DIMACS Series in Discrete Mathematics
and Theoretical Computer Science
Volume **75**, 2010

Introduction to Optimal Control for Discrete Time Models with an Application to Disease Modeling

Wandi Ding and Suzanne Lenhart

Abstract. This paper serves as an introduction to the theory of optimal control applied to systems of discrete time models with an emphasis on disease models. We outline the steps in solving such optimal control problems and discuss the necessary conditions. A simple disease example provides detailed methodology in charactering the optimal control through the use of Pontryagin's Maximum Principle. Numerical results are given to illustrate several cases.

1. Introduction

For many populations, births and growth occur in regular times each year (or each cycle). Discrete time models or difference equations are well suited to describe the life histories of organisms with discrete reproduction and/or growth. For example, the Beverton-Holt stock-recruitment [**7**] model for a population N_t at time t is

$$N_{t+1} = rN_t \left(1 + N_t \frac{r-1}{K}\right)^{-1}.$$

Another application involves a population which is divided into separate discrete age classes. At each time step, a certain proportion of each class may survive and enter the next age class. Individuals in the first age class originate through reproduction from other classes

$$\begin{aligned} N_1(t+1) &= f_1N_1(t) + f_2N_2(t) + f_3N_3(t) \\ N_2(t+l) &= p_1N_1(t) \\ N_3(t+1) &= p_2N_2(t). \end{aligned}$$

For more background on discrete models, see the paper by Yakubu [**13**] in this volume and see the book by Caswell [**3**] and the edited volume by Abello and Cormode [**1**].

1991 *Mathematics Subject Classification.* Primary 39A10, 49N05 .

Key words and phrases. Optimal control, discrete time models, disease model.

The authors were supported in part by National Science Foundation Grants EF-0832858 (National Institute for Mathematical and Biological Synthesis) and ITR-04274710.

For examples involving disease models in discrete time, see [**2, 4, 6**]. We remark that in discrete time models, the order of events within a time step is crucial, so one should keep that in mind when building a discrete model. See [**5**] for an epidemic which is discrete in time and space and in which the order of events is important.

In an optimal control problem, one adjusts controls in a dynamic system to achieve a goal. The underlying system can have a variety of types of equations such as ordinary differential equations, partial differential equations, difference equations, stochastic differential equations or integrodifference equations. In this paper, we are considering only systems of equations which are discrete in time.

In control of a single difference equation, with discrete time steps, we denote $u = (u_0, u_1, \dots, u_{T-1})$ as the control and $x = (x_0, x_1, \dots, x_T)$ the state. Given x_0, the state function satisfies the difference equation modeling the scenario. The control affects the state difference equation,

$$x_{k+1} = g(x_k, u_k, k)$$

for $k = 0, 1, 2, \dots, T-1$ and with x_0 as given. Both the control and the state usually affect the goal, which is called the objective functional. We seek to find an optimal control and corresponding state that achieve the maximum (or minimum) of our objective functional.

Let's start with a simple example of optimal control of a discrete time model to illustrate the ideas.

$$\begin{aligned} &\min_u \sum_{k=0}^{2} \frac{1}{2}\left[x_k^2 + Bu_k^2\right] \\ \text{subject to} \quad & x_{k+1} = x_k + u_k \quad \text{ for } k = 0, 1, 2, \\ & x_0 = 5. \end{aligned}$$

The state has 4 components, x_0, x_1, x_2, x_3 while the control has one fewer component, u_0, u_1, u_2. Here the goal is to minimize the square of the state terms and the square of the control terms. The coefficient B is a weight factor, that gives the relative importance of the two terms in the goal.

Now we formulate a control problem in more generality. Given a control $u = (u_0, u_1, \dots, u_{T-1})$ and initial state x_0, the state equation is given by the difference equation

$$x_{k+1} = g(x_k, u_k, k)$$

for $k = 0, 1, 2, \dots, T-1$. Note that the state has one more component than the control

$$x = (x_0, x_1, \dots, x_T).$$

We have the following objective functional, which represents our goal:

$$J(u) = \phi(x_T) + \sum_{k=0}^{T-1} f(x_k, u_k, k).$$

The term, $\phi(x_T)$, represents a type of 'salvage' term; for example, one may want the population to be large at the final time T. The objective functional can be maximized or minimized over controls u. In the minimization case, the goal is to find an optimal control u^* such that

$$J(u^*) = \min_u J(u),$$

where the minimization is over a class of vectors with bounded components (with bounds specified to fit the situation.)

Necessary conditions, that an optimal control and corresponding state must satisfy, can be derived similarly to the case of ordinary differential equations, using a generalization of Pontryagin's Maximum Principle [**9**]. To see more detail about the derivation of the necessary conditions, see the book by Lenhart and Workman [**8**]. The key idea is introducing the adjoint function to attach the difference equation to the objective functional, resulting in the formation of a function called the Hamiltonian. This principle converts the problem of finding the control to optimize the objective functional subject to the state difference equation with initial condition to finding the control to optimize Hamiltonian pointwise (with respect to the control).

Now we have the Hamiltonian at each time step k, where our adjoint function is $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_T)$:

$$H_k = f(x_k, u_k, k) + \lambda_{k+1} g(x_k, u_k, k), \quad \text{for } k = 0, 1, \ldots, T-1.$$

Notice the indexing on the adjoint; it is one step ahead of the other terms. The necessary condition states that the Hamiltonian is maximized at each step with respect to the control u_k at the optimal control u_k^*. The adjoint equations and corresponding final time conditions (transversality conditions) are also given. If we do not have any constraints on our control, the necessary conditions are

$$\begin{aligned} \lambda_k &= \frac{\partial H_k}{\partial x_k} \\ \lambda_T &= \phi'(x_T^*) \\ \frac{\partial H_k}{\partial u_k} &= 0 \text{ at } u^*. \end{aligned}$$

Notice that the adjoint function has final time conditions while the state function has initial time conditions. Suppose that the controls are bounded, which is quite usual in biological examples. Suppose $a \le u_k \le b$ for each k, then these bounds need to be imposed after you solve the optimality equation

$$\frac{\partial H_k}{\partial u_k} = 0 \text{ at } u^*$$

for each component of the control at each time step.

2. Simple Illustrative Example

Next we consider a simple example [**8**] to illustrate the solution technique. Our objective functional is

$$\begin{aligned} &\min_u \sum_{k=0}^{2} \frac{1}{2}\left[x_k^2 + u_k^2\right] \\ \text{subject to} \quad &x_{k+1} = x_k + u_k \quad \text{for } k = 0, 1, 2, \\ &x_0 = 5. \end{aligned}$$

At each time step, the control u_k is the input, that will result in the growth or decline of the state. What optimal control is expected? We are seeking to minimize the state and the size of control. We expect the optimal control to be negative or zero.

Starting with the Hamiltonian,

$$H_k = \frac{1}{2}\left[x_k^2 + u_k^2\right] + \lambda_{k+1}(x_k + u_k),$$

our necessary conditions are:

$$\begin{aligned} \lambda_k &= \frac{\partial H_k}{\partial x_k} = x_k + \lambda_{k+1} \quad \text{for } k = 0, 1, 2, \\ \lambda_3 &= 0, \\ 0 &= \frac{\partial H_k}{\partial u_k} = u_k + \lambda_{k+1} \quad \text{at } u_k^*. \end{aligned}$$

Thus the optimal control satisfies

$$u_k = -\lambda_{k+1},$$

which gives

$$x_{k+1} = x_k - \lambda_{k+1} \quad \text{for } k = 0, 1, 2.$$

Our transversality condition is

$$\lambda_3 = 0,$$

since we do not have a salvage term, meaning there is no dependence on the state at the final time in the objective functional.

Combining above conditions yields

$$x_3 = x_2$$

and four equations in $x_1, x_2, \lambda_1, \lambda_2$,

$$x_1 = 5 - \lambda_1, \qquad x_2 = x_1 - \lambda_2$$

$$\lambda_1 = x_1 + \lambda_2, \qquad \lambda_2 = x_2.$$

After solving these algebraic equations, the optimal state values are

$$x_1^* = 2, x_2^* = 1, x_3^* = 1$$

and the optimal control values are

$$u_0^* = -3, u_1^* = -1, u_2^* = 0.$$

We see in this simple case, only the controls are making changes in the states.

3. System Case

Next we state the necessary conditions in the case of a system of difference equations:

$$x_{j,k+1} = g_j(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k),$$

with

$$x_j = (x_{j,0}, x_{j,1}, \ldots, x_{j,T}).$$

for $k = 0, 1, 2, \ldots, T-1, j = 1, 2, \ldots, n$. Note that k is the index for the time steps and j is the index for the states. There are m controls, n states, and T time steps. Define the objective functional as

$$J(u) = \phi(x_{1,T}, \ldots, x_{n,T}) + \sum_{k=0}^{T-1} f(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k)$$

Now we have one adjoint variable for each state variable and form the Hamiltonian,

$$\begin{aligned} H_k &= f(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k) \\ &+ \sum_{j=1}^{n} \lambda_{j,k+1} g_j(x_{1,k}, \ldots, x_{n,k}, u_{1,k}, \ldots, u_{m,k}, k), \end{aligned}$$

we can obtain the necessary conditions,

$$\lambda_{j,k} = \frac{\partial H_k}{\partial x_{j,k}}$$

$$\lambda_{j,T} = \frac{\partial \phi}{\partial x_{j,T}}(x_{1,T}, \ldots, x_{n,T})$$

$$\frac{\partial H_k}{\partial u_{i,k}} = 0 \text{ at } (u^*_{1,k}, \ldots, u^*_{m,k}).$$

for $k = 0, 1, 2, \ldots, T-1$, $j = 1, 2, \ldots, n$ and $i = 0, 1, 2, \ldots, m$.

We illustrate the system case in the following disease example.

4. Disease Example

We illustrate the techniques of optimal control using a simple epidemic example. This volume contains many examples of epidemic models and the emphasis in this article is on control techniques.

Consider an SIR system, in which the state variables are S, susceptibles, I, infecteds and R, immune individuals. Our stae equations are:

$$\begin{aligned} S_{k+1} &= S_k(1-u_k) - \beta\Big(S_k(1-u_k)\Big)I_k, \\ I_{k+1} &= I_k + \beta S_k(1-u_k)I_k - d_2 I_k, \\ R_{k+1} &= R_k + u_k S_k, \end{aligned} \tag{4.1}$$

where $k = 1, 2, \ldots, T-1$, β is the transmission rate and d_2 is the additional death rate due to infection. The control variable is u with $0 \le u_k \le 1-d$, with $d > 0$. The control can be interpreted as the proportion to be vaccinated, so we can see that $u_k S_k$ individuals move from the susceptible class to the immune class at time step k. Notice the order of the events in this model. Here the vaccination happens first, meaning that a proportion of susceptibles $u_k S_k$ is moved from the susceptible class to the immune class. Then the interaction of the non-immune susceptibles with the infecteds, which is why the infectivity term has the format,

$$\beta S_k(1-u_k)I_k.$$

The remaining susceptibles, after the movement of the vaccinated susceptibles to the immune class,

$$S_k(1-u_k),$$

are interacting with the infecteds, I_k. Note that infected individuals have the disease and are able to transmit it.

Note that positivity of the components of a discrete model can be an issue in a discrete model. We are using a small number of time steps here since we are considering control actions reacting to an outbreak of a disease in short time. Thus the sizes of our parameters and the number of time steps insures the positivity of the S, I, R classes here.

The goal is to minimize our objective functional,

$$\sum_{k=1}^{T-1}(I_k + Bu_k^2 + B_1 u_k) + I_T,$$

where T is the final time. The constants B and B_1 are the cost coefficients. We are minimizing the number of infected individuals during the time steps $k = 1$ to $T-1$ and at the final time and also minimizing the cost of administering the control. We are assuming the cost of administering the control is quadratic for simplicity. See [**8**] for other formats of controls in objective functionals.

4.1. The adjoints. The Hamiltonian at time step k is

$$\begin{aligned}(4.2)\qquad H_k = I_k + Bu_k^2 + B_1 u_k &+ \lambda_{1,k+1}\Big(S_k(1-u_k) - \beta\Big(S_k(1-u_k)\Big)I_k\Big)\\ &+ \lambda_{2,k+1}\Big(I_k + \beta S_k(1-u_k)I_k - d_2 I_k\Big)\\ &+ \lambda_{3,k+1}\Big(R_k + u_k S_k\Big).\end{aligned}$$

The equations for the adjoint variables for $k = 1, 2, \ldots, T-1$ are

$$\begin{aligned}(4.3)\qquad \lambda_{1,k} = \frac{\partial H_k}{\partial S_k} &= \lambda_{1,k+1}\Big(1 - u_k - \beta(1-u_k)I_k\Big)\\ &+ \lambda_{2,k+1}\Big(\beta(1-u_k)I_k\Big) + \lambda_{3,k+1}u_k,\end{aligned}$$

$$\begin{aligned}(4.4)\qquad \lambda_{2,k} = \frac{\partial H_k}{\partial I_k} &= 1 + \lambda_{1,k+1}\Big(-\beta S_k(1-u_k)\Big)\\ &+ \lambda_{2,k+1}\Big(1 + \beta S_k(1-u_k) - d_2\Big) +\end{aligned}$$

$$(4.5)\qquad \lambda_{3,k} = \frac{\partial H_k}{\partial R_k} = \lambda_{3,k+1}.$$

Our transersality conditions give at time T, we have $\lambda_{1,T} = \lambda_{3,T} = 0$, since S_T and R_T are not in the objective functional. But notice that $\lambda_{2,T} = 1$ since

$$\phi(I_T) = I_T \text{ and } \frac{\partial \phi}{\partial I} = 1.$$

For $k = 1, 2, \ldots, T-1$, the control characterization is derived from

$$(4.6)\quad \frac{\partial H_k}{\partial u_k} = 2Bu_k + B_1 + \lambda_{1,k+1}\Big(-S_k + \beta S_k I_k\Big) + \lambda_{2,k+1}\Big(-\beta S_k I_k\Big) + \lambda_{3,k+1}S_k = 0,$$

subject to the lower and upper bounds for u. For $k = 1, 2, \ldots, T-1$, the characterization becomes

$$u_k^* = \min\Big(1-d, \max\Big(0, [\lambda_{1,k+1r}(S_k - \beta S_k I_k) + \lambda_{2,k+1}(\beta S_k I_k) - \lambda_{3,k+1}S_k - B_1]/2B\Big)\Big).$$

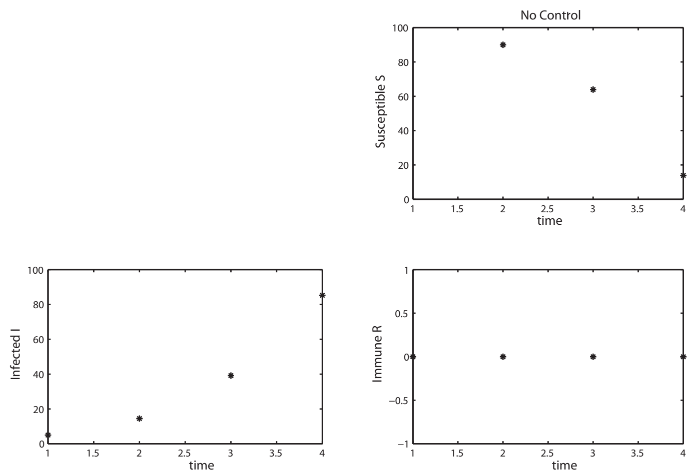

FIGURE 1. S, I, R without control

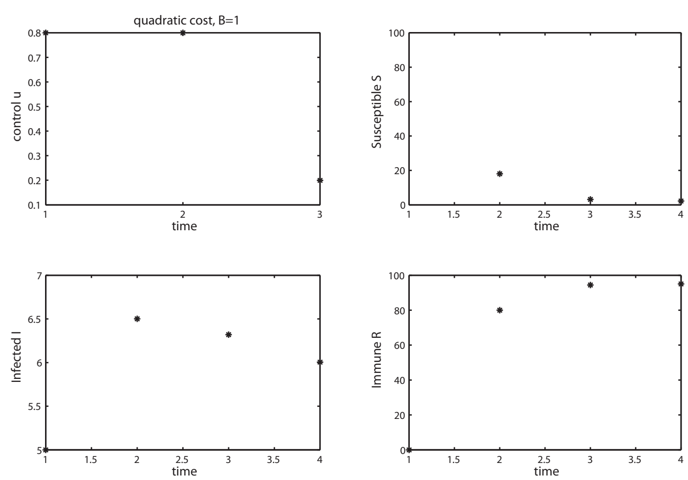

FIGURE 2. Case 1: S, I, R with $B = 1$

4.2. Numerical Examples. The optimality system consists of the state system, adjoint system, initial and final time conditions, and the control characterization. We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the control at the first iteration and then before the next iteration, we update the control by using the characterization. We continued until convergence of successive iterates is achieved.

We illustrate two cases of changing of the parameters in the objective functional.
Case 1. Quadratic Cost ($B_1 = 0$)

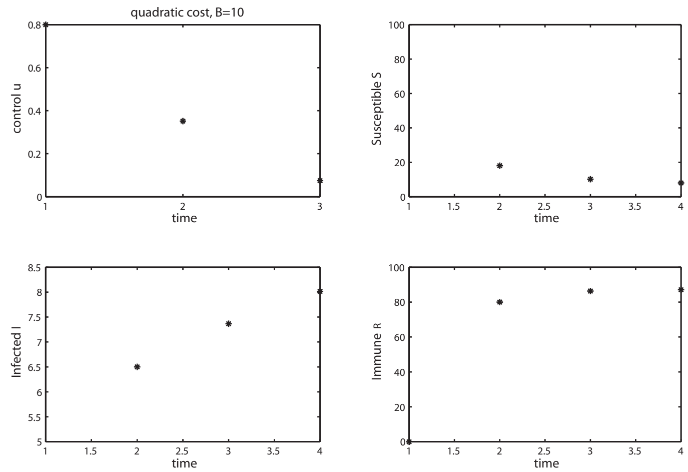

FIGURE 3. Case 1: S, I, R with $B = 10$

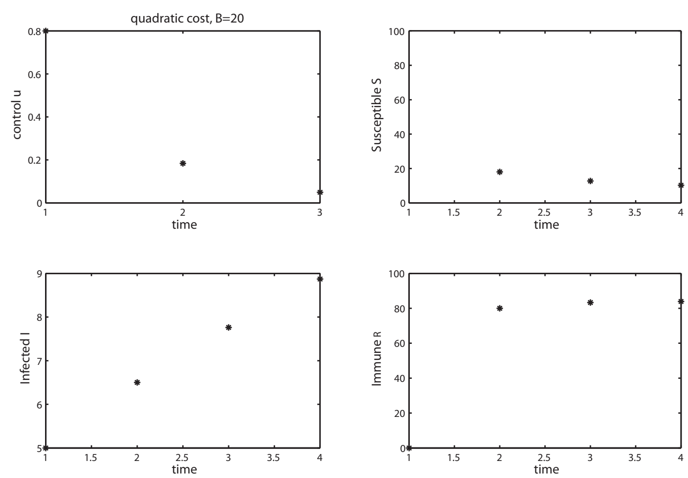

FIGURE 4. Case 1: S, I, R with $B = 20$

We choose $S(1) = 100, I(1) = 5, \beta = 0.02, d_2 = 0.1, d = 0.2$ and $T = 4$. First we give the numerical results without any control in Figure 1. Then we vary cost coefficient $B = 1, 10, 20, 50$ in J to see how it affects the control and the susceptible, infected and immune populations in Figures 2-5. We can see the number of infecteds are reduced significantly. As B gets bigger, it is more costly to apply the control, so the control effort is decreasing.

Case 2. Approximate Linear Cost

We still use the same parameters for $S(1), I(1), \beta, d_2, d$ and T. For Figure 6, we take $B = 0.01$ and $B_1 = 1$ in J to approximate the linear cost. The control for the first two time steps reaches the upper bound, then reduces to the lower bound

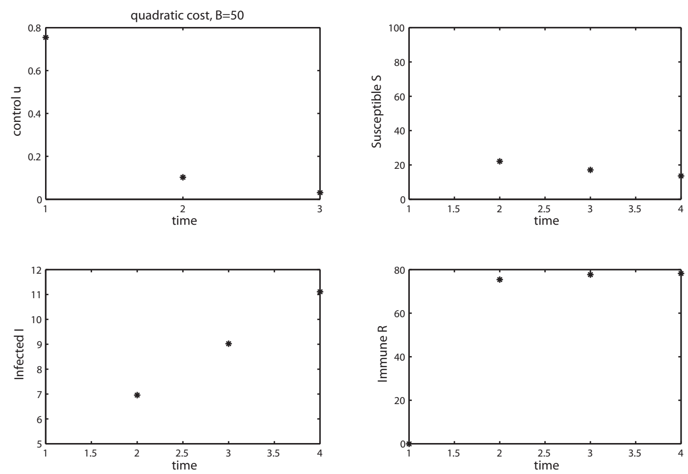

FIGURE 5. Case 1: S, I, R with $B = 50$

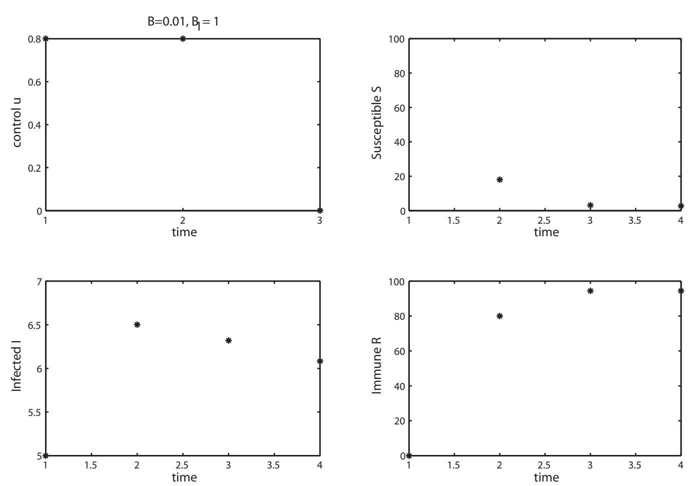

FIGURE 6. Case 2: S, I, R with $B = 0.01, B_1 = 1$

for the third time step. In Figure 7, we take $B_1 = 10, B = 0.01$ and see the control for the first time step reaches the upper bound, then reduces to the lower bound for the second and third time steps. So the control is a "bang-bang" control, which means the optimal control values are only at the upper and lower bounds. Note the differences in the optimal controls and the states between the two cases, illustrating that parameters in the objective functional make an impact.

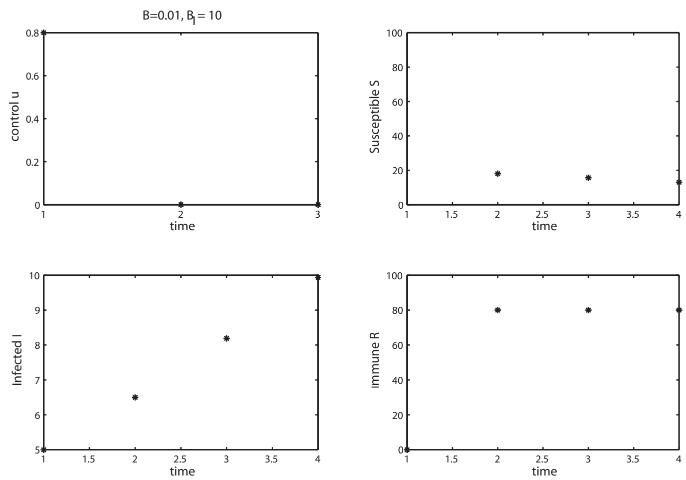

FIGURE 7. Case 2: S, I, R with $B = 0.01, B_1 = 10$

5. Conclusion

Here we have presented the techniques of optimal control on some simple examples of discrete models. Given a model with a control, the format of an optimal control depends on the format of the objective functional and the corresponding parameters. We illustrate in section 4 how changing parameters in an objective functional can affect the optimal controls.

For further examples of optimal control with discrete time, see [**10, 11, 12**].

References

[1] J. Abello and G. Cormode, editors, Discrete Methods in Epidiemology, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, American Mathematical Society, Providence RI, 2006,

[2] L. J. S. Allen and D. A. Flores and R. K. Ratnayake and J. R. Herbold, Discrete-time deterministic and stochastic models for the spread of rabies, Applied Mathematics and Computation, 132 (2002), 132, 271–292.

[3] H. Caswell, Matrix Population Models: Construction, Analysis and Interpretation, Sinauer Press, Sunderland, MA 2001.

[4] C. Castillo-Chavez and A. Yakubu, Discrete time S-I-S models with complex dynamics, Nonlinear Analysis TMA, 47 (2001), 4753-4762.

[5] W. Ding, L. J. Gross, K. Langston, S. Lenhart and L. S. Real, Rabies in Raccoons: Optimal Control for a Discrete Time Model on a Spatial Grid, J. of Biological Dynamics 1(4), 2007, 379-393.

[6] J. E. Franke and A-A. Yakubu, Disease-induced mortality in density-dependent discrete-time SIS epidemic models, J. Math. Biol. 57 (2008), 755-780.

[7] M. Kot, Elements of Mathematical Ecology, Cambridge, MA 2001.

[8] S. Lenhart and J. Workman, Optimal Control Applied to Biological Models, Boca Raton, Chapmal Hall/CRC, 2007.

[9] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelize, and E.F. Mishchenko, The Mathematical Theory of Optimal Processes, New York, Wiley, 1962.

[10] Sethi, S. P., and Thompson, G. L., *Optimal Control Theory: Applications to Management Science and Economics*, Kluwer, Boston, 2nd edition, 2000.

[11] A. Whittle, S. Lenhart and L. T. Gross, Optimal Control for Management of an Invasive Plant Species, Mathematical Biosciences and Engineering 4(1), 2007, 101-112.
[12] A. Whittle, S. Lenhart and J. White, Optimal Control of Gypsy Moth populations, the Bulletin of Mathematical Biology 70 (2008), 398-411.
[13] A-A. Yakubu, Introduction to discrete-time models, this volume to appear.

Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, Tennessee 37132
E-mail address: wding@math.mtsu.edu

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996
E-mail address: lenhart@math.utk.edu