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Optimal control of growth coefficient on a steady-state population model

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ABSTRACT

We study the control problem of maximizing the net benefit in the conservation of a single species with a fixed amount of resources. The existence of an optimal control is established and the uniqueness and characterization of the optimal control are investigated. Numerical simulations illustrate several cases, for both 1D and 2D domains, in which several interesting phenomena are found. Some open problems are discussed.

1. Introduction

How resource allocation affects the population dynamics of species remains an important issue in conservation biology. For instance, given a fixed amount of resources, how can we determine the optimal spatial arrangement of the favorable and unfavorable parts of the habitat for species to survive? This question was first addressed by Cantrell and Cosner [1,2] via the reaction–diffusion equation

$$u_t = \lambda \Delta u + m(x)u - u^2 \quad \text{in } \Omega,$$

subject to Dirichlet, Robin, or Neumann boundary conditions, where $u(x,t)$ is the density of the species at location x and time t, and the constant λ is the dispersal rate of the species and is assumed to be a positive constant. The coefficient $m(x)$ represents the intrinsic growth rate of the species and it measures the availability of the resources. Throughout this paper we will focus on the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

which means that the individuals do not cross the boundary, where $\partial\Omega$ stands for the boundary of the habitat Ω, and n is the outward unit normal vector on $\partial\Omega$.

Among other things, Cantrell and Cosner [1] showed that there exists a "bang-bang" type optimal spatial arrangement of the favorable and unfavorable parts of the habitat for species to survive, i.e., the corresponding optimal control function $m(x)$ must be a step function in Ω. Determining the exact shape of the optimal control is a much more challenging mathematical

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problem. When Ω is an interval, Cantrell and Cosner [2] showed that if the resource is so arranged that $m(x)$ is equal to some positive constant in one subinterval and is equal to some negative constant otherwise, then the optimal arrangement occurs when the subinterval with $m(x)$ positive is one of the two ends of the interval. Recently, it was further shown in [3] that any optimal control function m must be of "bang-bang" type and when the domain Ω is an interval, there are exactly two optimal controls, for which the control $m(x)$ is positive at one end of the interval and is negative in the remainder. The biological implication is that a single favorable region at one end of the habitat provides the best opportunity for the species to survive. For high-dimensional habitats, very little is known about the exact shape of the optimal control, and we refer to [4] for some recent analytical and numerical results in this direction.

From a biological point of view, it is more interesting to know how resource allocation affects population size of the species since the population abundance is clearly a good measurement of conservation effort. To this end, we consider the following control problem. Given $0 < \delta < |\Omega|$, define the control set

$$U = \left\{ m \in L^\infty(\Omega) \mid 0 \le m(x) \le 1, \int_\Omega m(x)\mathrm{d}x = \delta \right\}.$$

We seek to find $m^* \in U$, such that $J(m^*) = \max_m J(m)$, where the objective functional is defined by

$$J(m) = \int_\Omega [u - (Bm^2)]\mathrm{d}x, \tag{1}$$

subject to the state equation and boundary condition

$$\begin{cases} -\lambda \Delta u = mu - u^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where the state $u \in H^1(\Omega)$. The objective functional represents the net benefit, which is the size of the population less the cost of implementing the control (the cost of making the environment favorable to change the intrinsic growth rate). The coefficient $B \ge 0$ is the parameter which balances the two parts of the objective functional.

The first term in $J(m)$, i.e. $\int_\Omega u\mathrm{d}x$, represents the total population of the species, which not only serves as a good measurement for the conservation of a single species, but also plays an important role in preventing the invasion of alien species [5]. Roughly speaking, if a resident species has a high population size, it is usually harder for other species to invade its habitat. The second term $\int_\Omega Bm^2$ is a measurement of the cost of distributing the resources in the habitat. As a whole, $J(m)$ can be regarded as a way of determining the net benefit in the conservation of a single species with a fixed amount of resources.

By the super–sub-solution method and standard elliptic regularity [6], it is well known that if $m \in U$, then for every $\lambda > 0$, (2) has a unique positive solution $u \in W^{2,p}(\Omega)$ for every $p > 1$. We refer to Section 2 for more details. It is easy to show that $\|u(x)\|_{L^\infty(\Omega)} \le \|m(x)\|_{L^\infty(\Omega)}$, e.g. by the maximum principle [7]. Moreover, $u \to m$ in $L^p(\Omega)$ for every $p > 1$ as $\lambda \to 0$, and $u \to \delta/|\Omega|$ in $W^{2,p}(\Omega)$ for every $p > 1$ as $\lambda \to \infty$ [5].

There also has been some related work done on optimal control of elliptic PDEs. Much of the biological applications have focused on harvesting, but we refer the reader to [8] for general results for optimal control of elliptic equations. We note that the harvesting problems are related to the goal here since maximizing the yield relates to keeping the population high for harvesting. Leung and Stojanovic [9] studied the optimal harvesting control of a biological species, whose growth is governed by the diffusive Volterra–Lotka equation. The species concentration satisfied a steady-state equation with no-flux (Neumann) boundary condition. Leung [10] also studied the corresponding optimal control problem for steady-state prey–predator diffusive Volterra–Lotka systems and obtained similar results to the single equation case [9]. Cañada et al. [11] and Montero [12] studied an optimal control problem for a nonlinear elliptic equation of the Lotka–Volterra type with Dirichlet boundary condition. Shi and Kurata [13] studied a reaction–diffusion model with logistic growth and constant effort harvesting. By minimizing an intrinsic biological energy function which is different from the yield, they obtained an optimal spatial harvesting strategy which would benefit the population. Ding and Lenhart [14] considered an optimal fishery harvesting problem using a spatially explicit model with a semilinear elliptic PDE, Dirichlet boundary conditions and logistic population growth. They considered two objective functionals: maximizing the yield and minimizing the cost or the variation in the fishing effort (control). Minimizing variation was considered to avoid the "chattering" effect in Neubert's results in one spatial dimension [15]. The optimal control when minimizing the variation is characterized by a variational inequality instead of the usual algebraic characterization, which involves the solutions of an optimality system of nonlinear elliptic PDEs. For optimal control of competitive systems, see [16,17].

This paper is organized as follows: Section 2 is concerned with the uniqueness and existence of positive solutions to (2). In Section 3 we establish the existence of an optimal control for $J(m)$. Section 4 is devoted to the sensitivity function, which is the Gateaux derivative of the control-to-state map and is used in deriving the necessary conditions. In Section 5, we discuss the uniqueness and characterization of optimal control for large and small B. We further investigate the characterization of the optimal control in Section 6 for general domains and in Section 7 for 1D habitat. In both of these sections, an extra state variable is introduced to handle the integral constraint on the controls. Some interesting numerical results for the optimal control will be presented in Section 8. Finally in Section 9 we give a discussion of both of our results and numerous open problems.

2. Existence and uniqueness of a positive state solution

We present some results from [11,18] to guarantee the existence and uniqueness of a positive solution to (2). For a function $q \in L^\infty(\Omega)$, we define $\sigma_1(q, \lambda)$ to be the principal eigenvalue of the eigenvalue problem

$$\begin{aligned} &-\lambda \Delta u(x) + q(x)u(x) = \sigma u(x), \quad x \in \Omega; \\ &\frac{\partial u}{\partial n}(x) = 0, \quad x \in \partial\Omega. \end{aligned} \tag{3}$$

This principal eigenvalue can be expressed as

$$\sigma_1(q, \lambda) = \inf_{\substack{\phi \in H^1(\Omega) \\ \phi \not\equiv 0}} \frac{\int_\Omega \lambda |\nabla \phi|^2 \, dx + \int_\Omega q\phi^2 \, dx}{\int_\Omega \phi^2 \, dx}. \tag{4}$$

It is known that the algebraic multiplicity of $\sigma_1(q, \lambda)$ is equal to one and the associated eigenfunction is positive. We note the following properties for $\sigma_1(q, \lambda)$.

1. $\sigma_1(q, \lambda)$ is increasing with respect to q, i.e. if $q_1 < q_2$, then $\sigma_1(q_1, \lambda) < \sigma_1(q_2, \lambda)$;
2. $\sigma_1(q, \lambda)$ is continuous with respect to $q \in L^\infty(\Omega)$; also $\sigma_1(q, \lambda)$ depends continuously on q with respect to $L^p(\Omega),\ p < \infty$;
3. if $\sigma_1(q, \lambda) > 0$, then there exists $c_\lambda > 0$, such that
$$c_\lambda \lambda \int_\Omega |\nabla \phi|^2 \, dx \le \int_\Omega \lambda |\nabla \phi|^2 \, dx + \int_\Omega q\phi^2 \, dx.$$

We can show this easily by noting that if we take $0 < c_\lambda \le \frac{\sigma_1(q,\lambda)}{\sigma_1(q,\lambda)+\|q\|_\infty}$, then $0 < c_\lambda < 1$, and

$$\begin{aligned} (1 - c_\lambda)\left(\int_\Omega \lambda |\nabla \phi|^2 \, dx + \int_\Omega q\phi^2 \, dx\right) &\ge (1 - c_\lambda)\sigma_1(q, \lambda) \int_\Omega \phi^2 \, dx \\ &\ge c_\lambda \|q\|_\infty \int_\Omega \phi^2 \, dx \ge -c_\lambda \int_\Omega q\phi^2 \, dx. \end{aligned}$$

Property 3 follows immediately. Moreover, if there exist two positive constants M and μ_λ, such that

$$\|q\|_\infty \le M, \qquad \sigma_1(q, \lambda) \ge \mu_\lambda,$$

then the constant c_λ may be chosen independent of q. We can take $c_\lambda = \frac{\mu_\lambda}{\mu_\lambda + M}$, since

$$\frac{\mu_\lambda}{\mu_\lambda + M} \le \frac{\sigma_1(q, \lambda)}{\sigma_1(q, \lambda) + \|q\|_\infty}.$$

Finally, note that if $\int_\Omega m \, dx > 0$ and we take $\phi = 1$ in (4), we get $\sigma_1(-m, \lambda) < 0$ which implies $u = u(m, \lambda) > 0$. In fact, $\sigma_1(-q, \lambda) < 0$ if and only if there is a unique positive solution to (3) [18]. Hence for every $m \in U$, (3) has a unique positive solution up to a multiple.

From here on, we assume that $\lambda > 0$ is fixed, and we indicate only the dependence of the positive solution u to (2) on the control m by writing $u = u(m)$.

3. Existence of an optimal control

First we prove the existence of an optimal control for our objective functional.

Theorem 3.1. *There exists an optimal control $m^* \in U$ maximizing the objective functional $J(m)$.*

Proof. Since $J(m) \le C$ where C is a constant, we can choose a maximizing sequence $\{m^n\} \subset U$, s.t.

$$\lim_{n\to\infty} J(m^n) = \sup_{m \in U} J(m). \tag{5}$$

We can obtain an *a priori* estimate for u. If we let $u^n = u(m^n)$, and take $v = u^n$ as the test function in (1), then we have

$$\lambda \int_\Omega |\nabla u^n|^2 \, dx = \int_\Omega m^n (u^n)^2 - (u^n)^2 \, dx \le \int_\Omega m^n (u^n)^2 \, dx \le \delta, \tag{6}$$

since $\|u\|_{L^\infty} \le \|m\|_{L^\infty} \le 1$ and $m^n \in U$. So we have

$$\|u^n\|_{H^1(\Omega)} \le C. \tag{7}$$

Then there exists u^* in $H^1(\Omega)$ such that on a subsequence, $u^n \rightharpoonup u^*$ weakly in $H^1(\Omega)$. Since $H^1(\Omega) \subset\subset L^2(\Omega)$, we obtain

$$u^n \longrightarrow u^* \quad \text{strongly in } L^2(\Omega)$$

and $0 \le \|u^*\|_{L^\infty} \le \|m\|_{L^\infty} \le 1$ [22,23].

Notice that the sequence $\{m^n\}$ in U is uniformly bounded in $L^2(\Omega)$, so on an appropriate subsequence,

$$m^n \rightharpoonup m^* \quad \text{weakly in } L^2(\Omega).$$

Next we need to prove $u^* = u(m^*)$. The weak formulation of (1) for u^n gives

$$\lambda \int_\Omega \nabla u^n \cdot \nabla v \, dx = \int_\Omega m^n u^n v - (u^n)^2 v \, dx, \quad \forall v \in H^1(\Omega). \tag{8}$$

$$\begin{aligned} \left| \int_\Omega (u^n)^2 v - (u^*)^2 v \, dx \right| &\le \int_\Omega |u^n + u^*||(u^n - u^*)v| \, dx \\ &\le 2 \int_\Omega |(u^n - u^*)v| \, dx \longrightarrow 0, \end{aligned} \tag{9}$$

since $u^n \to u^*$ strongly in $L^2(\Omega)$. Thus, we have

$$\left| \int_\Omega (m^n u^n v - m^* u^* v) \, dx \right| \le \left| \int_\Omega m^n (u^n - u^*) v \, dx \right| + \left| \int_\Omega (m^n - m^*) u^* v \, dx \right| \longrightarrow 0, \tag{10}$$

since $m \in U$ and $u^* v \in L^2(\Omega)$.

Passing to the limit in (8), we have $u^* = u(m^*)$.

Finally, we need to verify that m^* is an optimal control, i.e.

$$J(m^*) \ge \sup_{m \in U} J(m). \tag{11}$$

This is true since

$$\begin{aligned} \sup_{m \in U} J(m) &= \lim_{n \to \infty} J(m^n) = \lim_{n \to \infty} \int_\Omega u^n - B(m^n)^2 \, dx \\ &\le \int_\Omega u^* - B(m^*)^2 \, dx = J(m^*), \end{aligned} \tag{12}$$

where we used lower semicontinuity of the objective functional with respect to weak L^2 convergence. Thus we verified (11). □

4. Sensitivity

In order to characterize the optimal control, we need to differentiate the objective functional with respect to the control m. We denote by $u = u(m)$ the unique, positive solution of (2). Since $u = u(m)$ is involved in $J(m)$, we must first prove the appropriate differentiability of the mapping $m \longrightarrow u(m)$ whose derivative is called the *sensitivity*.

Lemma 4.1 (*Sensitivity*). *Assume that for $m \in U$, the mapping $m \in U \longrightarrow u(m)$ is differentiable at m in the following sense: there exists $\psi \in H^1(\Omega)$, such that*

$$\frac{u(m + \epsilon l) - u(m)}{\epsilon} \rightharpoonup \psi \quad \text{weakly in } H^1(\Omega) \text{ as } \epsilon \to 0,$$

where $m + \epsilon l \in U,\ l \in L^\infty(\Omega)$. And the sensitivity $\psi = \psi(m, l)$ satisfies

$$\begin{aligned} -\lambda \Delta \psi &= m\psi - 2u\psi + lu, \\ \frac{\partial \psi}{\partial n} &= 0. \end{aligned} \tag{13}$$

Proof. Define $u^\epsilon = u(m + \epsilon l)$, using (2), we have

$$-\lambda \Delta u^\epsilon = (m + \epsilon l) u^\epsilon - (u^\epsilon)^2, \tag{14}$$

then subtracting (14) from (2) and dividing by ϵ, we have

$$-\lambda \Delta \frac{u^\epsilon - u}{\epsilon} = m \frac{u^\epsilon - u}{\epsilon} - \frac{(u^\epsilon)^2 - u^2}{\epsilon} - l u^\epsilon. \tag{15}$$

Multiplying both sides by $\frac{u^\epsilon-u}{\epsilon}$ and integrating in Ω, we obtain

$$\int_\Omega \lambda \left|\nabla \frac{(u^\varepsilon - u)}{\varepsilon}\right|^2 \mathrm{d}x = \int_\Omega m \left(\frac{u^\varepsilon - u}{\varepsilon}\right)^2 - \left(\frac{u^\varepsilon - u}{\varepsilon}\right)^2 (u^\varepsilon + u) + lu^\varepsilon \cdot \frac{u^\varepsilon - u}{\varepsilon}\mathrm{d}x. \tag{16}$$

Recall

$$\sigma_1(-m, \lambda) = \inf_{\substack{\phi\in H^1(\Omega)\\ \phi\not\equiv 0}} \frac{\int_\Omega \lambda|\nabla\phi|^2 - m\phi^2 \mathrm{d}x}{\int_\Omega \phi^2 \mathrm{d}x}, \tag{17}$$

notice

$$\sigma_1(-m + u^\varepsilon + u, \lambda) > \sigma_1(-m + u, \lambda) = 0, \tag{18}$$

since $u^\epsilon > 0$ and $\sigma_1(q, \lambda)$ is an increasing function of q. Then using Property 3 of σ_1, there exists $c_\lambda > 0$ such that

$$c_\lambda\lambda \int_\Omega \left|\nabla \frac{u^\varepsilon - u}{\varepsilon}\right|^2 \mathrm{d}x \le \lambda \int_\Omega \left|\nabla \frac{u^\varepsilon - u}{\varepsilon}\right|^2 \mathrm{d}x + \int_\Omega (-m + u^\varepsilon + u) \left(\frac{u^\varepsilon - u}{\varepsilon}\right)^2 \mathrm{d}x \tag{19}$$

$$= \int_\Omega lu^\varepsilon \frac{u^\varepsilon - u}{\varepsilon}\mathrm{d}x. \tag{20}$$

In the following we will get an *a priori* estimate of $\|\frac{u^\epsilon-u}{\epsilon}\|_{L^2}$. Recall

$$\sigma_1(-m + u^\epsilon + u, \lambda) = \inf_{\substack{\phi\in H^1(\Omega)\\ \phi\not\equiv 0}} \frac{\int_\Omega \lambda|\nabla\phi|^2 \mathrm{d}x + \int_\Omega(-m + u^\epsilon + u)\phi^2 \mathrm{d}x}{\int_\Omega \phi^2 \mathrm{d}x}. \tag{21}$$

Taking $\phi = \frac{u^\epsilon-u}{\epsilon}$, we have

$$\sigma_1(-m + u^\epsilon + u, \lambda) \le \frac{\int_\Omega \lambda|\nabla \frac{u^\varepsilon-u}{\epsilon}|^2 \mathrm{d}x + \int_\Omega(-m + u^\epsilon + u)\left(\frac{u^\epsilon-u}{\epsilon}\right)^2 \mathrm{d}x}{\int_\Omega \left(\frac{u^\epsilon-u}{\varepsilon}\right)^2 \mathrm{d}x}, \tag{22}$$

i.e.

$$\sigma_1(-m + u^\epsilon + u, \lambda) \int_\Omega \left(\frac{u^\varepsilon - u}{\varepsilon}\right)^2 \mathrm{d}x \le \int_\Omega \lambda \left|\nabla \frac{u^\varepsilon - u}{\varepsilon}\right|^2 + (-m + u^\varepsilon + u) \left(\frac{u^\varepsilon - u}{\varepsilon}\right)^2 \mathrm{d}x$$

$$= \int_\Omega lu^\varepsilon \frac{u^\varepsilon - u}{\varepsilon} \mathrm{d}x, \tag{23}$$

where we used (16). Then this gives

$$0 < \sigma_1(-m + u^\epsilon + u, \lambda) \int_\Omega \left(\frac{u^\epsilon - u}{\epsilon}\right)^2 \mathrm{d}x \le \|l\|_{L^2} \|u^\epsilon\|_{L^\infty} \left(\int_\Omega \left(\frac{u^\epsilon - u}{\epsilon}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\le \|l\|_{L^2} \|m + \epsilon l\|_{L^\infty} \left(\int_\Omega \left(\frac{u^\epsilon - u}{\epsilon}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}, \tag{24}$$

which yields

$$\left\|\frac{u^\epsilon - u}{\epsilon}\right\|_{L^2} \le \frac{\|l\|_{L^2} \|m + \epsilon l\|_{L^\infty}}{\sigma_1(-m + u^\epsilon + u, \lambda)}. \tag{25}$$

Since $\lim_{\epsilon\to 0} \sigma_1(-m + u^\epsilon + u, \lambda) = \sigma_1(-m + 2u, \lambda)$ and there exists $\eta_\lambda > 0$ such that $\sigma_1(-m + 2u, \lambda) > \eta_\lambda$, we obtain

$$\left\|\frac{u^\epsilon - u}{\epsilon}\right\|_{L^2} \le \frac{\|l\|_{L^2} \|m + \epsilon l\|_{L^\infty}}{\eta_\lambda}. \tag{26}$$

This, combined with (19), gives

$$\left\|\frac{u^\epsilon - u}{\epsilon}\right\|_{H^1} \le \frac{2\|m + \epsilon l\|_\infty}{\min\{c_\lambda\lambda, \eta_\lambda\}} \|l\|_{L^2}. \quad \square \tag{27}$$

5. Uniqueness and characterization of optimal control for large and small B

5.1. Characterization of an optimal control for large B

Theorem 5.1. *If $B > \frac{|\Omega|}{2\delta}$, then $m^* = u^* = \frac{\delta}{|\Omega|}$ is an optimal control and corresponding state.*

Proof. Recall that u satisfies

$$\lambda \Delta u + u(m - u) = 0$$

with $\frac{\partial u}{\partial n} = 0$ on the boundary of Ω. Integrating the equation we obtain

$$\int_\Omega um\, \mathrm{d}x = \int_\Omega u^2\, \mathrm{d}x$$

which implies that

$$\|u\|_{L^2} \le \|m\|_{L^2}.$$

Applying Holder's inequality to the left-hand side of the above integral equation and using the last inequality above, we see that

$$\int_\Omega u \le |\Omega|^{1/2} \|m\|_{L^2}.$$

From Holder's inequality for $\int_\Omega m\, \mathrm{d}x$, we obtain the following lower bound on the L^2 norm of our controls,

$$\|m\|_{L^2} \ge \frac{\delta}{|\Omega|^{1/2}}.$$

Note that the upper bound on the L^2 norm of our controls is $|\Omega|^{1/2}$.

Now, since $J(m) = \int_\Omega (u - Bm^2)\, \mathrm{d}x$, it follows that

$$J(m) \le |\Omega|^{1/2} \|m\|_{L^2} - B\|m\|_{L^2}^2.$$

Thus we have a bound on $J(m)$ that is quadratic in $\|m\|_{L^2}$, and that bound achieves its maximum at $\|\hat{m}\|_{L^2} = \frac{|\Omega|^{1/2}}{2B}$.

For $B > \frac{|\Omega|}{2\delta}$, we have

$$\frac{\delta}{|\Omega|^{1/2}} > \frac{|\Omega|^{1/2}}{2B},$$

and there is no possible control in U which will achieve the above bound, $\frac{|\Omega|^{1/2}}{2B}$. Considering the bounds on $\|m\|_{L^2}$ for our controls, we can conclude that, $J(m)$ is maximized for $\|m\|_{L^2} = \frac{\delta}{|\Omega|^{1/2}}$, so that

$$J(m) \le \delta - B\frac{\delta^2}{|\Omega|}$$

for all possible $m \in U$. Since $m^*(x) = u^*(x) = \frac{\delta}{|\Omega|}$ achieves this bound, it will always be an optimal pair for $B > \frac{|\Omega|}{2\delta}$. □

5.2. Uniqueness for B large

The question remains — is this optimal pair unique? We show that it is for large B.

Theorem 5.2. *Let Ω be a domain in $\mathbb{R}^N$, with $N \le 4$. For B sufficiently large, the optimal control maximizing $J(m)$ is unique.*

Proof. For $m, l \in U$ and $0 \le \epsilon \le 1$, we will show that

$$g(\epsilon) = J(\epsilon l + (1 - \epsilon)m) = J(m + \epsilon(l - m))$$

is strictly concave, which implies the uniqueness of the optimal control.

Denoting $u^\epsilon = u(m + \epsilon(l - m))$ and $u^{\epsilon+\eta} = u(m + (\epsilon + \eta)(l - m))$, we obtain

$$g'(\epsilon) = \lim_{\eta \to 0} \int_\Omega \left(\frac{u^{\epsilon+\eta} - u^\epsilon}{\eta} - 2B(m + \epsilon(l - m))(l - m) - \eta B(l - m)^2 \right) \mathrm{d}x.$$

By our previous work, $\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\epsilon} \to \psi^{\epsilon}$ in L^2 where ψ^{ϵ} is the sensitivity, satisfying

$$-\lambda\Delta\psi^{\epsilon} = (m+\epsilon(l-m))\psi^{\epsilon} - 2u^{\epsilon}\psi^{\epsilon} + (l-m)u^{\epsilon} \quad \text{in } \Omega, \qquad (28)$$
$$\frac{\partial\psi^{\epsilon}}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Thus we have

$$g'(\epsilon) = \int_{\Omega} (\psi^{\epsilon} - 2B(m+\epsilon(l-m))(l-m))\ \mathrm{d}x.$$

To estimate the terms in $g''(\epsilon)$, we must estimate the quotients

$$\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta},$$

where $\psi^{\epsilon+\eta}$ satisfies

$$-\lambda\Delta\psi^{\epsilon+\eta} = (m+(\epsilon+\eta)(l-m))\psi^{\epsilon+\eta} - 2u^{\epsilon+\eta}\psi^{\epsilon+\eta} + (l-m)u^{\epsilon+\eta} \quad \text{in } \Omega, \qquad (29)$$
$$\frac{\partial\psi^{\epsilon+\eta}}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Taking the difference and simplifying, we obtain

$$\begin{aligned}-\lambda\Delta\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right) &= (m+\epsilon(l-m))\left(\frac{\psi^{\epsilon+\eta}-\psi^{\eta}}{\eta}\right) + (l-m)\psi^{\epsilon+\eta}\\ &\quad + (l-m)\left(\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right) - 2\left(\frac{u^{\epsilon+\eta}\psi^{\epsilon+\eta}-u^{\epsilon}\psi^{\epsilon}}{\eta}\right),\end{aligned}$$

and then

$$\begin{aligned}-\lambda\Delta\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right) &= (m+\epsilon(l-m)-2u^{\epsilon+\eta})\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right) + (l-m)\psi^{\epsilon+\eta}\\ &\quad + (l-m)\left(\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right) - 2\left(\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right)\psi^{\epsilon}.\end{aligned}$$

We have by Property 3 of $\sigma_1(-m-\epsilon(l-m)+2u^{\epsilon},\lambda)$, there exists $c_\lambda > 0$, such that for ϵ small enough

$$\begin{aligned}&c_\lambda\lambda\int_{\Omega}\left|\nabla\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)\right|^2\ \mathrm{d}x\\ &\le \int_{\Omega}\lambda\left|\nabla\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)\right|^2\ \mathrm{d}x + \int_{\Omega}(-m-\epsilon(l-m)+2u^{\epsilon+\eta})\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)^2\ \mathrm{d}x\\ &= \int_{\Omega}\left[(l-m)\left(\psi^{\epsilon+\eta}+\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right) - 2\psi^{\epsilon}\left(\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right)\right]\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)\ \mathrm{d}x.\end{aligned}$$

Also, since

$$\sigma_1(-m-\epsilon(l-m)+2u^{\epsilon+\eta},\lambda) = \inf_{\substack{\phi\in H^1(\Omega)\\ \phi\neq 0}} \frac{\int_{\Omega}\lambda|\nabla\phi|^2 + (-m+\epsilon(l-m)+2u^{\epsilon+\eta})\phi^2\mathrm{d}x}{\int_{\Omega}\phi^2\ \mathrm{d}x}, \qquad (30)$$

we have

$$\begin{aligned}&\sigma_1(-m-\epsilon(l-m)+2u^{\epsilon+\eta},\lambda)\int_{\Omega}\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)^2\mathrm{d}x\\ &\le \int_{\Omega}\lambda\left|\nabla\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right|^2 + (-m-\epsilon(l-m)+2u^{\epsilon+\eta})\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)^2\mathrm{d}x\\ &= \int_{\Omega}\left[(l-m)\left(\psi^{\epsilon+\eta}+\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right) - 2\psi^{\epsilon}\left(\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right)\right]\left(\frac{\psi^{\epsilon+\eta}-\psi^{\epsilon}}{\eta}\right)\ \mathrm{d}x.\end{aligned}$$

Similar to the results in (27), we see

$$\left\|\frac{u^{\epsilon+\eta}-u^{\epsilon}}{\eta}\right\|_{H^1} \le \frac{2\|m+(\epsilon+\eta)(l-m)\|_\infty\|l-m\|_{L^2}}{\min\{c_\lambda\lambda,\sigma_1(-m-\epsilon(l-m)+u^{\epsilon+\eta}+u^{\epsilon},\lambda)\}} \le C_1\|l-m\|_{L^2},$$

which implies that

$$\|\psi^\epsilon\|_{H^1} \le \frac{2\|m+\epsilon(l-m)\|_\infty \|l-m\|_{L^2}}{\min\{c_\lambda \lambda, \sigma_1(-m-\epsilon(l-m)+2u^\epsilon, \lambda)\}} \le C_2\|l-m\|_{L^2},$$

and

$$\|\psi^{\epsilon+\eta}\|_{H^1} \le \frac{2\|m+(\epsilon+\eta)(l-m)\|_\infty \|l-m\|_{L^2}}{\min\{c_\lambda \lambda, \sigma_1(-m+(\epsilon+\eta)(l-m)+2u^{\epsilon+\eta}, \lambda)\}} \le C_3\|l-m\|_{l^2}.$$

Recall that, by the Sobolev Embedding Theorems for $\phi \in H^1(\Omega)$ and Ω open and bounded in $\mathbb{R}^N$ ($N \le 4$), there exists some $C_4 = C_4(\Omega) > 0$ such that

$$\|\phi\|_{L^4} \le C_4\|\phi\|_{H^1}.$$

Thus, we estimate the quotient of our sensitivities,

$$\begin{aligned}
&\min\{c_\lambda\lambda, \sigma_1(-m-\epsilon(l-m)+2u^{\epsilon+\eta}, \lambda)\} \left\| \frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta} \right\|_{H^1}^2 \\
&\quad \le \int_\Omega \left[(l-m)\left(\psi^{\epsilon+\eta} + \frac{u^{\epsilon+\eta}-u^\epsilon}{\eta}\right) - 2\psi^\epsilon\left(\frac{u^{\epsilon+\eta}-u^\epsilon}{\eta}\right)\right]\left(\frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta}\right) dx \\
&\quad \le \|l-m\|_{L^2}\left(\|\psi^{\epsilon+\eta}\|_{L^4} + \left\|\frac{u^{\epsilon+\eta}-u^\epsilon}{\eta}\right\|_{L^4}\right)\left\|\frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta}\right\|_{L^4} + 2\|\psi^\epsilon\|_{L^4}\left\|\frac{u^{\epsilon+\eta}-u^\epsilon}{\eta}\right\|_{L^4}\left\|\frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta}\right\|_{L^2} \\
&\quad \le C_5\|l-m\|_{L^2}^2 \left\|\frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta}\right\|_{H^1}
\end{aligned}$$

where C_5 depends on the terms in C_1, C_2, C_3, C_4. Finally, by the work above,

$$\int_\Omega \frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta} dx \le C_6 \left\|\frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta}\right\|_{H^1} \le C_6\|l-m\|_{L^2}^2. \tag{31}$$

We conclude

$$g''(\epsilon) = \lim_{\eta\to 0}\int_\Omega \left(\frac{\psi^{\epsilon+\eta}-\psi^\epsilon}{\eta} - 2B(l-m)^2\right) dx \le (C_7 - 2B)\|l-m\|_{L^2}^2$$

and for B large enough, we have the desired concavity. Hence, for large B, we have a unique optimal control. □

5.3. Some results and conjectures for B small

For B sufficiently small, while we have not found a characterization of the optimal control, we can say a bit more. In this section, we show that if B is small enough, the constant solution $m = u = \frac{\delta}{|\Omega|}$ is no longer an optimal control. Based on numerical observations, we conjecture that the optimal control is also no longer unique for B small enough and λ large enough.

Lemma 5.3. *Let $0 < \lambda_1 < \cdots < \lambda_k < \cdots$ denote the non-zero eigenvalues of $-\Delta$, subject to zero Neumann boundary conditions, and let the corresponding orthonormal eigenfunctions be denoted by ϕ_i for $i = 1, 2, \ldots$. Then if*

$$B < \frac{\lambda\lambda_1}{\left[\lambda\lambda_1 + \frac{\delta}{|\Omega|}\right]^2}$$

then $m = \frac{\delta}{|\Omega|}$ is not an optimal control.

Proof. The idea is to perturb m slightly around the constant $\frac{\delta}{|\Omega|}$ by letting

$$m_\epsilon = \frac{\delta}{|\Omega|} + \epsilon g \in U$$

for $\epsilon > 0$ small, $\int_\Omega g\, dx = 0$ and $g \in C^1(\bar{\Omega})$.

Let $u = u(m_\epsilon)$ be the solution of (2). We want to show that u has the form

$$u = \frac{\delta}{|\Omega|} + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3).$$

In fact, if we let u_1 be the unique solution of

$$-\lambda\Delta u_1 - \frac{\delta}{|\Omega|}(g - u_1) = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_1}{\partial n}|_{\partial\Omega} = 0$$

and let u_2 be uniquely determined by

$$-\lambda \Delta u_2 - u_1(g - u_1) + \frac{\delta}{|\Omega|} u_2 = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_2}{\partial n}|_{\partial\Omega} = 0,$$

then we are able to obtain by the following result that u does in fact have the form we desire. Notice that $\int_\Omega g\,dx = 0$ implies that $\int_\Omega u_1\,dx = 0$.

Claim. There exist positive constants κ and ϵ_0 such that for $0 < \epsilon < \epsilon_0$,

$$\left\| u - \left(\frac{\delta}{|\Omega|} + \epsilon u_1 + \epsilon^2 u_2 \right) \right\|_{L^\infty(\Omega)} \le \kappa \epsilon^3. \tag{32}$$

To establish our assertion, we write u as

$$u = \frac{\delta}{|\Omega|} + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3$$

for some function u_3. Substituting both u and m_ϵ into the differential equation (2) we see that u_3 satisfies

$$\lambda \Delta u_3 + u_3 \left[-\frac{\delta}{|\Omega|} + \epsilon(g - 2u_1) - 2\epsilon^2 u_2 - \epsilon^3 u_3 \right] = -u_2(g - 2u_1) + \epsilon u_2^2 \quad \text{in } \Omega,$$

and the zero Neumann boundary condition. Since $g \in C^1$, then u_1, u_2, and u_3 are classical solutions. Suppose that $u_3(\bar{x}) = \max_{\bar{\Omega}} u_3$ for some $\bar{x} \in \bar{\Omega}$. By Proposition 2.2 of [19] we have

$$u_3(\bar{x}) \left\{ -\frac{\delta}{|\Omega|} + \epsilon[g(\bar{x}) - 2u_1(\bar{x})] - 2\epsilon^2 u_2(\bar{x}) - \epsilon^3 u_3(\bar{x}) \right\} \ge -u_2(\bar{x})[g(\bar{x}) - 2u_1(\bar{x})].$$

Since the L^∞ bounds of g, u_1 and u_2 are independent of ϵ, from the above inequality it follows that there exist positive constants κ_1 and ϵ_1 such that $u_3(\bar{x}) \le \kappa_1$ for $0 < \epsilon < \epsilon_1$. That is, $\max_{\bar{\Omega}} u_3 \le \kappa_1$. Similarly, we can show that there exist positive constants κ_2 and ϵ_2 such that $\min_{\bar{\Omega}} u_3 \ge -\kappa_2$ for $0 < \epsilon < \epsilon_2$. This proves (32).

Clearly, by (32) we can now obtain

$$\begin{aligned} J(m_\epsilon) &= \int_\Omega u - B m_\epsilon^2 dx \\ &= \int_\Omega \left(\frac{\delta}{|\Omega|} + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3) \right) - B \left(\left(\frac{\delta}{|\Omega|} \right)^2 + 2\epsilon g \frac{\delta}{|\Omega|} + \epsilon^2 g^2 \right) dx \\ &= J\left(\frac{\delta}{|\Omega|} \right) + \epsilon^2 \left(\int_\Omega u_2 - B g^2\, dx \right) + O(\epsilon^3). \end{aligned}$$

Integrating both sides of the equation for u_2, we see that

$$\int_\Omega u_2\, dx = \int_\Omega \frac{|\Omega|}{\delta} u_1(g - u_1)\, dx$$

so that

$$J(m_\epsilon) = J\left(\frac{\delta}{|\Omega|} \right) + \epsilon^2 \left(\int_\Omega \frac{|\Omega|}{\delta} u_1(g - u_1) - B g^2 dx \right) + O(\epsilon^3).$$

Now, we can express g and u_1 as

$$g = \sum_i g_i \phi_i$$
$$u_1 = \sum_i a_i \phi_i.$$

Using the equation for u_1, we can see that

$$a_i = \frac{g_i}{\frac{\lambda |\Omega| \lambda_i}{\delta} + 1}.$$

Substituting all of the above into the expression for $J(m)$, we obtain

$$J(m_\epsilon) = J\left(\frac{\delta}{|\Omega|} \right) + \epsilon^2 \left(\sum_{i=1}^{\infty} \left\{ \frac{\lambda \lambda_i}{[\lambda \lambda_i + \frac{\delta}{|\Omega|}]^2} - B \right\} g_i^2 \right) + O(\epsilon^3)$$

which is greater than $J(\frac{\delta}{|\Omega|})$ provided that ϵ is sufficiently small, $g_1 \neq 0$ and $g_i = 0$ for all $i > 1$, and

$$B < \frac{\lambda\lambda_1}{\left[\lambda\lambda_1 + \frac{\delta}{|\Omega|}\right]^2}.$$

Thus, the constant function $\frac{\delta}{|\Omega|}$ is not an optimal control when B is suitably small. □

Remark. From the numerical computations one can see that for B small enough, it appears that the optimal control may not be unique (but could be unique up to symmetry). For example, in one dimension, if we let $\Omega = [0, L]$ and $\phi(x) = L - x$, then if (u, m) is an optimal pair, $(u \circ \phi, m \circ \phi)$ is also an optimal pair. So, the existence of an asymmetric optimal control implies that the control is not unique. However, this does also seem to depend on λ, as for λ sufficiently small, regardless of the value of B, the numerics indicate a symmetric optimal control. In other words, the profiles of the optimal controls depend on both parameters B and λ in a rather intricate manner.

6. Necessary conditions

In order to obtain a characterization of the optimal control m more easily, we need to find another way to put the control integral constraint,

$$\int_\Omega m \, \mathrm{d}x = \delta,$$

into our problem. We introduce an extra state variable w, denoted by $w(m)$, such that

$$\begin{cases} \Delta w = m, & x \in \Omega, \\ \dfrac{\partial w}{\partial n} = \dfrac{\delta}{|\partial\Omega|}, & x \in \partial\Omega. \end{cases} \tag{33}$$

Introducing an extra state variable to handle an integral constraint on the controls is commonly done in one dimension [20], but doing this in multiple dimensions is a new feature.

Our control problem becomes the problem of maximizing $J(m)$ as given in (1) over all admissible controls in $U_1 = \{m \in L^\infty(\Omega) \mid 0 \le m \le 1\}$, such that a corresponding state solution pair u, w exists satisfying (2) and (33). Now we have a system of sensitivities, ψ_1 and ψ_2, where ψ_1 satisfies (13). We can derive ψ_2 in the same fashion as before, by taking $w^\epsilon = w(m+\epsilon l)$ with $m + \epsilon l \in U_1$. Thus, w^ϵ satisfies

$$\begin{cases} \Delta w^\epsilon = m + \epsilon l, & x \in \Omega \\ \dfrac{\partial w^\epsilon}{\partial n} = \dfrac{\delta}{|\partial\Omega|}, & x \in \partial\Omega. \end{cases} \tag{34}$$

It follows that

$$\begin{cases} \Delta \left(\dfrac{w^\epsilon - w}{\epsilon}\right) = l, & x \in \Omega \\ \dfrac{\partial}{\partial n}\left(\dfrac{w^\epsilon - w}{\epsilon}\right) = 0, & x \in \partial\Omega. \end{cases} \tag{35}$$

Note that $\int_\Omega l \, \mathrm{d}x = 0$.

Standard estimates imply that $\frac{w^\epsilon - w}{\epsilon} \rightharpoonup \psi_2$ in $H^1(\Omega)$, and that ψ_2 satisfies

$$\begin{cases} \Delta \psi_2 = l, & x \in \Omega \\ \dfrac{\partial \psi_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{36}$$

Theorem 6.1. *Given an optimal control m and corresponding states, u, w, there exists a solution p_1, p_2 to the adjoint system, with $p_1 \in H^2(\Omega)$ and p_2 constant, satisfying*

$$\begin{cases} -\lambda\Delta p_1 - (m - 2u)p_1 = 1, & x \in \Omega, \\ \dfrac{\partial p_1}{\partial n} = 0, & x \in \partial\Omega, \\ \Delta p_2 = 0, & x \in \Omega, \\ \dfrac{\partial p_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Furthermore, we have

$$m^* = \min\left\{\max\left\{0, \frac{up_1 + p_2}{2B}\right\}, 1\right\}.$$

Proof. Suppose $m(x)$ is an optimal control. Let $l \in L^\infty(\Omega)$ such that $m + \epsilon l$ is an admissible control for small $\epsilon > 0$, with $\int_\Omega l \, dx = 0$. There exists a solution, p_1, p_2, to the adjoint system [18]. The directional derivative of J with respect to the control at m in the direction of l satisfies

$$\begin{aligned}
0 &\geq \lim_{\epsilon\to 0^+} \frac{J(m+\epsilon l) - J(m)}{\epsilon} \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\int_\Omega u^\epsilon - B(m+\epsilon l)^2 \, dx - \left(\int_\Omega u - Bm^2 dx\right)\right] \\
&= \lim_{\epsilon\to 0^+} \left[\int_\Omega \frac{u^\epsilon - u}{\epsilon} \, dx - \int_\Omega B(2ml + \epsilon l^2) \, dx\right] \\
&= \int_\Omega \psi_1 \, dx - \int_\Omega 2mBl \, dx. \qquad (37)
\end{aligned}$$

Using the adjoint equations and the sensitivities, we have

$$\begin{aligned}
0 &\geq \int_\Omega \psi_1 \, dx - \int_\Omega 2mBl \, dx \qquad (38) \\
&= \int_\Omega (-\lambda\Delta p_1 - (m-2u)p_1)\psi_1 + \psi_2 \Delta p_2 \, dx - \int_\Omega 2mBl \, dx \\
&= \int_\Omega (-\lambda\Delta \psi_1 - (m-2u)\psi_1)p_1 + p_2 \Delta \psi_2 \, dx - \int_\Omega 2mBl \, dx \\
&= \int_\Omega lup_1 + p_2 l \, dx - \int_\Omega 2mBl \, dx \\
&= \int_\Omega (up_1 + p_2 - 2mB) l \, dx. \qquad (39)
\end{aligned}$$

On the interior of the control set, $\{x \mid 0 < m(x) < 1\}$, we can take variations l with support on that set, that can have positive and negative values. So we can conclude on that set,

$$up_1 + p_2 - 2mB = 0, \qquad (40)$$

and so

$$m^* = \frac{up_1 + p_2}{2B}.$$

Next consider variations l with support contained in $\{x \mid 0 \leq m(x) < 1\}$. Those variations would be non-negative on the set $A = \{x \mid m(x) = 0\}$, and using the above expression on the interior of the control set, the integral inequality above becomes

$$0 \geq \int_A (up_1 + p_2) l \, dx.$$

On set A, this gives

$$up_1 + p_2 \leq 0.$$

Similarly consider variations l with support contained in the set $\{x \mid 0 < m(x) \leq 1\}$, and obtain

$$up_1 + p_2 - 2B \geq 0,$$

on the set where $m(x) = 1$.

Combining the above cases, we can obtain

$$m^* = \min\left\{\max\left\{0, \frac{up_1 + p_2}{2B}\right\}, 1\right\}. \quad \square$$

7. 1D case

In the 1D case, Pontryagin's Maximum Principle [21] can be used to derive the necessary conditions. In this section, we use this principle to rederive the necessary conditions for the optimal control and compare with the PDE case. Again, we have the control problem of maximizing J, where

$$J(m) = \int_0^L \left(u - Bm^2\right)\, \mathrm{d}x \tag{41}$$

subject to

$$-\lambda u'' = mu - u^2, \qquad u'(0) = 0, \qquad u'(L) = 0 \tag{42}$$

where the control set is

$$U = \left\{ m : [0, L] \longrightarrow \mathbb{R} \mid 0 \le m(x) \le 1, \int_0^L m(x)\, \mathrm{d}x = \delta > 0 \right\},$$

and we assume that λ is a positive constant.

Let $z(t) = \int_0^t m(s)\, \mathrm{d}s$, then $z' = m$, $z(0) = 0$, $z(L) = \delta$. Moreover, let $u_1 = u$, $u_1' = u_2$, then we get $u_2' = u_1'' = -\frac{mu_1 - u_1^2}{\lambda}$. We can rewrite the state equation and the new variable equation as the following system:

$$\begin{cases} u_1' = u_2 \\ u_2' = -\dfrac{mu_1 - u_1^2}{\lambda} \\ z' = m \end{cases} \tag{43}$$

with

$$u_2(0) = u_2(L) = 0, \qquad z(0) = 0, \qquad z(L) = \delta.$$

Pontryagin's Maximum Principle [21] solves our problem through considering the Hamiltonian function H:

$$H = u_1 - Bm^2 + p_1 u_2 - p_2 \frac{mu_1 - u_1^2}{\lambda} + p_3 m. \tag{44}$$

Then on the interior of the control set,

$$\frac{\partial H}{\partial m} = 0 \Rightarrow \frac{\partial H}{\partial m} = -2Bm - p_2 \frac{u_1}{\lambda} + p_3 = 0. \tag{45}$$

Our three adjoint variables, corresponding to the three state variables, satisfy

$$p_1' = -\frac{\partial H}{\partial u_1} \Rightarrow p_1' = -\left(1 - p_2 \left(\frac{m - 2u_1}{\lambda}\right)\right) \tag{46}$$

$$p_2' = -\frac{\partial H}{\partial u_2} \Rightarrow p_2' = -p_1 \tag{47}$$

$$p_3' = -\frac{\partial H}{\partial z} \Rightarrow p_3' = 0 \Rightarrow p_3 = C \tag{48}$$

where p_2 and p_3 have no transversality boundary conditions. Thus we have

$$p_1' = -\left(1 - p_2 \left(\frac{m - 2u_1}{\lambda}\right)\right), \qquad p_1(0) = p_1(L) = 0 \tag{49}$$

$$p_2' = -p_1 \tag{50}$$

$$p_3 = C. \tag{51}$$

Therefore, on the interior of the control set,

$$-2Bm - p_2 \frac{u_1}{\lambda} + C = 0 \tag{52}$$

which, by taking the bounds into account, gives

$$m^* = \min\left\{1, \max\left\{0, \frac{C - \frac{p_2 u_1}{\lambda}}{2B}\right\}\right\}. \tag{53}$$

The optimality system is

$$\begin{cases} u_1' = u_2, \\ u_2' = -\dfrac{mu_1 - u_1^2}{\lambda}, \\ z' = m, \\ p_1' = -\left(1 - p_2\left(\dfrac{m - 2u_1}{\lambda}\right)\right), \\ p_2' = -p_1, \\ p_3 = C, \end{cases} \tag{54}$$

with

$$u_2(0) = u_2(L) = 0, \qquad z(0) = 0, \qquad z(L) = \delta;$$
$$p_1(0) = p_1(L) = 0.$$

To compare with the PDE case and to solve numerically, we convert to second order ODEs, taking $u_1 = u$ and $\frac{p_2}{\lambda} = \bar{p}$,

$$-\lambda u'' = mu - u^2, \qquad u'(0) = 0, \qquad u'(L) = 0, \tag{55}$$

$$-\lambda \bar{p}'' = -1 + (m - 2u)\bar{p}, \qquad \bar{p}'(0) = 0, \qquad \bar{p}'(L) = 0. \tag{56}$$

Letting $p = -\bar{p}$, p satisfies the adjoint problem derived in the PDE setting.

To get the $z(L) = \delta$, i.e. the integral condition on the control, we must find the constant C.

8. Numerical results

In the following sections we present numerical results obtained for 1D and 2D problems (i.e. $m = m(x)$ and $u = u(x)$, or $m = m(x, y)$ and $u = u(x, y)$). We assume throughout that $|\Omega| = 1$ and $\delta = 0.5$.

We use iterative schemes coded in MATLAB to obtain our results. An initial guess for the control, m_0, is chosen and an initial guess for the second adjoint, p_2, is fixed. Corresponding solutions to the state and adjoint equations u_0 and $p_{1,0}$ are obtained, and the first approximation for the control is obtained by $m_1 = \min\{\max\{0, \frac{u_0 p_{1,0} + p_2}{2B}\}, 1\}$. This process is repeated until the sequence $m_n = \min\{\max\{0, \frac{u_{n-1} p_{1,n-1} + p_2}{2B}\}, 1\}$ converges to within a given tolerance in L^2 norm. Finally, we check that our optimal control candidate has integral within some tolerance of δ. If not, we appropriately modify our choice of the constant p_2 and begin the process again, repeating until we arrive at an optimal control whose integral is sufficiently close to δ.

We use the MATLAB solver bvp4c in the 1D case to solve for u_n and $p_{1,n}$ given m_n. In two dimensions, we use the standard 5 point stencil (second order) finite difference approximation for creating a matrix operator A approximating the Laplacian. On rows of A corresponding to boundary nodes, we include second order approximations for the Neumann boundary conditions instead. In this case, for each approximation of the control m_n, we also need initial guesses u_n^0 and $p_{1,n}^0$ for the solutions u_n and $p_{1,n}$ that correspond to m_n. We then iterate

$$u_n^{i+1} = (\mu A)^{-1} \text{rhs1}(i)$$
$$p_{1,n}^{i+1} = (\mu A - m_n I)^{-1} \text{rhs2}(i)$$

until we reach convergence to within some tolerance in L^2 norm to an approximation of the actual solutions u_n and $p_{1,n}$ corresponding to m_n. Here rhs1(i) is the vector of grid values for $(u_n^i m_n - (u_n^i)^2)$ modified on boundary nodes to include the boundary conditions, and rhs2(i) is a vector of value one in each entry, except for those entries corresponding to boundary nodes which are modified to include boundary conditions. We then use u_n and $p_{1,n}$ within the iterative scheme described initially to update our approximation for the optimal control.

It is important to note that (in both the 1D and 2D cases) if the initial guess for m has any symmetry for which the Laplacian is invariant, then the approximation of u and p_1 will also have that symmetry (recall p_2 is constant). Because the form of the optimal control is $m = \min\{\max\{0, \frac{up_1 + p_2}{2B}\}, 1\}$, and we use this to update our guess for the optimal m, the initial symmetry is carried through the iterations, and the solver can get "stuck" away from the actual optimal control. This is easily avoided however by choosing an initial guess for m that does not have such symmetries.

8.1. 1D variation in space

We fix $\lambda = 0.1$. As was proven earlier, for $B > \frac{|\Omega|}{2\delta} = 1$, we obtain numerical confirmation that the (unique) optimal control is indeed $m(x) = \frac{\delta}{|\Omega|} = 0.5$.

For $B < \frac{|\Omega|}{2\delta}$, we begin to see a marked difference in the solution, and we are no longer guaranteed uniqueness. For example, if we take $B = 0.5$ and again $\lambda = 0.1$, we see that an optimal control and state could be represented by either

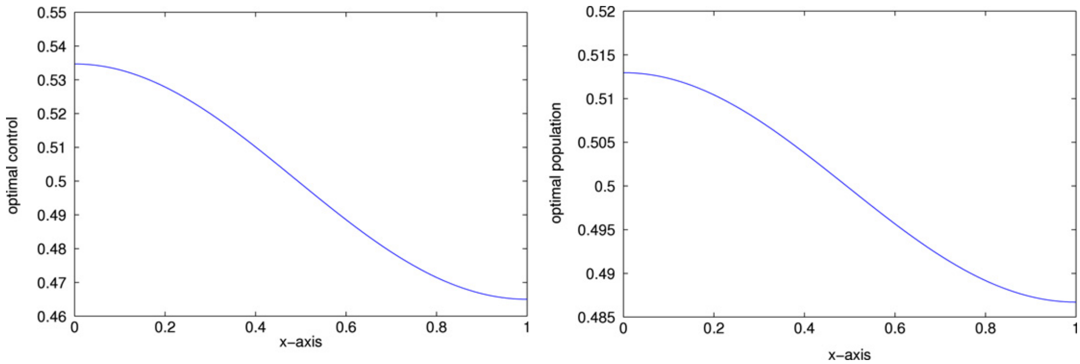

Fig. 1. An optimal control and corresponding state in 1D for $\lambda = 0.1, B = 0.5$.

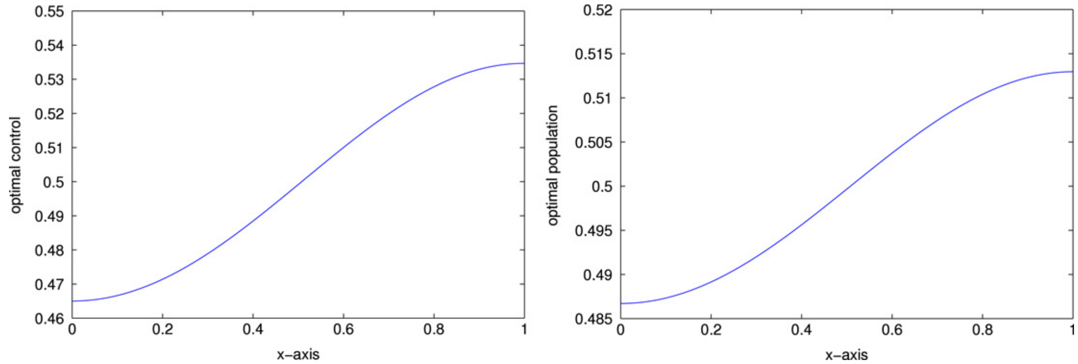

Fig. 2. Another optimal control and corresponding state in 1D for $\lambda = 0.1, B = 0.5$.

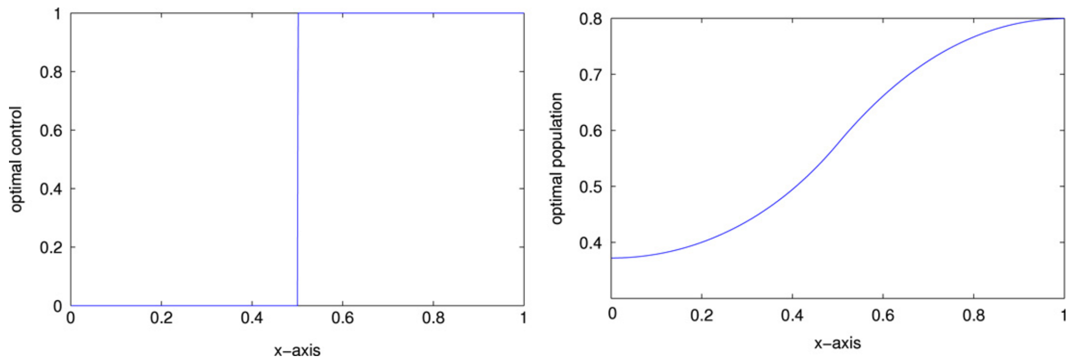

Fig. 3. An optimal control and corresponding state in 1D for $\lambda = 0.1, B = 0.001$.

Fig. 1 or Fig. 2, as both solutions yield the same value for $J(m)$. We do not yet know whether or not there are other possible optimal controls besides these two.

In general, if an optimal control in one dimension is not symmetric about the center of the domain, then it will not be unique. If (m, u) is an optimal pair, and m is not symmetric about the center of Ω, then let $f : \Omega \to \Omega$ be the reflection about the center of Ω, $\hat{m} = m \circ f \neq m$, and $\hat{u} = u \circ f \neq u$. It is easy to show that $(\hat{m}, \hat{u})$ also satisfies our state equation and $J(\hat{m}) = J(m)$. Thus, $(\hat{m}, \hat{u})$ will be another optimal pair.

As B gets smaller, it appears that the control becomes close to bang-bang. For example, with $B = 0.001$ and $\lambda = 0.1$, we obtain the optimal control and state shown in Fig. 3 (with $m = 1$ on $.5 + \epsilon < x \leq 1$ and $m = 0$ for $.5 - \epsilon > x \geq 0$ for some small $\epsilon > 0$), or its reflection about the center of Ω. Again, it is not yet known whether or not there are other possible optimal controls for this case.

Finally, we remark that these results are not independent of the choice of diffusion rate λ. For λ small enough, numerical results indicate that the optimal control may now have a symmetry that was not present for larger λ. We hope to further investigate this in a later paper. To illustrate, we conclude with the example $B = 0.1$ and $\lambda = 0.011$. An optimal control

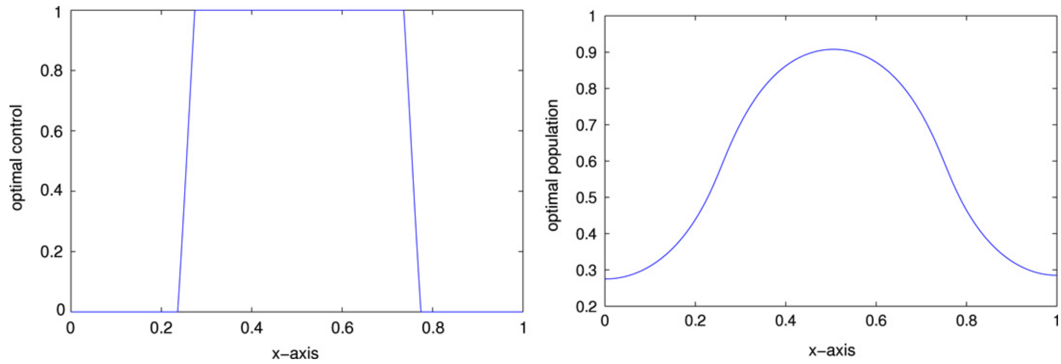

Fig. 4. Optimal control and state in 1D for $\lambda = 0.011$, $B = 0.1$.

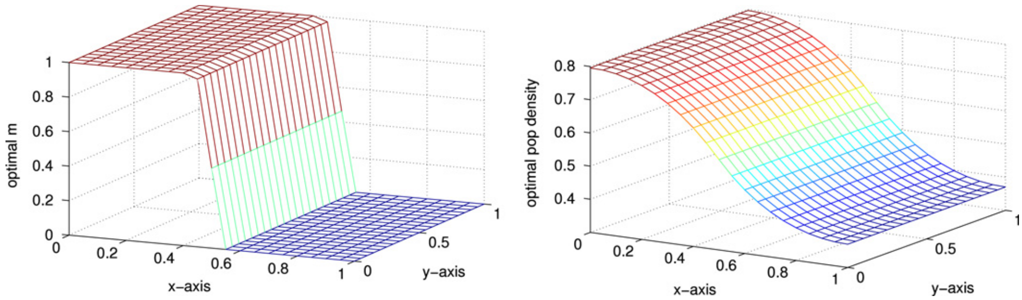

Fig. 5. An optimal control and state in 2D for $\lambda = 0.1$, $B = 0.1$, $\delta = 0.5$.

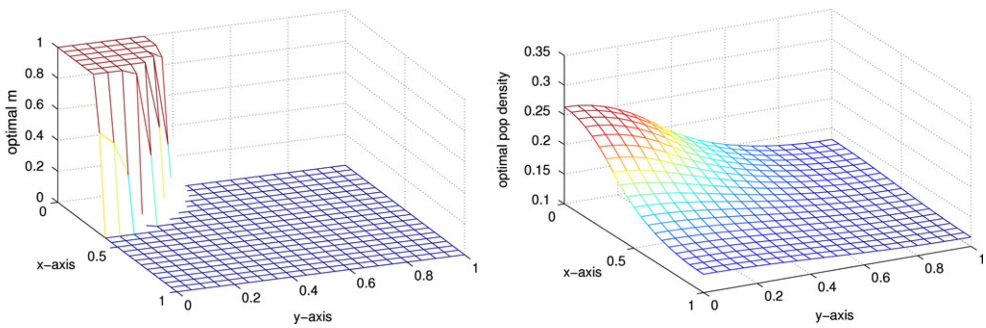

Fig. 6. An optimal control and state in 2D for $\lambda = 0.1$, $B = 0.1$, $\delta = 0.1$.

(now with $m = 1$ on $.25 + \epsilon < x < .75 - \epsilon$ for some small $\epsilon > 0$) and state obtained numerically in this case are shown in Fig. 4.

8.2. 2D variation in space

The case of two dimensions turns out to be quite similar to that of one dimension. For the sake of comparison, we again present the numerical results for $\lambda = 0.1$, $\delta = 0.5$, and $\Omega = [0, 1] \times [0, 1]$. First for $B = 10$, the optimal control and corresponding state are given by $m = u = \delta/|\Omega| = 0.5$, again confirming our theoretical results for $B > \frac{|\Omega|}{2\delta} = 1$.

For $B < \frac{|\Omega|}{2\delta} = 1$ and $\lambda = 0.1$, we again see m concentrated near a boundary edge, and we no longer have uniqueness of the optimal control or state. For example, for $B = 0.1$, an optimal control and state, respectively are shown in Fig. 5. We have at least three other optimal controls and states given by the same basic shape as that of Fig. 5, with concentration near a different boundary edge instead (giving four possible optimal controls and states). Again, we do not yet know if these four are the only possible optimal controls or if there exist others as well.

In varying δ, we see some different behavior in two dimensions. For example, in the case with $\delta = 0.1$, with $B = 0.1$, an optimal control and state are pictured in Fig. 6. Notice that now we have concentration near a corner rather than along an edge. Similarly if $\delta = 0.9$, and again $B = 0.1$, an optimal control and state are pictured in Fig. 7. Of course, again these are

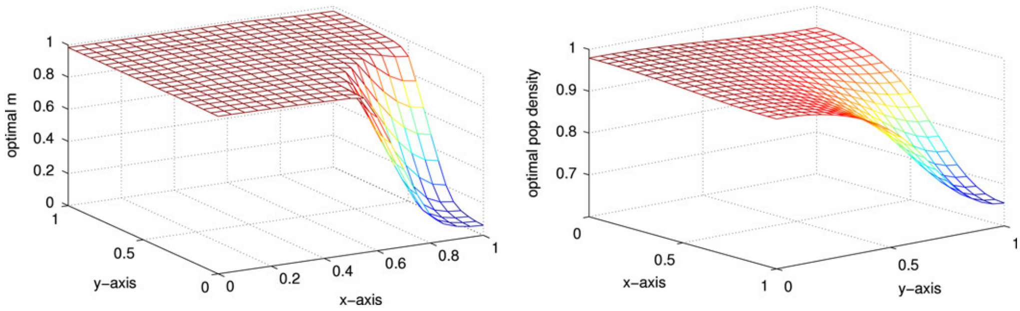

Fig. 7. An optimal control and state in 2D for $\lambda = 0.1$, $B = 0.1$, $\delta = 0.9$.

not unique, as reflection around the line $x = 0.5$, or around $y = 0.5$, or both, yields three more possible optimal controls in either case.

9. Discussions

We studied the control problem of maximizing the total payoff in the conservation of a single species with a fixed amount of resource. The existence of an optimal control is established and uniqueness and characterization of the optimal control is investigated. Some necessary conditions are provided for the characterization of the optimal control. We introduced an extra state variable to handle the integral constraint for the control to get the characterization in the multi-dimensional space. For 1D case, we present a simpler version of this technique. Extensive numerical simulations are done for both 1D and 2D habitats, in which we found several interesting phenomena:

(i) For 1D habitat, the characterization of the optimal control depends on the choice of the diffusion rate λ. For small λ the optimal control seems to be symmetric (Fig. 4), and so may be unique. This is in strong contrast to the case when λ is suitably larger, where the optimal control (Figs. 1–3) is not unique and non-symmetric.
(ii) For rectangular domains, the shape of the optimal control depends on the choice of the amount of total resources, δ. When the amount is small, the optimal control is concentrated at one of the corners of the rectangle (Fig. 6). This is very different from the situation where the amount of total resources is suitably large, for which the optimal control concentrates at a boundary edge of the rectangle (Fig. 5).

Many interesting questions still remain open and we briefly discuss a few of them here:
1. When B is small, numerical simulations indicate that the optimal control is close to "bang-bang". Can one show that the optimal control is exactly "bang-bang" for $B = 0$?
2. It was shown in [5] that the total population size $\int_\Omega u dx$, as a function of the diffusion rate λ, is not monotone. In fact, $\int_\Omega u dx$ is exactly minimized at $\lambda = 0$ and $\lambda = \infty$ and maximized at some value of $\lambda = \lambda^* \in (0, \infty)$. From our numerical simulations we think that there exists some connection between λ^* and the symmetry of the optimal control for 1D habitat.
3. For a high-dimensional habitat, we see that the profile of the optimal control may depend on the amount of total resources. Also, will the geometry of the boundary play some role in determining the optimal control? For example, when the total amount of resource is small, is it the best strategy to arrange resources near the most curved part of the boundary? Such questions seem to be rather challenging even for the simplest domains.

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