Optimal Harvesting of a Spatially Explicit Fishery Model

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1. Motivation: Benefits of marine reserves?

Neubert (*Ecology Letters, 2003*) studied the fishery management problem:

Maximize the yield

$$J(E) = \int_0^L qE(X)N(X)\,dX,\ 0 \le E(X) \le E_{\max}$$

Subject to

$$-D\frac{d^2N}{dX^2} = rN\Big(1-\frac{N}{K}\Big) - qE(X)N,\ 0 < X < L,$$
$$N(0) = N(L) = 0.$$

2. Neubert's Results

- No-take marine reserves are always part of an optimal harvest designed to maximize yield
- The sizes and locations of the optimal reserves depend on a dimensionless length parameter
- For small values of this parameter, the maximum yield is obtained by placing a large reserve in the center of the habitat
- For large values of this parameter, the optimal harvesting strategy is a spatial "chattering control" with infinite sequences of reserves alternating with areas of intense fishing
  (*big variation in fishing efforts*)

3. Our Fishery Model: Steady-State

$$\begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $u(x)$ is the fish density, r is the growth rate, $h(x)$ is the harvesting depending on the location of fish, $\Omega \in R^n$, smooth and bounded domain.
Note $u \equiv 0$ is a solution. BUT we seek solutions that are positive in Ω.

4. Two Optimal Control Problems

Control Set I:

$$U_1 = \{h(x) \in L^2(\Omega)|\ 0 \le h(x) \le h_{\max} \text{ a.e.}\}$$

Goal I: Maximizing the yield and minimizing the cost of fishing

$$J_1(h) = \int_\Omega h(x)u(x)\,dx - \int_\Omega (B_1 + B_2h)h\,dx,\ h \in U_1.$$

Control Set II:

$$U_2 = \{h(x) \in H_0^1(\Omega)|\ 0 \le h(x) \le h_{\max} \text{ a.e.}\}$$

Goal II: Maximizing the yield and minimizing the variation of the fishing effort

$$J_2(h) = \int_\Omega h(x)u(x)\,dx - A\int_\Omega |\nabla h|^2\,dx,\ h \in U_2.$$

5. Optimality System I

- State equation

$$\begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega,\\ u = 0, & x \in \partial\Omega; \end{cases}$$

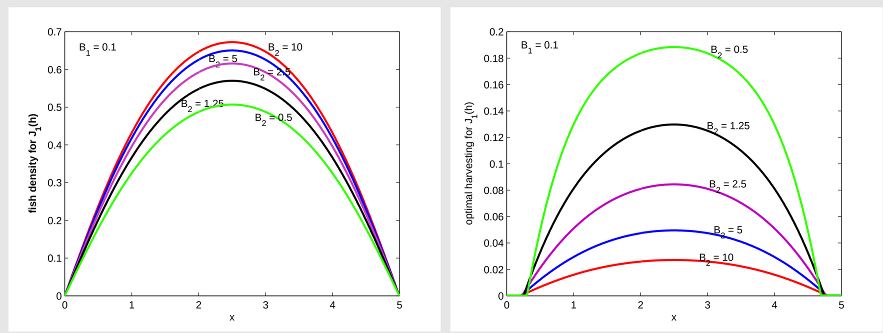
- Adjoint equation

$$\begin{cases} -\Delta p - r(1-2u)p + hp = h, & x \in \Omega,\\ p = 0, & x \in \partial\Omega; \end{cases}$$

- Characterization of optimal control

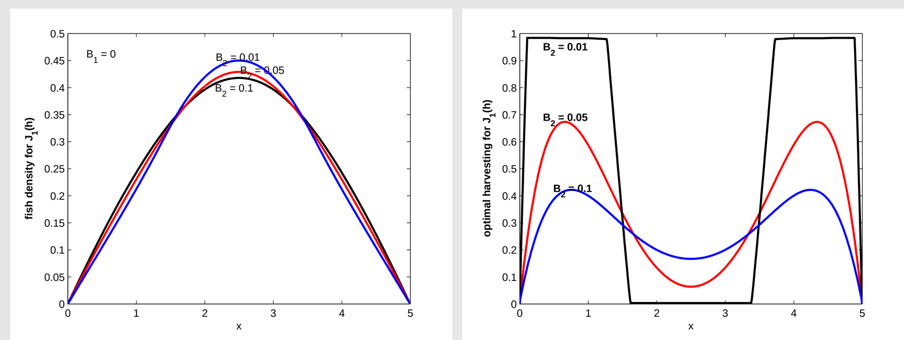
$$h(x) = \min\{\max\{0, \frac{u - pu - B_1}{2B_2}\}, h_{\max}\}.$$

6. Numerical Examples for J_1: 1-D case, B_2 effect - set $B_1 = 0.1$, vary $B_2 = 0.5, 1.25, 2.5, 5, 10$



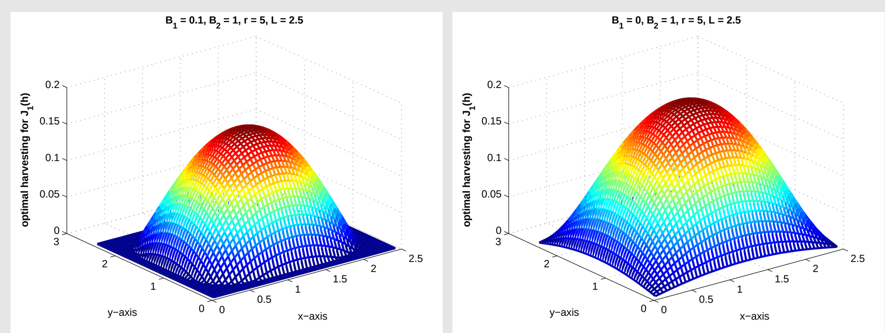
Fish Density and Optimal Harvesting

7. Numerical Examples for J_1: 1-D case, small B_2 - set $B_1 = 0$, vary $B_2 = 0.1, 0.05, 0.01$



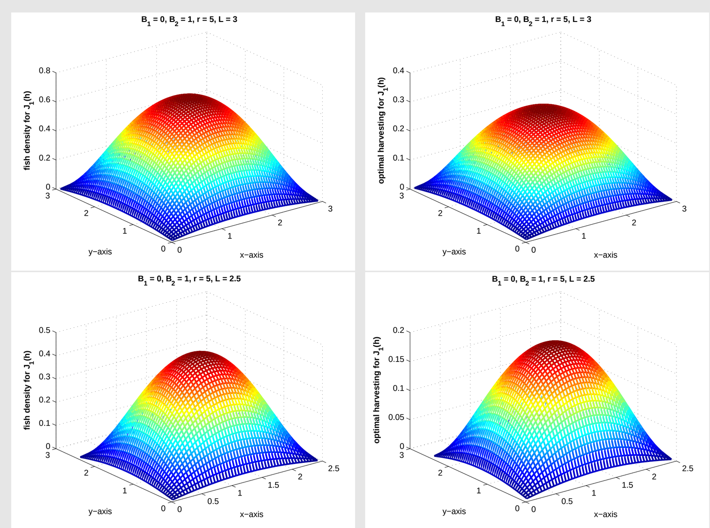
Fish density and optimal harvesting

8. Numerical Examples for J_1: 2-D case, B_1 effect



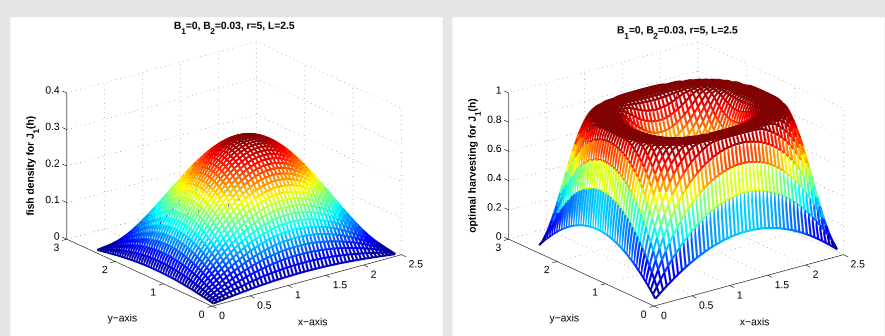
Optimal harvesting for $B_1 = 0.1,\ 0$

9. Numerical Examples for J_1: 2-D case, domain size effect: $(0,3)\times(0,3)$ v.s $(0,2.5)\times(0,2.5)$



Fish Density and Optimal Harvesting

10. Numerical Examples for J_1: 2-D case, small B_2, $B_1 = 0,\ B_2 = 0.03$



Fish density and optimal harvesting

11. Maximizing the yield with No-flux Boundary Condition

If $B_1 = B_2 = 0$ in $J_1(h)$, and we have Neumann (No-flux) boundary condition, then the optimal control and optimal state are

$$h^*(x) = \frac{1}{2},\ u^*(x) = \frac{1}{2}.$$

12. Generalize Neubert's Results

Maximizing the yield in Multidimension

If $B_1 = B_2 = 0$ in $J_1(h)$, then the optimal control is given by

$$h(x) = \begin{cases} 0, & \text{if } p > 1;\\ h_{\max}, & \text{if } p < 1;\\ \frac{r}{2}, & \text{if } p = 1. \end{cases}$$

13. Optimality System II

- State equation

$$\begin{cases} -\Delta u = ru(1-u) - h(x)u, & x \in \Omega,\\ u = 0, & x \in \partial\Omega; \end{cases}$$

- Adjoint equation

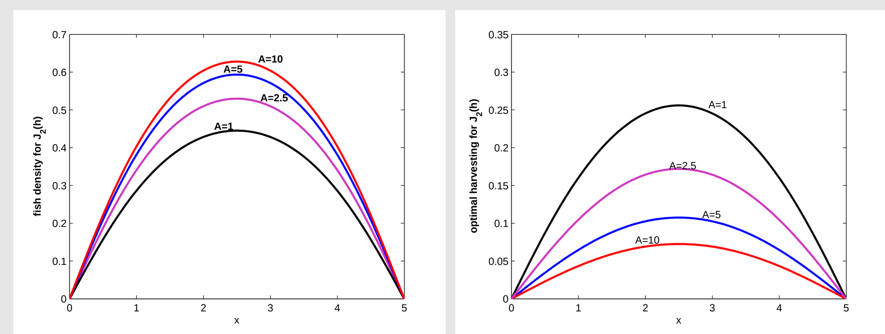
$$\begin{cases} -\Delta p - r(1-2u)p + hp = h, & x \in \Omega,\\ p = 0, & x \in \partial\Omega; \end{cases}$$

- Characterization of optimal control

$$\min\{\max(pu - u - 2A\Delta h, h - h_{\max}), h - 0\} = 0.$$

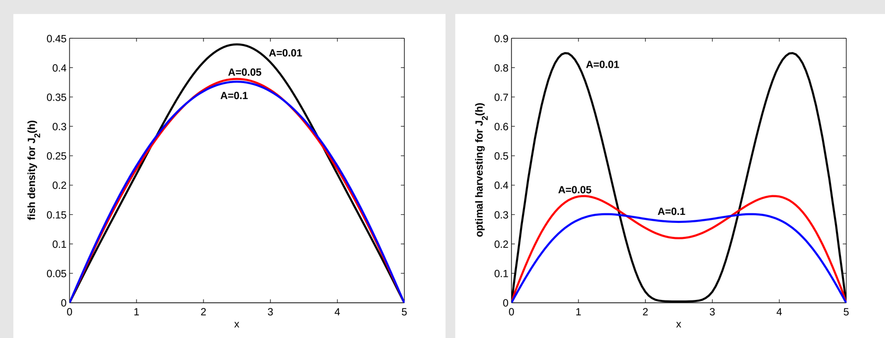
$$\begin{cases} pu - u - 2A\Delta h = 0, & 0 < h < h_{\max}\\ pu - u - 2A\Delta h > 0, & h = 0\\ pu - u - 2A\Delta h < 0, & h = h_{\max} \end{cases}$$

14. Numerical Examples for J_2: vary $A = 1, 2.5, 5, 10$



Fish density and optimal harvesting

15. Numerical Examples for J_2: vary $A = 0.1, 0.05, 0.01$



Fish density and optimal harvesting

16. Conclusion

- If we want to maximize yield and minimize cost, then increasing the cost coefficients B_1 or B_2, will decrease optimal harvesting
- With small B_1 and B_2, the harvest control is concentrated near the boundary
- If we only want to maximize yield, then reserve is part of the optimal harvesting strategy
- The problem of maximizing yield only with Neumann boundary condition gives a simple optimal control, a singular case
- For J_1, the optimal benefit increases when domain size increases
- If we want to maximize yield and minimize variation in fishing effort, then increasing the variation coefficient A will reduce optimal harvesting