the following sense: there exists $\psi \in H_0^1(\Omega)$, such that*

$$\frac{u(h_0 + l\epsilon) - u(h_0)}{\epsilon} \rightharpoonup \psi \textit{ weakly in } H_0^1(\Omega) \textit{ as } \epsilon \to 0,$$

where $h_0 + \epsilon l \in U_1, l \in L^\infty(\Omega)$. And the sensitivity $\psi = \psi(h_0;\, l)$ satisfies

$$\begin{cases} -\Delta\psi = r\psi(1-2u) - h_0(x)\psi - lu, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \tag{19}$$

Proof. Because $\sigma_1(-r + h_0) < 0$, we have $u(h_0) > 0$ on Ω. Define $u^\epsilon = u(h_0 + l\epsilon)$, using (1), we have

$$-\Delta u^\epsilon = ru^\epsilon(1 - u^\epsilon) - (h_0(x) + \epsilon l)u^\epsilon, \tag{20}$$

then subtracting (1) from (20) and divide by ϵ, we have

$$-\Delta\frac{u^\epsilon-u}{\epsilon}=r\frac{u^\epsilon-u}{\epsilon}-r\frac{(u^\epsilon)^2-u^2}{\epsilon}-h_0(x)\frac{u^\epsilon-u}{\epsilon}-lu^\epsilon. \tag{21}$$

Multiplying both sides by $\frac{u^\epsilon-u}{\epsilon}$ and integrating in Ω, we obtain

(22)

$$\int_\Omega\left|\nabla\frac{u^\epsilon-u}{\epsilon}\right|^2dx=\int_\Omega r\left(\frac{u^\epsilon-u}{\epsilon}\right)^2dx-r\int_\Omega(u^\epsilon+u)\left(\frac{u^\epsilon-u}{\epsilon}\right)^2dx$$
$$-\int_\Omega h_0(x)\left(\frac{u^\epsilon-u}{\epsilon}\right)^2dx-\int_\Omega lu^\epsilon\frac{u^\epsilon-u}{\epsilon}\,dx.$$

that is,

$$\int_\Omega\left|\nabla\frac{u^\epsilon-u}{\epsilon}\right|^2dx+\int_\Omega(-r+h_0+r(u^\epsilon+u))\left(\frac{u^\epsilon-u}{\epsilon}\right)^2dx$$
$$=-\int_\Omega lu^\epsilon\frac{u^\epsilon-u}{\epsilon}\,dx.$$

Using (1), that is, $-\Delta u+(-r+h_0+ru)u=0$, we have $\sigma_1(-r+h_0+ru)=0$. Let ϵ_1 sufficiently small, such that for $|\epsilon|<\epsilon_1$, by the continuity of $\sigma_1(q)$, the assumption $\sigma_1(-r+h_0)<0$ implies $\sigma_1(-r+h_0+\epsilon_1||l||_\infty)<0$. Thus, there exists $u_{\epsilon_1}>0$, with $u_{\epsilon_1}=u(h_0+\epsilon_1||l||_\infty)$,

$$\sigma_1(-r+h_0+r(u_{\epsilon_1}+u))>\sigma_1(-r+h_0+ru)=0.$$

Because u is decreasing with respect to h, we have $u^\epsilon\geq u_{\epsilon_1}$. By the monotonicity of $\sigma_1(q)$, we obtain

$$\sigma_1(-r+h_0+r(u^\epsilon+u))\geq\sigma_1(-r+h_0+r(u_{\epsilon_1}+u))>0.$$

Using property 3 of $\sigma_1(q)$, there exists $c_1>0$, such that

$$c_1\int_\Omega\left|\nabla\frac{u^\epsilon-u}{\epsilon}\right|^2dx\leq\int_\Omega\left|\nabla\frac{u^\epsilon-u}{\epsilon}\right|^2dx \tag{23}$$
$$+\int_\Omega(-r+h_0+r(u^\epsilon+u))\left(\frac{u^\epsilon-u}{\epsilon}\right)^2dx$$

$$
\begin{aligned}
&= -\int_\Omega l u^\epsilon \frac{u^\epsilon - u}{\epsilon}\, dx \\
&\le \left(\int_\Omega l^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega \left(\frac{u^\epsilon - u}{\epsilon}\right)^2\right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega l^2\, dx\right)^{\frac{1}{2}} \left(C_P \int_\Omega \left|\nabla \frac{u^\epsilon - u}{\epsilon}\right|^2\right)^{\frac{1}{2}} \\
&\le C_1 \left(\int_\Omega \left|\nabla \frac{u^\epsilon - u}{\epsilon}\right|^2\right)^{\frac{1}{2}},
\end{aligned}
$$

where $C_1 = ||l||_\infty |\Omega|^{\frac{1}{2}} C_P^{\frac{1}{2}}$ and C_P is the constant from Poincaré's inequality.

Then our bound

$$
\left|\left|\frac{u^\epsilon - u}{\epsilon}\right|\right|_{H_0^1(\Omega)} \le C_2, \tag{24}
$$

implies there exists $\psi \in H_0^1(\Omega)$ with the desired weak convergence of the quotients to ψ.

To get the equation for ψ, we multiply $\phi \in H_0^1(\Omega)$ on both sides of (21) and integrate in Ω,

$$
\begin{aligned}
\int_\Omega -\Delta \frac{u^\epsilon - u}{\epsilon} \phi\, dx &= \int_\Omega r \frac{u^\epsilon - u}{\epsilon} \phi\, dx - \int_\Omega r \frac{u^\epsilon - u}{\epsilon} (u^\epsilon + u) \phi\, dx \\
&\quad - \int_\Omega h_0(x) \frac{u^\epsilon - u}{\epsilon} \phi\, dx - \int_\Omega l u^\epsilon \phi\, dx.
\end{aligned} \tag{25}
$$

Because $\frac{u^\epsilon - u}{\epsilon} \rightharpoonup \psi$ weakly in $H_0^1(\Omega)$, $0 \le h_0(x) \le h_{\max}, 0 \le u^\epsilon, u \le 1$, and $u^\epsilon \to$ u strongly in $L^2(\Omega)$ as $\epsilon \to 0$, we obtain

$$
\begin{aligned}
\int_\Omega -\Delta \psi \phi\, dx &= \int_\Omega r(1-2u)\psi\phi\, dx \\
&\quad - \int_\Omega h_0(x)\psi\phi\, dx - \int_\Omega l u \phi\, dx, \forall \phi \in H_0^1(\Omega),
\end{aligned} \tag{26}
$$

that is, ψ satisfies (19).

Remark. Note that the sensitivity result depends only on the PDE and the boundary condition and not on the objective functional. Hence, Lemma 4.1 can be applied in both control problems.

Now we are ready to characterize the optimal control, by deriving the optimality system through differentiating $J_1(h)$ with respect to h at an optimal control.

Theorem 4.2. *Assume $B_2 > 0$, and for an optimal control h in U_1, $\sigma_1(-r+h) < 0$, then there exists a solution p in $H^2(\Omega)\cap H_0^1(\Omega)$ to the adjoint problem*

$$\begin{cases} -\Delta p - r(1-2u)p + hp = h, & x \in \Omega, \\ p = 0, & x \in \partial\Omega. \end{cases} \tag{27}$$

Furthermore $h(x)$ satisfies

$$h(x) = \min\left\{\max\left\{0, \frac{u - pu - B_1}{2B_2}\right\}, h_{\max}\right\}.$$

Proof. Suppose $h(x)$ is an optimal control. Let $l \in L^\infty(\Omega)$ such that $h + \epsilon l \in U_1$ for small $\epsilon > 0$. The derivative of $J_1(h)$ with respect to h in the direction of l satisfies

$$\begin{aligned} 0 &\geq \lim_{\epsilon\to 0^+} \frac{J_1(h+\epsilon l) - J_1(h)}{\epsilon} \\ &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\int_\Omega (h+\epsilon l)u^\epsilon\, dx - \int_\Omega \left(B_1(h+\epsilon l) + B_2(h+\epsilon l)^2\right)\, dx \right.\\ &\qquad \left. - \int_\Omega hu\, dx + \int_\Omega \left(B_1 h + B_2 h^2\right)\, dx\right] \\ &= \lim_{\epsilon\to 0^+} \int_\Omega (h\frac{u^\epsilon - u}{\epsilon} + lu^\epsilon)\, dx - \int_\Omega (B_1 l + B_2(2hl + \epsilon l^2))\, dx \\ &= \int_\Omega (h\psi + lu)\, dx - \int_\Omega (B_1 l + 2B_2 hl)\, dx. \end{aligned} \tag{28}$$

We used ideas from Cantrell and Cosner [2003] to show 0 is not an eigenvalue of $L\psi = \sigma\psi$, where $Lv = -\Delta\ v + (-r + h(x) + 2ru)\ v$, from the Fredholm alternative, $Lv - 0 \cdot v = h(x)$ has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ for $h(x) \in L^2(\bar{\Omega})$. Then choosing $v(x) = p(x)$, we have the existence and uniqueness of the adjoint.

The state equation (1) can be rewritten as

$$\begin{cases} -\Delta u + (-r + h)u + ru^2 = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{29}$$

Because $\sigma_1(-r + h) < 0$, we have $u > 0$, and then $\psi = u$ is a solution to

$$\begin{cases} -\Delta\psi + (-r + h + ru)\psi = \sigma\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases} \tag{30}$$

with $\sigma = 0$. So $\sigma = 0$ is an eigenvalue of (30) and because $u > 0$ in Ω, it must be the principal eigenvalue for (30), denoted by $\sigma_1(-r + h + ru)$. Notice the problem

$$\begin{cases} -\Delta\psi + (-r + h + 2ru)\psi = \sigma\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases} \tag{31}$$

has a principal eigenvalue $\sigma_1(-r + h + 2ru)$ with

$$\sigma_1(-r + h + 2ru) > \sigma_1(-r + h + ru) = 0,$$

because $-r + h + 2ru > -r + h + ru$ and using the monotonocity of $\sigma_1(q)$. Because $\sigma_1(-r + h + 2ru)$ is the smallest eigenvalue of (31), 0 cannot be an eigenvalue for (31); hence 0 is not an eigenvalue of L. This completes the existence and uniqueness of the solution of the adjoint problem.

Let p be the solution to adjoint problem (27), then from (28), we have

$$0 \geq \int_\Omega [\psi(-\Delta p - r(1-2u)p + hp) + lu]\, dx - \int_\Omega (B_1 l + 2B_2 hl)\, dx = \int_\Omega [\nabla p \nabla \psi + p(-r(1-2u)\psi + h\psi) + lu]\, dx - \int_\Omega (B_1 l + 2B_2 hl)\, dx. \tag{32}$$

Using the sensitivity PDE (19) (with h as h_0), we obtain

$$0 \geq \int_\Omega (-p\ lu + lu)\, dx - \int_\Omega (B_1 l + 2B_2 hl)\, dx = \int_\Omega l(-p\ u + u - B_1 - 2B_2 h)\, dx. \tag{33}$$

Then on the set $0 < h < h_{\max}$, we choose variation l with support on this set and l to be any sign, which gives $-pu + u - B_1 - 2B_2 h = 0$. On the set where $h = 0$, we choose $l \geq 0$, which implies $-pu + u - B_1 - 2B_2 h \leq 0$. Similarly where $h = h_{\max}$, we choose $l \leq 0$, which implies $-pu + u - B_1 - 2B_2 h \geq 0$. This can be written in the compact form as

$$h^* = \min\left\{\max\left\{0, \frac{u - pu - B_1}{2B_2}\right\}, h_{\max}\right\}. \tag{34}$$

Next, we give the generalization of Neubert's result (Neubert [2003]) (for maximizing the yield only) to multidimensions.

Theorem 4.3. *If $B_1 = B_2 = 0$ in $J_1(h)$, then the optimal control is given by*

$$h(x) = \begin{cases} 0, & \text{if } p > 1; \\ h_{\max}, & \text{if } p < 1; \\ \frac{r}{2}, & \text{if } p = 1. \end{cases} \tag{35}$$

Proof. The proof in Theorem 4.2 is valid for $B_2 = 0$ except when we solve for h in (34). In (33), if we set $B_1 = 0, B_2 = 0$, inequality (33) reduces to

$$0 \geq \int_\Omega l(1-p)u\,dx,$$

which is a generalization of Neubert's [2003] case in multidimension.

On the set where $p < 1$, then variation satisfies $l \leq 0$, so conclude $h = h_{\max}$. Where $p > 1$, then variation satisfies $l \geq 0$, which gives $h = 0$. Where $p = 1$, using adjoint PDE (27), we get $u = \frac{1}{2}$, and then using state PDE (1), we get $h = \frac{r}{2}$. This gives the same result as Neubert's [2003] with $r = 1$.

The problem of maximizing $J_1(h)$ with the state PDE in (1) with Neumann boundary condition gives a simple optimal control, a singular case.

Theorem 4.4. *If* $B_1 = B_2 = 0$ in $J_1(h)$, *and our state problem is*

$$\begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & x \in \partial\Omega, \end{cases} \tag{36}$$

then the optimal control and state are

$$h^*(x) = \frac{1}{2},\ u^*(x) = \frac{1}{2}.$$

Proof. From the state PDE (1),

$$\begin{aligned} J_1(h) &= \int_\Omega hu\,dx = \int_\Omega \Delta u + u(1-u)\,dx \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \eta}\,dS + \int_\Omega u(1-u)\,dx = \int_\Omega u(1-u)\,dx, \end{aligned} \tag{37}$$

where we used integration by parts,

$$\int_\Omega \Delta u\,dx = \int_\Omega 1 \sum_{i=1}^{n} \left(u_{x_i}\right)_{x_i} dx = \int_\Omega 0\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta}\,ds,$$

and the Neumann boundary condition. The integral is maximized at $u^* = \frac{1}{2}$, because $u(1-u)$ is maximized there. Then using state PDE (1), we have $h^* = \frac{1}{2}$.

5. Uniqueness of optimality system I. In the case $B_2 > 0$, the state equation (1) and the adjoint equation (27) together with the characterization of the optimal control (34) is called optimality system I (OS_1), which is given by

$$\text{(38)} \quad \begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega; \\ -\Delta p - r(1-2u)p + hp = h, & x \in \Omega, \\ p = 0, & x \in \partial\Omega; \\ h(x) = \min\left\{ \max\left\{0, \dfrac{u - pu - B_1}{2B_2}\right\}, h_{\max}\right\}. \end{cases}$$

We know that the solutions of the optimality system exist by Theorems 3.1 and 4.2. We now prove that the solutions of (OS_1) are unique, which gives a characterization of the unique optimal control in terms of the unique solutions of (OS_1).

To prove the uniqueness of the solutions to (OS_1), we need a bound of the adjoint p in $L^\infty(\Omega)$ depending on B_2.

Lemma 5.1. *For $B_2 \neq 0$, given u, p, h solving (38) with u positive in Ω, for dimension $n = 1, 2, 3$, the adjoint p satisfies*

$$\text{(39)} \qquad ||p||_{L^\infty(\Omega)} \le \frac{C_8}{B_2},$$

where C_8 doesn't depend on B_2.

Proof. Because $u > 0$, using the monotonicity of $\sigma_1(q)$, we have $\sigma_1(-r + h + 2ru) > \sigma_1(-r + h + ru) = 0$. Then by property 3 of $\sigma_1(q)$

and using weak formulation of adjoint equation in (38), there exists $c_2 > 0$, such that

$$(40) \qquad \begin{aligned} c_2 \int_\Omega |\nabla p|^2 \, dx \le \int_\Omega |\nabla p|^2 \, dx \\ + \int_\Omega (-r + h + 2ru) p^2 \, dx = \int_\Omega hp \, dx \\ \le \frac{1}{4\epsilon} \int_\Omega h^2 \, dx + \epsilon \int_\Omega p^2 \, dx \\ \le \frac{1}{4\epsilon} h_{\max}^2 |\Omega| + \epsilon C_P \int_\Omega |\nabla p|^2 \, dx, \end{aligned}$$

where we use $0 \le h \le h_{\max}$, and C_P is the constant from Poincaré's inequality. If we choose $\epsilon = \frac{c_2}{2C_P}$, we have

$$(41) \qquad \int_\Omega |\nabla p|^2 \, dx \le C_1,$$

where $C_1 = \frac{C_P}{c_2^2} h_{\max}^2 |\Omega|$.

By standard elliptic regularity (Chapter 6, Evans [1998]) and using the adjoint equation in (38),

$$(42) \qquad ||p||_{H^2(\Omega)} \le C_4 ||(r - h - 2ru)p + h||_{L^2(\Omega)} \le C_5,$$

where C_4 only depends on Ω and dimension n, and C_5 only depends on r, C_P, c_2, Ω, n. Using the characterization of an optimal control h in (38), estimate (41) together with Poincaré's inequality, we have

$$(43) \qquad ||h||_{L^2(\Omega)} \le \left|\left| \frac{u - pu - B_1}{B_2} \right|\right|_{L^2(\Omega)} \le \frac{C_6}{B_2},$$

where C_6 only depends on Ω, n, r, C_P, c_2. Go back to (40), refine estimate of p by (43), we have

$$(44) \qquad \int_\Omega p^2 \, dx \le C_P \int_\Omega |\nabla p|^2 \, dx \le \frac{C_7 C_P}{B_2^2},$$

where $C_7 = \frac{C_P C_6^2}{c_2^2}$. From Li and Yong [1995], for $n = 1, 2, 3,\ H^2(\Omega) \subset\subset C(\bar{\Omega})$, we have

$$||p||_{L^\infty(\Omega)} \leq C_0 ||p||_{H^2(\Omega)}, \tag{45}$$

where C_0 only depends on Ω and dimension n. Using a refinement of (42) with B_2 dependence, (44) and (43), we obtain

$$||p||_{L^\infty(\Omega)} \leq \bar{C_0}(||p||_{L^2(\Omega)} + ||h||_{L^2(\Omega)}) \leq \frac{C_8}{B_2}, \tag{46}$$

where C_8 doesn't depend on B_2 and only depends on Ω, n, r, C_P, c_2.

Theorem 5.2. *For $n = 1, 2, 3$, if B_2 is sufficiently large, then solutions of the optimality system I (OS_1) with positive u components are unique.*

Proof. Suppose u, p, h and $\bar{u}, \bar{p}, \bar{h}$ are two solutions of (OS_1). From (34), using $pu - \bar{p}\bar{u} = p(u - \bar{u}) + (p - \bar{p})\bar{u}$ we have

$$\begin{aligned} |h - \bar{h}| &\leq \left| \frac{u - pu - B_1}{2B_2} - \frac{\bar{u} - \bar{p}\bar{u} - B_1}{2B_2} \right| \\ &\leq \frac{1}{2B_2} \left(|(1 - p)(u - \bar{u})| + |p - \bar{p}| \right). \end{aligned} \tag{47}$$

Choosing test functions $u - \bar{u}$ in the state PDEs and using $hu - \bar{h}\bar{u} = h(u - \bar{u}) + (h - \bar{h})\bar{u}$, gives

$$\begin{aligned} \int_\Omega |\nabla(u - \bar{u})|^2\, dx - \int_\Omega r(u - \bar{u})^2\, dx + \int_\Omega r(u + \bar{u})(u - \bar{u})^2\, dx \\ + \int_\Omega (u - \bar{u})^2 h + \bar{u}(u - \bar{u})(h - \bar{h})\, dx = 0, \end{aligned} \tag{48}$$

and choosing test functions $p - \bar{p}$ in the adjoint PDEs and using $up - \bar{u}\bar{p} = (u - \bar{u})p + \bar{u}(p - \bar{p})$, gives

$$(49)\quad \int_\Omega |\nabla(p-\bar{p})|^2\,dx - \int_\Omega r(p-\bar{p})^2\,dx + 2r\int_\Omega \left[(u-\bar{u})p(p-\bar{p})\,dx + \bar{u}(p-\bar{p})^2\right]dx + \int_\Omega (h-\bar{h})p(p-\bar{p}) + \bar{h}(p-\bar{p})^2\,dx - \int_\Omega (h-\bar{h})(p-\bar{p})\,dx = 0.$$

Adding (48) and (49), the following equation results:

$$(50)\quad \begin{aligned} &\int_\Omega |\nabla(u-\bar{u})|^2\,dx + \int_\Omega \left(-r+h+r(u+\bar{u})\right)(u-\bar{u})^2\,dx \\ &\quad + \int_\Omega |\nabla(p-\bar{p})|^2\,dx + \int_\Omega (-r+\bar{h}+2\bar{u}r)(p-\bar{p})^2\,dx \\ &= \int_\Omega -\bar{u}(u-\bar{u})(h-\bar{h})\,dx - \int_\Omega 2r(u-\bar{u})p(p-\bar{p})\,dx \\ &\quad - \int_\Omega (h-\bar{h})p(p-\bar{p})\,dx + \int_\Omega (h-\bar{h})(p-\bar{p})\,dx. \end{aligned}$$

Because $u, \bar{u} > 0$, and $u, \bar{u}$ satisfy the state equation in (38), we have

$$\sigma_1(-r+h+r(u+\bar{u})) > \sigma_1(-r+h+ru) = 0,$$
$$\sigma_1(-r+\bar{h}+2\bar{u}r) > \sigma_1(-r+\bar{h}+r\bar{u}) = 0.$$

Using property 3 of $\sigma_1(q)$, (47) and $0 \le u, \bar{u} \le 1$, there exists $c_3 > 0$, such that

$$(51)\quad \begin{aligned} &c_3\left(\int_\Omega |\nabla(u-\bar{u})|^2\,dx + \int_\Omega |\nabla(p-\bar{p})|^2\,dx\right) \\ &\le \int_\Omega |\nabla(u-\bar{u})|^2\,dx + \int_\Omega \left(-r+h+r(u+\bar{u})\right)(u-\bar{u})^2\,dx \\ &\quad + \int_\Omega |\nabla(p-\bar{p})|^2\,dx + \int_\Omega (-r+\bar{h}+2\bar{u}r)(p-\bar{p})^2\,dx \end{aligned}$$

$$\leq \int_\Omega \frac{||1-p||_{L^\infty}}{2B_2}(u-\bar{u})^2\ dx + \int_\Omega \frac{1}{2B_2}|(u-\bar{u})(p-\bar{p})|\,dx$$
$$+ 2r||p||_{L^\infty}\int_\Omega |(u-\bar{u})(p-\bar{p})|\,dx + (1+||p||_{L^\infty})$$
$$\times\left(\int_\Omega \frac{||1-p||_{L^\infty}}{2B_2}|(u-\bar{u})(p-\bar{p})|\ dx + \int_\Omega \frac{1}{2B_2}(p-\bar{p})^2\,dx\right).$$

Using the Cauchy–Schwartz inequality, Poincaré's inequality, and (39), we have

$$\begin{aligned} c_3\left(\int_\Omega |\nabla(u-\bar{u})|^2\,dx + \int_\Omega |\nabla(p-\bar{p})|^2\,dx\right) \\ \leq \frac{C_9}{B_2}\int_\Omega \left(|\nabla(u-\bar{u})|^2 + |\nabla(p-\bar{p})|^2\right)\ dx. \end{aligned} \tag{52}$$

If we take B_2 sufficiently large, so that

$$c_3 > \frac{C_9}{B_2},$$

we conclude $u=\bar{u},\ p=\bar{p}, h=\bar{h}$, that is, we have the uniqueness of OS_1, which implies the uniqueness of the optimal control.

6. Numerical examples for J_1. We solve the optimality system I (38) numerically by the following iteration method, which is implemented using MATLAB:

(i) **Initialization:** Choose initial guesses for fish density u_0 and harvesting h_0.

(ii) **Discretization:** Use the finite difference method to discretize state and adjoint equations to nonlinear algebraic systems.

(iii) **Iteration:** h_n is known
- (a) Solve discretized PDE of (1) for state u: with the discretized Laplacian term on the left and the nonlinear terms to the right, we solve a linear system in which the corresponding matrix is tridiagonal for 1-D case and block tridiagonal for 2-D case.
- (b) Solve discretized PDE of (27) for adjoint p similarly.

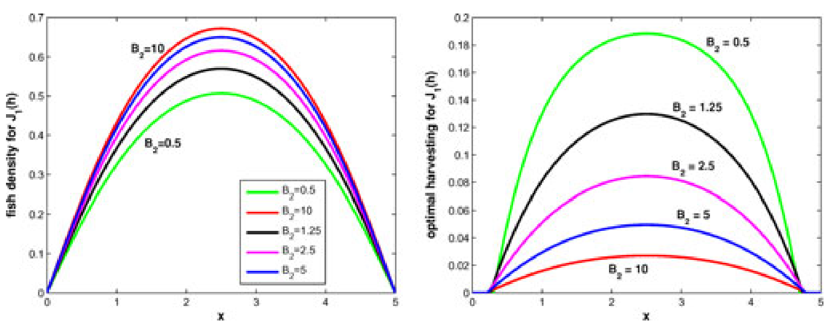
(c) Update the control by entering new fish density and adjoint values into the characterization of optimal control (34).

(iv) **Repeat** step 3 if successive iterates are not sufficient close.

We make a few remarks about the algorithm (Lenhart and Workman [2007]). A central difference scheme is used to discretize the Laplacian operator. A convex combination between the previous control values and values given by the current characterization is used in updating h, which helps to speed up convergence.

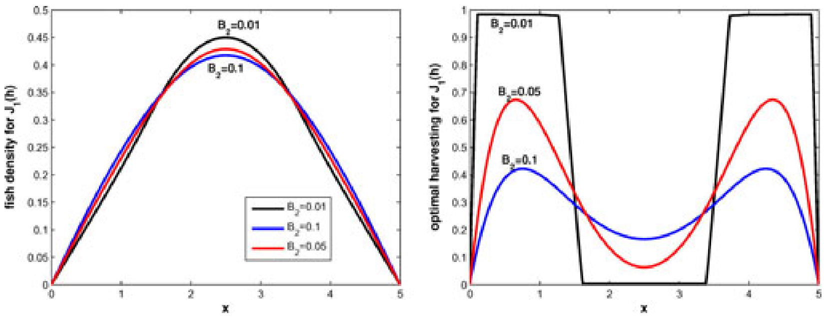
Next we give some numerical examples to illustrate the results. For the 1-D case, the interval length is 5. We vary values for B_1 and B_2 to see how they affect the corresponding fish density and optimal harvesting. Also, we investigate how the domain size will affect the optimal benefit (the fish density, optimal harvesting, and J_1 value). We choose $h_{\max} = 0.99$ because we don't want to deplete the fish stock. In Figures 1 and 2, we set $r = 1$.

First we set $B_1 = 0.1$ and vary B_2 to see how the change in labor cost will affect the fish density and harvesting strategy. Choosing $B_2 = 0.5$, 1.25, 2.5, 5, 10, in Figure 1, we see that when labor cost increases, that is, B_2 increases, we harvest less and the corresponding fish density is increasing. In this scenario, we harvest more on the center of the habitat where the fish density is high. Notice that there is no harvesting in a small neighborhood about the boundary.



(a) Fish density u (b) Optimal harvesting h

FIGURE 1. Fish density and optimal harvesting: $B_1 = 0.1, B_2 = 0.5$, 1.25, 2.5, 5, 10.



(a) Fish density u (b) Optimal harvesting h

FIGURE 2. Fish density and optimal harvesting: $B_1 = 0, B_2 = 0.1$, 0.05, 0.01.

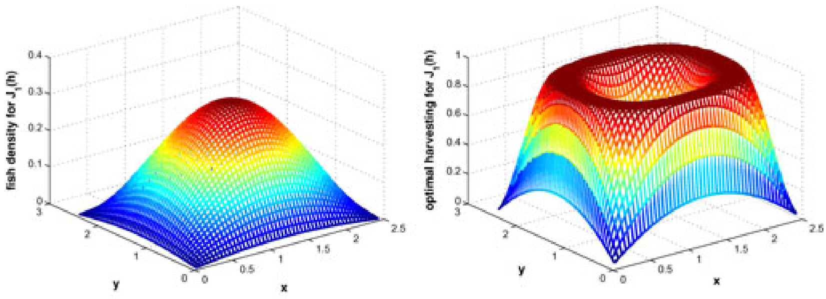
Then we set $B_1 = 0$, but choose very small values for B_2 so that we can compare to Neubert's results (Neubert [2003]). In Figure 2, we let $B_2 = 0.1, 0.05, 0.01$ and see the tendency toward a reserve in the middle of the region, and this result is similar to Neubert's result (in Neubert [2003]) when the length parameter $l = 5$. Note that the harvesting occurs closer to the edge of the habitat but tapers off right at the edge. In this case, the cost of control is very low, which gives different results from the case when B_2 is larger.

Next we illustrate the case of two dimensional space. In Figure 3, we set $B_1 = 0$ and choose $B_2 = 0.03$. We can observe a similar situation as in the 1-D case in that there tends to be a reserve in the center of the region.

We take $r = 5$ in Figures 4–6. In Figures 4 and 5, we take the domain to be $(0, 2.5) \times (0, 2.5)$. In Figure 6, the domain is $(0, 3) \times (0, 3)$.

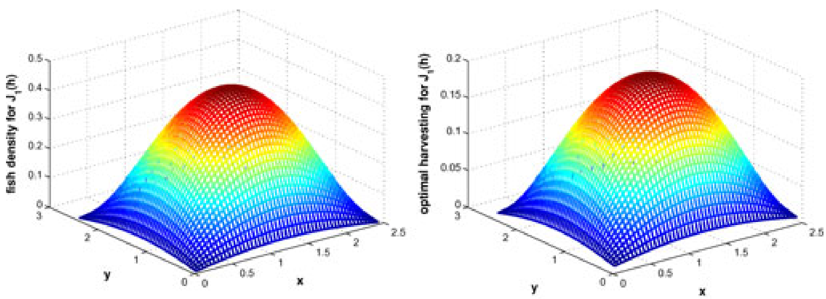
We set $B_1 = 0$ and $B_2 = 1$ in Figure 4, then we keep $B_2 = 1$ and take $B_1 = 0.1$ in Figure 5. Comparing these two figures, we can see larger B_1 causes the the optimal harvesting to decrease, which can be seen from (34). When the fish density is too low in a neighborhood of the boundary, (34) indicates the harvesting to be 0, which can also be seen from Figure 1(b).

In Figure 6, we still have $B_1 = 0, B_2 = 1$ but a bigger domain size. Comparing with Figure 4, as domain size increases, the fish density



(a) Fish density u (b) Optimal harvesting h

FIGURE 3. Fish density and optimal harvesting: $B_1 = 0$, $B_2 = 0.03$, $r = 5$, $L = 2.5$.

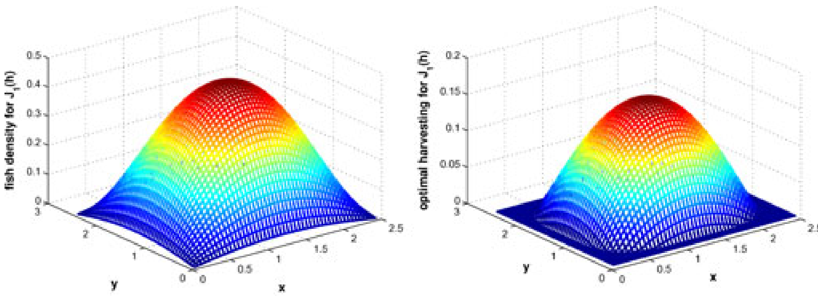


(a) Fish density u (b) Optimal harvesting h

FIGURE 4. Fish density and optimal harvesting: $B_1 = 0, B_2 = 1, r = 5, L = 2.5$.

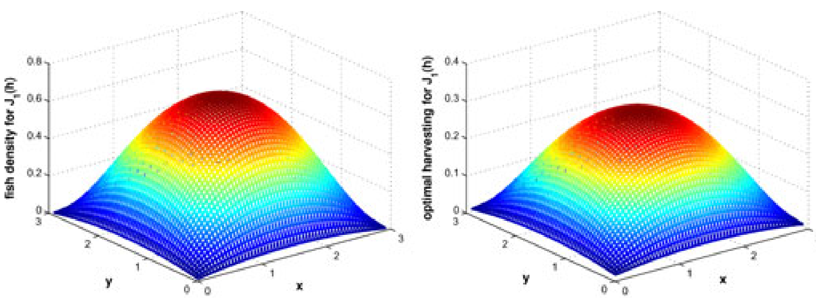
and the optimal harvesting both increase. We calculated the J_1 value in each case, giving 1.78 and 6.22 for the smaller and the larger domain respectively. Montero [2001] studied a control problem of a biological growing species in a bounded domain, modeled by a logistic elliptic equation with Dirichlet boundary condition and a payoff-cost functional of quadratic type. He showed mathematically the optimal benefit increases when the domain increases (in case $B_1 = 0$), and here we verified this property numerically.

7. Second objective functional. In this section, we will deal with the second objective functional (5). We will show the existence



(a) Fish density u (b) Optimal harvesting h

FIGURE 5. Fish density and optimal harvesting: $B_1 = 0.1$, $B_2 = 1$, $r = 5$, $L = 2.5$.



(a) Fish density u (b) Optimal harvesting h

FIGURE 6. Fish density and optimal harvesting: $B_1 = 0, B_2 = 1, r = 5, L = 3$.

of an optimal control, derive the characterization of an optimal control, and prove the uniqueness. Finally we will give some numerical examples.

7.1 Existence of an optimal control for J_2

Theorem 7.1. *There exists an optimal control $h^* \in U_2$ maximizing the objective functional $J_2(h)$.*

Proof. The proof is similar to Theorem 3.1, using a maximizing sequence argument. But in addition to an H_0^1 a priori estimate for u, we also need an H_0^1 estimate for $h \in U_2$.

Because $\{J_2(h) \mid h \in U_2\}$ is bounded above by $h_{\max}\,|\Omega|$, there exists a maximizing sequence $\{h^n\}$ such that

$$\lim_{n\to\infty} J_2(h^n) = \sup_{h\in U_2} J_2(h),$$

where

$$J_2(h^n) = \int_\Omega h^n u^n - A|\nabla h^n|^2\,dx.$$

Notice the L^∞ bounds on the states and the controls give the bounds, $\int_\Omega h^n u^n\,dx$ so

$$\int_\Omega |\nabla h^n|^2\,dx \le C_1,$$

which implies the H_0^1 boundedness of the h^n sequence.

Thus, there exist h^*, $u^* \in H_0^1(\Omega)$, such that on a subsequent $u^n \rightharpoonup u^*$ and $h^n \rightharpoonup h^*$, both weakly in $H_0^1(\Omega)$, and the limits satisfy $u^* = u(h^*)$. There is a slight change when we need to verify h^* is an optimal control, that is,

$$J_2(h^*) \ge \sup_{h\in U_2} J_2(h). \tag{53}$$

$$\begin{aligned}\sup_{h\in U_2} J_2(h) &= \lim_{n\to\infty} J_2(h^n) \\ &= \lim_{n\to\infty}\int_\Omega h^n u^n - A|\nabla h^n|^2\,dx \\ &\le \int_\Omega h^* u^*\,dx - \underline{\lim}_{n\to\infty}\int_\Omega A|\nabla h^n|^2\,dx \\ &\le \int_\Omega h^* u^*\,dx - \int_\Omega A|\nabla h^*|^2\,dx = J_2(h^*).\end{aligned} \tag{54}$$

where we used (5) and lower semicontinuity of $\nabla\, h^n$ in L^2 norm with respect to weak convergence.

7.2 Derivation of optimality system for J_2. We provide a similar analysis for the sensitivity and adjoint equation. Because the sensitivity equation does not depend on the form of the objective functional, Lemma 4.1 is valid for this problem. The nonhomogeneous term of the adjoint only comes from the dependence of the objective functional on the state, which is the same in J_1 and J_2. But we have a different characterization for an optimal control $h \in U_2$.

When we differentiate the map, $u \to \mathrm{J}_2(\mathrm{u})$, the gradient of the control will be in the resulting inequality, and thus we do not just get a characterization of u with bounds. We obtain a characterization of u and its derivatives with the bounds on the controls taken into account. Thus our characterization will be a variational inequality.

To clarify the characterization of our optimal control in U_2, we make the following definition involving a variational inequality with upper and lower obstacles (Chipot [1984], Kinderlehrer and Stampacchia [2000]).

Definition 7.2. *A function $h \in U_2$ is a weak solution of the following variational inequality (VI)*

$$\min\{\max(pu - u - 2A\Delta h, h - h_{\max}), h - 0\} = 0,$$

if for all $v_0 \in U_2$,

$$\int_\Omega 2A\nabla h \cdot \nabla(v_0 - h) + (pu - u)(v_0 - h)\, dx \geq 0. \tag{55}$$

Now we derive the characterization of an optimal control using this VI.

Theorem 7.3. *For an optimal control h* in $U_2, \sigma_1(-r + h) < 0$, *there exists a solution p in $H^2(\Omega) \cap H_0^1(\Omega)$ to the adjoint problem*

$$\begin{cases} -\Delta p - r(1 - 2u)p + hp = h, & x \in \Omega, \\ p = 0, & x \in \partial\Omega, \end{cases} \tag{56}$$

Furthermore $h(x)$ satisfies the following variational inequality (VI)

$$\min\{\max(pu - u - 2A\Delta h, h - h_{\max}), h - 0\} = 0. \tag{57}$$

Proof. Suppose $h(x)$ is an optimal control. Let $l \in H_0^1(\Omega)$ such that $h + \epsilon l \in U_2$ for small $\epsilon > 0$. The derivative of $J_2(h)$ with respect to h in the direction of l satisfies

$$
\begin{aligned}
(58)\quad 0 &\geq \lim_{\epsilon \to 0^+} \frac{J_2(h+\epsilon l) - J_2(h)}{\epsilon} \\
&= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\left[\int_\Omega (h+\epsilon l) u^\epsilon \, dx - A \int_\Omega |\nabla (h+\epsilon l)|^2 \, dx \right. \\
&\quad \left. - \left(\int_\Omega hu \, dx - \int_\Omega A |\nabla h|^2 \, dx \right) \right] \\
&= \lim_{\epsilon \to 0^+} \int_\Omega \left(h \frac{u^\epsilon - u}{\epsilon} + l u^\epsilon \right) dx - A \int_\Omega \left(2 \nabla h \cdot \nabla l + \epsilon |\nabla l|^2 \right) dx \\
&= \int_\Omega (h\psi + lu) \, dx - 2A \int_\Omega \nabla h \cdot \nabla l \, dx,
\end{aligned}
$$

where ψ is the sensitivity from Lemma 4.1.

Using similar arguments as Theorem 4.2, we have the existence and uniqueness of the solution of the adjoint problem. Let p in $H^2(\Omega) \cap H_0^1(\Omega)$ be the solution to adjoint problem (56), then we have

(59)

$$
\begin{aligned}
0 &\geq \int_\Omega [\psi(-\Delta p - r(1-2u)p + hp) + lu] \, dx - 2A \int_\Omega \nabla h \cdot \nabla l \, dx \\
&= \int_\Omega [\nabla p \nabla \psi + p(-r(1-2u)\psi + h\psi) + lu] \, dx - 2A \int_\Omega \nabla h \cdot \nabla l \, dx.
\end{aligned}
$$

Using the sensitivity PDE (19) (with h as h_0), our inequality becomes

$$
(60)\qquad 0 \geq \int_\Omega (-plu + lu) \, dx - 2A \int_\Omega \nabla h \cdot \nabla l \, dx.
$$

On the set where $h = 0$, we choose variation l with support on this set and $l \geq 0$, which implies that $pu - u - 2A\Delta h \geq 0$ in an appropriate weak H^1 sense. Where $0 < h < h_{\max}$, we can take l to have arbitrary sign, that is, $pu - u - 2A\Delta h = 0$ a.e. Where $h = h_{\max}$, the variation $l \leq 0$ implies $pu - u - 2A\Delta h \leq 0$.

The characterization of h can be written in the compact form as

$$\min\{\max(pu - u - 2A\Delta h, h - h_{\max}), h - 0\} = 0, \tag{61}$$

which is interpreted as the weak solution to the VI given in Definition (54).

The state equation (1), adjoint equation (56) together with the characterization of the optimal control (57) is called optimality system II (OS_2), which is given by

$$\begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega; \\ -\Delta p - r(1-2u)p + hp = h, & x \in \Omega, \\ p = 0, & x \in \partial\Omega; \\ \min\{\max(pu - u - 2A\Delta h, h - h_{\max}), h - 0\} = 0. \end{cases} \tag{62}$$

We note that the weak solutions of (OS_2) exist by Theorems 7.1 and 7.3.

7.3 Uniqueness of optimality system for J_2. The adjoint equation (56) is the same as (27), so p is L^∞ bounded. Due to VI characterization of an optimal control, the estimate of the L^∞ norm of p in terms of A is more delicate than the estimate in Lemma 5.1.

Lemma 7.4. *Given u, p, h solving (62) with u positive in Ω, for $n =$ 1, 2, 3, the adjoint p satisfies*

$$||p||_{L^\infty(\Omega)} \leq \frac{C_5}{A}, \tag{63}$$

where C_5 doesn't depend on A.

Proof. For $h \in U_2$, the RHS of adjoint equation, $(r - h - 2ru)p + h$, in (62) is in $H_0^1(\Omega)$; by a standard elliptic regularity result Evans [1998], $p \in H^3(\Omega)$. From (41), we have an L^2 estimate on p independent of A. Now we use an L^2 estimate on an optimal h to refine our L^2 estimate of p, which will be used to obtain an estimate of $||p||_{L^\infty}$ in terms of A.

From standard approximation of a variational inequality by a semilinear approximating equation (Bergounioux and Lenhart [2004], Kinderlehrer and Stampacchia [2000]), we obtain

$$2A \int_\Omega |\nabla h|^2 \, dx \leq \int_\Omega (pu - u)h, \tag{64}$$

which gives

$$A \int_\Omega |\nabla h|^2 \, dx \leq \frac{C_P}{4A} \int_\Omega (pu - u)^2 \, dx \leq \frac{C}{A}, \tag{65}$$

where C is independent of A because the L^∞ norm of u and the L^2 norm of p are independent of A. By Poincaré's inequality,

$$\int_\Omega h^2 \, dx \leq \frac{C_1}{A^2}, \tag{66}$$

with C_1 depending on C and C_P. We use this L^2 estimate of h to refine the L^2 estimate of p. From (40),

$$c_2 \int_\Omega |\nabla p|^2 \, dx \leq \int_\Omega hp \, dx$$

implies

$$\int_\Omega |\nabla p|^2 \, dx \leq \frac{C_P C_1}{c_2^2 A^2}, \tag{67}$$

which gives

$$\int_\Omega p^2 \, dx \leq C_P \int_\Omega |\nabla p|^2 \, dx \leq \frac{C_2}{A^2}, \tag{68}$$

where C_2 depends on the constants from (67). Similar to (42), using the RHS of the adjoint is H^1 and that p and u are L^∞ bounded, we obtain

$$||p||_{H^3(\Omega)} \leq C_3 \big(||p||_{H^1(\Omega)} + ||h||_{H^1(\Omega)} \big) \leq \frac{C_4}{A}. \tag{69}$$

From the Sobolev embedding theorem (Li and Yong [1995]), for $n = 1, 2, 3,\ H^3(\Omega) \subset\subset C(\bar{\Omega})$, we have

$$||p||_{L^\infty(\Omega)} \leq C_6 ||p||_{H^3(\Omega)} \leq \frac{C_5}{A}. \tag{70}$$

Theorem 7.5. *If A is sufficiently large, the dimension $n = 1$, 2, 3, then the solution of the optimality system (OS_2) with positive u component is unique.*

Proof. Suppose u, p, h and $\bar{u}, \bar{p}, \bar{h}$ are two solutions of (OS_2). In (55), choosing $v_0 = \bar{h}$, we have

$$\int_\Omega 2A\nabla h \cdot \nabla(\bar{h} - h) + (pu - u)(\bar{h} - h)\,dx \geq 0. \tag{71}$$

In (55), replacing h by $\bar{h}$ and choosing $v_0 = h$, similarly gives

$$\int_\Omega 2A\nabla \bar{h} \cdot \nabla(h - \bar{h}) + (\bar{p}\bar{u} - \bar{u})(h - \bar{h})\,dx \geq 0. \tag{72}$$

Adding (71) and (72) gives

$$-2A\int_\Omega |\nabla(h - \bar{h})|^2\,dx + \int_\Omega (\bar{p}\bar{u} - pu + u - \bar{u})(h - \bar{h})\,dx \geq 0,$$

which rearranges to

$$\begin{aligned}\int_\Omega 2A|\nabla(h - \bar{h})|^2 + [(p - \bar{p})\bar{u} + p(u - \bar{u})](h - \bar{h})\,dx \\ \leq \int_\Omega (u - \bar{u})(h - \bar{h})\,dx.\end{aligned} \tag{73}$$

Adding (73), (48), and (49), we obtain

$$\begin{aligned}\int_\Omega |\nabla(u - \bar{u})|^2\,dx + \int_\Omega \left(-r + h + r(u + \bar{u})\right)(u - \bar{u})^2\,dx \\ + \int_\Omega |\nabla(p - \bar{p})|^2\,dx + \int_\Omega (-r + \bar{h} + 2\bar{u}r)(p - \bar{p})^2\,dx \\ + 2A\textstyle\int_\Omega |\nabla(h - \bar{h})|^2\,dx \\ \leq \int_\Omega -\bar{u}(u - \bar{u})(h - \bar{h})\,dx - \int_\Omega 2r(u - \bar{u})p(p - \bar{p})\,dx\end{aligned} \tag{74}$$

$$
\begin{aligned}
&- \int_\Omega (h-\bar h)p(p-\bar p)\,dx + \int_\Omega (h-\bar h)(p-\bar p)\,dx \\
&- \int_\Omega (p-\bar p)\bar u(h-\bar h)\,dx - \int_\Omega p(u-\bar u)(h-\bar h)\,dx \\
&+ \int_\Omega (u-\bar u)(h-\bar h)\,dx.
\end{aligned}
$$

Because $u, \bar u > 0$, and $u, \bar u$ satisfy state equation in (62), these eigenvalue inequalties result in

$$
\begin{aligned}
&\sigma_1(-r+h+r(u+\bar u)) > \sigma_1(-r+h+ru) = 0, \\
&\sigma_1(-r+\bar h+2\bar u r) > \sigma_1(-r+\bar h+r\bar u) = 0.
\end{aligned}
$$

Using property 3 of $\sigma_1(q)$, $0 \le u, \bar u \le 1$, and $\epsilon-$Cauchy inequality, there exists $c_3 > 0$, such that

(75)

$$
\begin{aligned}
c_3 &\left(\int_\Omega |\nabla(u-\bar u)|^2\,dx + \int_\Omega |\nabla(p-\bar p)|^2\,dx \right) + 2A \int_\Omega |\nabla(h-\bar h)|^2\,dx \\
&\le \int_\Omega |\nabla(u-\bar u)|^2\,dx + \int_\Omega \left(-r+h+r(u+\bar u)\right)(u-\bar u)^2\,dx \\
&\quad + \int_\Omega |\nabla(p-\bar p)|^2\,dx + \int_\Omega (-r+\bar h+2\bar u r)(p-\bar p)^2\,dx \\
&\quad + 2A \int_\Omega |\nabla(h-\bar h)|^2\,dx \\
&\le 2 \int_\Omega \left(\epsilon(u-\bar u)^2 + \frac{1}{4\epsilon}(h-\bar h)^2 \right) dx \\
&\quad + r||p||_{L^\infty} \int_\Omega \left((u-\bar u)^2 + (p-\bar p)^2\right) dx \\
&\quad + \frac{||p||_{L^\infty}}{2} \int_\Omega \left((h-\bar h)^2 + (p-\bar p)^2\right) dx \\
&\quad + 2 \int_\Omega \left(\epsilon(p-\bar p)^2 + \frac{1}{4\epsilon}(h-\bar h)^2 \right) dx \\
&\quad + \frac{||p||_{L^\infty}}{2} \int_\Omega \left((u-\bar u)^2 + (h-\bar h)^2\right) dx.
\end{aligned}
$$

Using (63) and Poincaré's inequality, the estimates simplify to

(76)

$$
\begin{aligned}
c_3 &\left(\int_\Omega |\nabla(u-\bar{u})|^2\,dx + \int_\Omega |\nabla(p-\bar{p})|^2\,dx\right) + 2A\int_\Omega |\nabla(h-\bar{h})|^2\,dx \\
&\leq \left(2\epsilon + r\frac{C_5}{A} + \frac{C_5}{2A}\right) C_P \int_\Omega |\nabla(u-\bar{u})|^2\,dx \\
&\quad + \left(r\frac{C_5}{A} + \frac{C_5}{2A} + 2\epsilon\right) C_P \int_\Omega |\nabla(p-\bar{p})|^2\,dx \\
&\quad + \left(\frac{1}{\epsilon} + \frac{C_5}{A}\right) C_P \int_\Omega |\nabla(h-\bar{h})|^2\,dx.
\end{aligned}
$$

If we choose $\epsilon = \frac{c_3}{4C_P}$ and A is sufficiently large, such that

$$c_3 > (2r+1)\frac{C_5 C_P}{A} \text{ and } 2A > \frac{C_5 C_P}{A} + \frac{4C_P^2}{c_3},$$

then we have $u = \bar{u},\ p = \bar{p}, h = \bar{h}$, that is, we have the uniqueness of OS_2, which implies the uniqueness of the optimal control.

7.4 Numerical examples for J_2. We need to solve the optimality system II (62) numerically. The variational inequality characterization of the optimal control is equivalent to the following minimization problem (Glowinski et al. [1981], Joshi et al. [2005]):

$$\min_{0\leq h\leq h_{\max}} \frac{1}{2}\int_\Omega 2A|\nabla h|^2\,dx - \int_\Omega (u-pu)h\,dx,\ \ h\in H_0^1(\Omega). \tag{77}$$

In the following, we set $A_1 = 2A$ and $f = u - p\ u$. We solve (62) by an iterative method that is implemented using MATLAB:

(i) **Initialization:** Choose initial guesses for fish density u_0 and harvest h_0;

(ii) **Iteration:** h_n is known.

1. Solve the state equation in (62) for u,

2. Solve the adjoint equation in (62) for p,
3. Update the control by solving the minimization problem (77);

(iii) **Repeat** step 3 if successive iterates are not sufficiently close.

Steps 1 and 2 are completed as in Section 6. A central difference scheme is used to discretize the Laplacian operator. A convex combination between the previous control values and values given by the current characterization is used in updating h.

To solve the variational inequality in (62), we discretize the energy functional (77) with the trapezoidal integration rule and use the steepest descent method to solve it. Indeed the functional is quadratic and constraints are bounded constraints. The discretized problem turns out to be

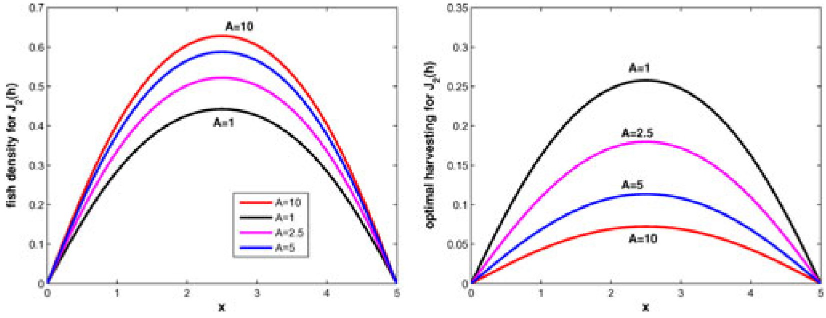
$$\min \frac{1}{2} X^T M X - F^T X, \; 0 \le X_i \le h_{\max}, \; i = 1, \ldots, N, \tag{78}$$

where $X = \{h(x_i)\}_{i=1}^N$ is the discretized control function, X^T denotes the transpose of X, and F is the discretized vector for $f = u - pu$. Also, M is the discretized 1-D Laplacian matrix:

$$M = \frac{A_1}{(\Delta x)^2} \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & -1 & 2 \end{bmatrix}. \tag{79}$$

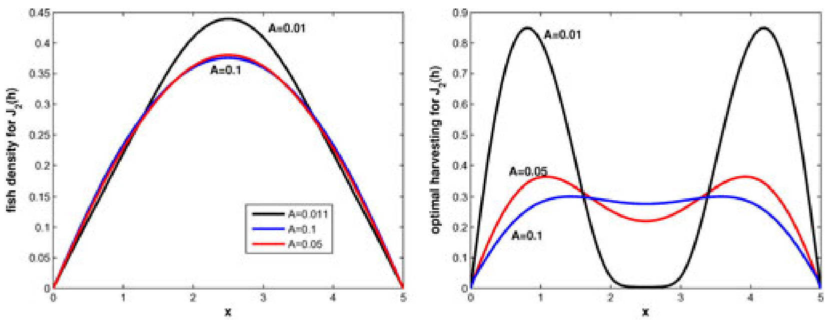
We give numerical examples for 1-D case, the interval length is 5. In Figure 7, we vary $A = 1$, 2.5, 5, 10, and set $r = 1$, $h_{\max} = 0.99$. We observe that increasing the variation in fishing effort will reduce optimal harvesting, and the corresponding fish density is increasing.

In Figure 8, we choose small $A = 0.1$, 0.05, 0.01, and set $r = 1$, $h_{\max} = 0.99$. We can see similar scenarios as in $J_1(h)$ when B_2 is small. Again there will be a reserve in the center of the habitat for optimal harvesting.



(a) Fish density u (b) Optimal harvesting h

FIGURE 7. Fish density and optimal harvesting: $r = 1, A = 1,\ 2.5,\ 5,\ 10.$



(a) Fish density u (b) Optimal harvesting h

FIGURE 8. Fish density and optimal harvesting: $r = 1, A = 0.1,\ 0.05,\ 0.01.$

8. Conclusion. We summarize our conclusions:

(i) If we want to maximize yield and minimize cost (J_1), then increasing the cost coefficients, B_1 or B_2, will decrease optimal harvesting;

(ii) With small B_1 and B_2, the harvest control is concentrated near the boundary;

(iii) If we only want to maximize yield, then a reserve is part of the optimal harvesting strategy;

(iv) The problem of maximizing yield only (J_1) with Neumann boundary condition gives a simple optimal control, a singular case;

(v) For J_1, the optimal benefit increases when domain size increases;

(vi) If we want to maximize yield and minimize variation in fishing effort, then increasing the variation coefficient A will reduce optimal harvesting.

Our results also illustrate how to incorporate "low variation" together with a goal of maximizing the yield. As the cost of implementing the control, we used an H^1 norm to minimize the variation in the fishing effort. This type of objective functional and control set leads to a variational inequality as the control characterization instead of the usual algebraic characterization. In both cases we completely characterized the optimal control in terms of the optimality system and implemented an iterative numerical scheme to illustrate the optimal harvesting strategy.

The time dependent problem can be tackled in the future and in that case the nonuniqueness of the state solution is not an issue.

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