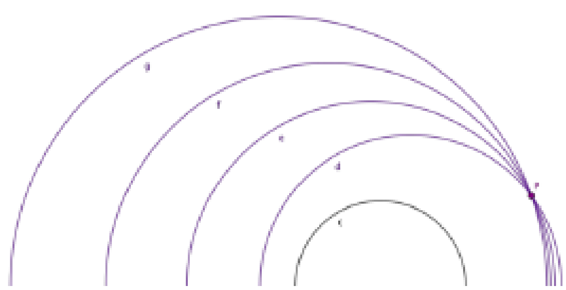
Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2): Geometry for Secondary Mathematics Teaching

Emina Alibegović
Alyson Lischka

Version Summer 2019



The Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE(S^2)) project is partially supported by funding from a collaborative grant of the National Science Foundation under Grant Nos. DUE-1726707,1726804, 1726252, 1726723, 1726744, and 1726098. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Contents

MODULE(S^2) Geometry Course Overview

In order to teach geometry one needs to understand its place in the wider mathematical context, its development and possible approaches to its study. There is much that can be done with and learned about geometry, but our path for this course will be focused on the needs of our students as future teachers.

We are used to the Euclidean approach to developing geometry involving an axiomatic approach and sometimes that of straight edge and compass constructions. Although we understand the need for our students to understand the axiomatic structure of geometry and to develop their proving skills, we also see competing needs for the time in a geometry class for preservice teachers. For many of our teachers, the geometry course they will be teaching necessitates understanding of transformations that are not always included in traditional Euclidean geometry class.

For this reason we chose to split the course into three modules hoping to address content relevant to mathematics teachers practice.

- Module 1: Axiomatic Development
- Module 2: Transformations
- Module 3: Analytic Geometry and Similarity

The work of a teacher consists of more than just possessing the knowledge of mathematics. Teachers are supposed to help others attain knowledge they are required to attain. For example, the teacher is tasked with creating, selecting, and modifying tasks; a skill not often addressed in the mathematics classroom. Another relevant skill is ability to understand, find flaws, and help students find errors in their arguments. In this course we are providing opportunities for students to develop and practice these skills in addition to learning geometry.

Instructional Notes and Expectations

In effort to create a learning environment that is aligned with current recommendations for teaching and learning mathematics in K-12 settings, the following practices are suggested for instruction in this course:

- Use of Technology: We encourage the investigative use of dynamic geometry programs for the exploration of ideas generated in class and through assignments. In addition, there are places where students will be directed to particular internet sites to read about the concepts of study. However, we discourage students from general internet searches when completing homework and instead encourage them to problem solve and communicate with peers in order to work as mathematicians work when pursuing new ideas. Reminder of this policy is often necessary throughout the course.
- Note-taking Assignment: As there is no textbook for this course, we recommend assigning note-taking as part of the regular work of the course. Each day, one student should be assigned as the official note-taker, who then submits notes completed in a template you provide. These notes can be used as part of a homework or participation grade for the course and posted in your learning management system for all students to have access. If you use a template for this activity, it is easily combined into a book authored by your class at the end of the semester.
- Handouts: In-class activities for students and homework assignments are listed as handouts. However, most can be easily incorporated into a digital display for the class or shared electronically through your learning management system.
- Homework: In many cases, homework assignments are structured so that they generate discussion for the next or an upcoming lesson. Assigning homework so that it can be submitted through a learning management system prior to the class in which it will be discussed provides the instructor an opportunity to peruse the work and adequately anticipate questions for the following class discussions. In some cases, considering how students respond to questions in homework prior to a class session can aid in assigning students to groups that will then move forward in their thinking based on shared ideas.
- Video Assignments: Two of the culminating assignments require access to videos and annotating tools on the web. In order to engage with video animations of mathematics classrooms you will need to invite students to open accounts at specified websites.

Module I

Axiomatic Development of Geometry

In Module 1 our goal is to develop appreciation for and understanding of the structure of mathematics in general and geometry in particular. Most of the time, people go about it through a careful development of Euclidean geometry, starting with a set of axioms, often small, then proving everything one possibly can, then expanding the set of axioms. We have opted against this approach since it is a time consuming endeavor, and there are several topics on which we felt pre-service teachers need to have a good grasp.

We will try to lead students through some of the interesting proofs, but this will be a shaky territory for both them and us. Namely, we haven't developed the machinery very carefully, so there could be some discomfort on our side with deciding what to demand be proven and what to leave out. On students' side, it may be frustrating not to know what is allowed and what isn't, what is known, and what isn't. You may want to leave it to students to decide which set of axioms they prefer to work with once we discuss several of them. We will try to indicate where it may be necessary to allow students to get away without proof as to not to burden the discussion unnecessarily. On the other hand, it is important to provide some opportunities for students to practice their proof writing skills, so we will take some time to do that as well. We like to think that it is much more important for students to develop a sense when their argument is flawed or incomplete and to say things like "If I knew (blank), then I can prove (blank)", and we will work toward that end.

- **Big ideas** In this module we will help students understand the structure of axiomatic systems. We will consider the impact different axioms have on the structure of geometry with a culminating experience involving the parallel postulate so that students understand the diversity of alternatives to Euclidean geometry.
- **Goals for studying the topic**:
  - Understand how geometry is built.
  - Become aware of and appreciate geometries other than Euclidean.
  - Be able to articulate differences between neutral, Euclidean, spherical, and hyperbolic geometries.
  - Develop skills for writing proofs; understand the role of axioms and rules of logic.
  - Develop ability to make sense of other's reasoning.
- **Rationale** A common complaint of high school students is that they are asked to prove things that are entirely obvious to anyone who is willing to look. In other words, they do not understand the need for proof. Part of the reason is that they have no sense that alternatives are possible. In this module, we introduce student to non-Euclidean geometries both through considerations of axioms, but also more intuitively with a goal of easier transfer to the K-12 classroom. While we believe that it is important for students to understand the structure of axiomatic systems and what constitutes mathematical reasoning, we also believe it is important for them to develop intuition and familiarity with different geometries.
- **Connections to Secondary Mathematics** In this module the student will start to understand the axiomatic system the Common Core State Standards advocates using in 6-12 curriculum: transformational approach. Unfortunately, as far as we know the axiomatic system of this nature that would be appropriate for secondary school has not been yet developed in detail, so we cannot choose it to help students develop the knowledge of curriculum. In order to remedy that, we will consider variety of systems in order to help them realize that while the choice impacts what results we can obtain, it is possible to develop equivalent sets of axioms which yield the same geometry.
- **Overview of content**
  - Lesson Where should we live?: In this lesson, students are introduced to the notion that Euclidean geometry isn't the only valuable one. Urban geography gives rise to Taxicab geometry as an alternative system to the one students are used to. Through this module students will get acquainted with other geometries, most notably spherical and hyperbolic. students will also encounter modeling which is an important strand in the standards for high school geometry.
  - Lesson What is Geometry: students are asked to reflect on what the building blocks of geometry are. Through this discussion we will start talking about definitions, undefined terms, and start developing our skills in writing definitions. Additionally, we will discuss proving and what constitutes a good proof.

◦ Lesson Where do the differences come from?: We continue to challenge students' notions of what geometry is. In this lesson, we will start exploring models of different axiomatic systems without being very formal about it. The student will talk about lines and notion of straightness on the plane (with both Euclidean and Taxicab distance), as well as on a sphere.

◦ Lesson Sets of Axioms We finally discuss axiomatic systems in order to more formally explain where all these differences are coming from. We discuss them from the mathematical point of view, but also from the point of the view of the user.

◦ Lesson Consequences of our Choices We elucidate further differences by discussing parallel lines. We discuss the parallel postulate, its historical and mathematical significance, and prove some important results in Euclidean geometry.

◦ Lesson What do students understand about axiomatic systems? We conclude the Module by considering representation of practice and engaging students in developing their own classroom discussion as it relates to the axiomatic system

- **Overview of mathematical and teaching practices** While we hope that all of our lessons will be in line with the *Standards of Mathematical Practice* as outlined in the CCSS, in this module we will particularly pay attention to:

SMP 2 Reason abstractly and quantitatively

SMP 3 Construct viable arguments and critique the reasoning of others

SMP 4 Model with mathematics

SMP 6 Attend to precision

SMP 7 Look for and make use of structure

- **Expectations for assignments**

◦ students can be tasked with writing a course textbook. After each lesson, a student is asked to write out the accomplishments of the day. The goal is to leave a reference for all students, give everyone an opportunity to practice writing, and provide feedback to each other.

◦ Writing Assignments are to be graded and feedback provided. These assignments may be commented on before a final version is submitted for evaluation.

◦ Homework Assignments are generally a preparation for class or planning information for the instructor. The instructor may choose whether to review and provide feedback.

◦ Instructors may choose to have a class discussion board where questions can be posted and encouraged. The following prompt may serve well to encourage questioning and discussions throughout the course: "An important part of doing mathematics is to learn to ask one's own questions. You might not be able to answer them, but you can always learn more through seeking the answer to your questions. Which questions has this work made you ask? Record all the questions that you'd like to investigate on the class website."

Lesson	Projected Length	Geometric Content	In-Class Activities	Homework	Connections and Notes
1: Where should we live?	90 min	Equidistance, perpendicular bisectors, Taxicab Geometry introduction, mathematical modeling	Choosing a Place to Live	Introductory Survey	• Choosing a Place to Live will be used to generate discussion in the lesson 2 and 3 activity discussions • Introductory Survey Question 6 will be revisited in Lesson 3 Homework: Constructions • Introductory Survey Question 7 will be used for discussion of the activity in Lesson 2 and 3
2: What is Geometry?	90 min	Foundations of a Geometry (undefined terms), shortest paths (straightness), Spherical Geometry	The Building Blocks	Shortest Paths	• Shortest Paths Question 1c will be used to generate discussion in Lesson 3 • Shortest Paths Question 2 will be used to generate discussion in Lesson 3
3: Where do the Differences Come From?	180 min	Circles, attending to precision, definitions, Euclids Postulates, interpretation and model	Euclids Assumptions	Constructions Distance on a Sphere (optional)	• Constructions Question 2 will be revisited in Lesson 6
4: Sets of Axioms	180 min	Incidence Axioms, Hilberts Axioms, proof, Hyperbolic Geometry	Incidence Geometry Handout: Hilberts Axioms	An Alternative Set of Axioms Angle Proofs	• You may choose to assign Incidence Geometry #5 as homework • An Alternative Set of Axioms cannot be assigned until after students have completed Incidence Geometry and started discussion of Hilberts Axioms
5: Consequences of Our Choices	180 min	Differences between neutral, Euclidean, hyperbolic, and spherical geometries; definitions, right angles, parallel lines, angles formed by transversals	Right Angles Parallel Lines Sum of the Angles in a Triangle	Video Simulation of Practice VanHiele Readings Triangles (optional)	• Video Simulation of Practice can be assigned as soon as Lesson 5 begins • You may choose to assign the Sum of the Angles in a Triangle as homework
6: What do Students Understand about Axiomatic Systems?	90 min	Summary of different axiomatic systems, examples of axiomatic systems in secondary classrooms	How do students understand axiomatic systems?	Written Simulation of Practice	
7: Culminating Activities				Mid-term Exam	

1 Where should we live?

Overview

Length

1 Class Meeting, ~ 90 minutes

Summary

In this lesson, we will connect the work we do in geometry to the world around us. We will see how the modeling and problem solving processes can play out in a geometrical context. We consider a situation many of us are familiar with: choosing a place to live. Such decisions come with a myriad of constraints, and we will attempt to solve this problem in the most general case.

In addition to engaging students in the modeling and problem solving processes, we include this problem to provide the glue to the unit. It will surface the ideas that will recur throughout the unit. We will introduce the difference between a plane equipped with Euclidean distance and one equipped with Taxicab distance. We will start thinking about circles and how they are constructed, as well as lines, specifically perpendicular bisectors.

Goals

- Students will engage in the modeling and problem solving processes.
- Students will explore the differences between a plane equipped with the Euclidean distance and a plane equipped with the Taxicab distance.
- Students will be introduced to the construction of lines, perpendicular bisectors, and circles.

Connection to Standards

CCSSM Standard	Connection to Lesson
MP4: Model with Mathematics	PSMTs will use mathematical modeling to engage with a problem involving choosing a location for housing based on the locations of places of employment. This task will introduce basic geometric objects to be used in discussion as the unit progresses.
HSG.MG.A.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).	Students will apply knowledge of perpendicular bisectors to determine possible locations for searching for housing.
HSG.CO.D.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.	Students will use tools to construct a perpendicular bisector in order to locate possible choices for home locations.
Concepts Beyond CCSSM	**Connection to Lesson**
Taxicab Geometry	In the process of considering a best possible place to live, students will be urged to acknowledge that one cannot travel off the given roads and therefore will consider a taxicab geometry system. This will take place only informally in this lesson but serve as introduction to exploration that will take place in future lessons.

Materials

- Compasses and Rulers (optional)
- Activity: Choosing a place to live
- Homework: Introduction Survey

Lesson Description

In this lesson, you will have an opportunity to make important observations about the ways your students interact with mathematics and with each other. Consider this activity to be setting the stage for conversations about the norms of engagement in your classroom. If you have a particular way of forming groups, use those methods. We tend to make random assignments on the first day as we do not have much information about students at this point. Students will work in groups throughout this lesson.

You may want to start by asking students if they moved recently. Ask them to discuss with their table groups what sorts of questions they ask themselves when they look for a place to live. Have the groups share their thoughts before you introduce the problem. Gather their responses for all to see on a board or poster and have discussion about which of the factors might be most important to consider in choosing a place to live.

Activity: Choosing a place to live

Shefali and Vanya have started dating a few months back and things are going pretty well. Vanya isn't too fond of their housemate, and has been advocating for moving in together. Shefali just got a new job, and isn't opposed to the idea in hopes to reduce her commute time.

1. Shefali works at the University of Utah (underlined in blue), while Vanya works southwest of Liberty Park (location circled in purple). They started considering where they should search for a place to live if they wanted their commutes to be roughly equal.

2. Both Shefali and Vanya commute on their bicycles. Turns out Shefali is far better cyclist than Vanya. She can usually ride the same distance in about two thirds the time Vanya takes. What would be a fair place for them to look for the house considering this newly available information?

3. (optional) What isn't necessarily clear from the map is that Shefali's place of work is about 400 feet higher in elevation than Vanya's. What are your recommendations now?

Note. This problem is easily modified to be relevant to your students. Change the map so that it is a map of your town. Choose the places of work so that you introduce the conversations that may be relevant to your town. These may include the issues of cost, gentrification, segregation, and others.

1. You may want to start by asking students if they moved recently. Ask them to discuss with their table groups what sorts of questions they ask themselves when they look for a place to live. Have the groups share their thoughts before you introduce the problem. Gather their responses for all to see on a board or poster and have discussion about which of the factors might be most important to consider in choosing a place to live.
2. Distribute the task and engage groups in working on question 1 together. Plan for approximately 30 minutes for groups to discuss all three questions, though they may not reach a conclusion on question 3. You may wish to provide tools such as compasses and rulers for those who want to be more precise in their work. We are hoping that various groups will attempt the problem in different ways. As you circulate the room, pay attention to different strategies students are using. Look for the following approaches:

   (a) Approximate the places of work with two points, remove the grid, and consider the unrestricted problem. In this case students should be looking for the perpendicular bisector. In the event that they only find a midpoint, ask questions that would encourage them to think about *all* the places that would be fair to both Shefali and Vanya.

   (b) Similar approach may be to include the grid, but more of a Cartesian grid, and approximate the two places with specific points, then continue in the similar manner as above.

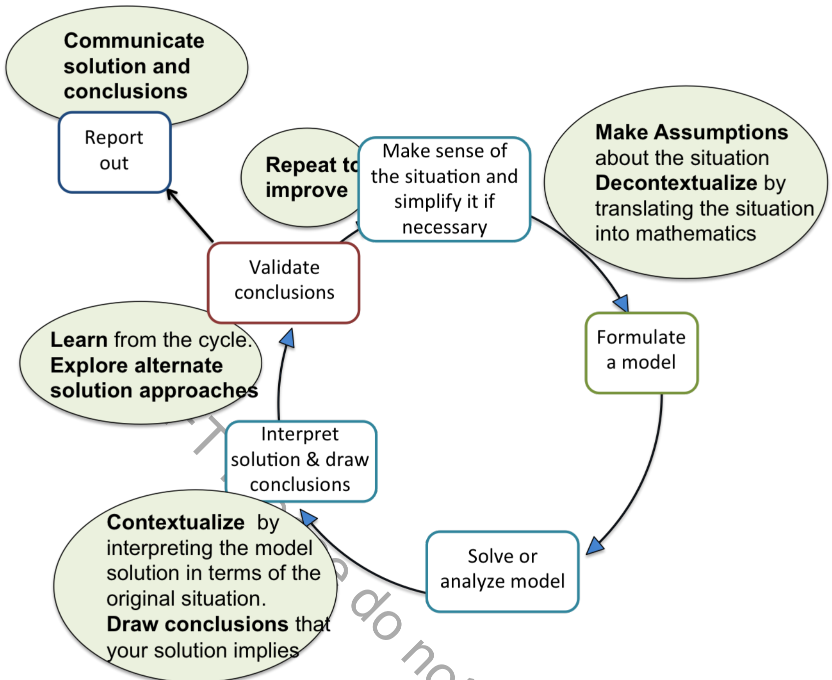
   (c) Include the actual physical constraints into the conversation - we do live in a city, we can only commute along the streets. They still may choose specific points on the Cartesian grid and use the vertical and horizontal lines as constraints (possibly through points with integer coordinates), or they might choose to use the actual town grid.

3. After about 10 minutes or when you notice the conversation dying down, bring students into a whole class discussion and have each group share out their strategies for solving Question 1. You might encourage groups to summarize their thinking prior to initiating the whole class discussion.

   (a) You should start with groups that used Euclidean distance, then follow up with those that used Taxicab distance. Have students compare their "perpendicular bisectors", and discuss the differences. Push them to articulate the reasons for why those look different in the two different circumstances.

   (b) Students should discuss the merits of the different approaches. Notice that we engaged in several important mathematical practices involved in the modeling cycle: translating a problem into a mathematical situation, reducing complexity of a problem, decontextualizing, then contextualizing once the solution has been obtained, and revisiting and validating solutions. Discuss this process with reference to the Modeling Cycle. Consider asking some of the following questions:

- Describe precisely what decisions you had to make in solving this problem.
- If you were to explain to a high school student what you had to do to get to the mathematics in this problem, how would you explain that?
- How did you decide which factors to consider and which to ignore?



4. After the whole class discussion of Question 1, have students get back into small groups to discuss and determine a solution to Question 2. This question tries to point out the difference between equal and equitable. Although it may be "fair" for both to travel the same distance the fact that commuting is easier for one of them should play into deciding where they should search for a place to live.
   (a) Depending on your population, you may replace bicycle commuting with car and/or public transportation.
   (b) Notice that the goal here is not to arrive at any particular solution. In fact, we are interested in sensible solutions and would like to celebrate different approaches. Students all too often believe that "the right answer" is what is beautiful and comforting about mathematics and disregard the fact that sometimes there just isn't one single correct answer.
5. After students have had time to discuss in small groups, have each group communicate their solution to the whole class.
6. Consider discussion of question 3 as time permits in your class. It is not necessary to reach a consensus on questions 2 or 3 before moving into the next lesson. Discussion of the task alone will provide the necessary foundation for moving into the next lesson.

Homework: Introduction Survey

- Students should fill out a form in which they provide an email address with which they can access Google Docs and Google Sites. Listed below as *Introduction Survey* Handout is the text of the survey but you should alter this to fit your needs.
- Students should download Geogebra (we find Classic to be easiest to work with) to their computers and have them available in class. If you choose to require students to type the course book with LaTeX, it may be most convenient to start a template file for them where they can work on it together. For example, `https://www.overleaf.com` works well.

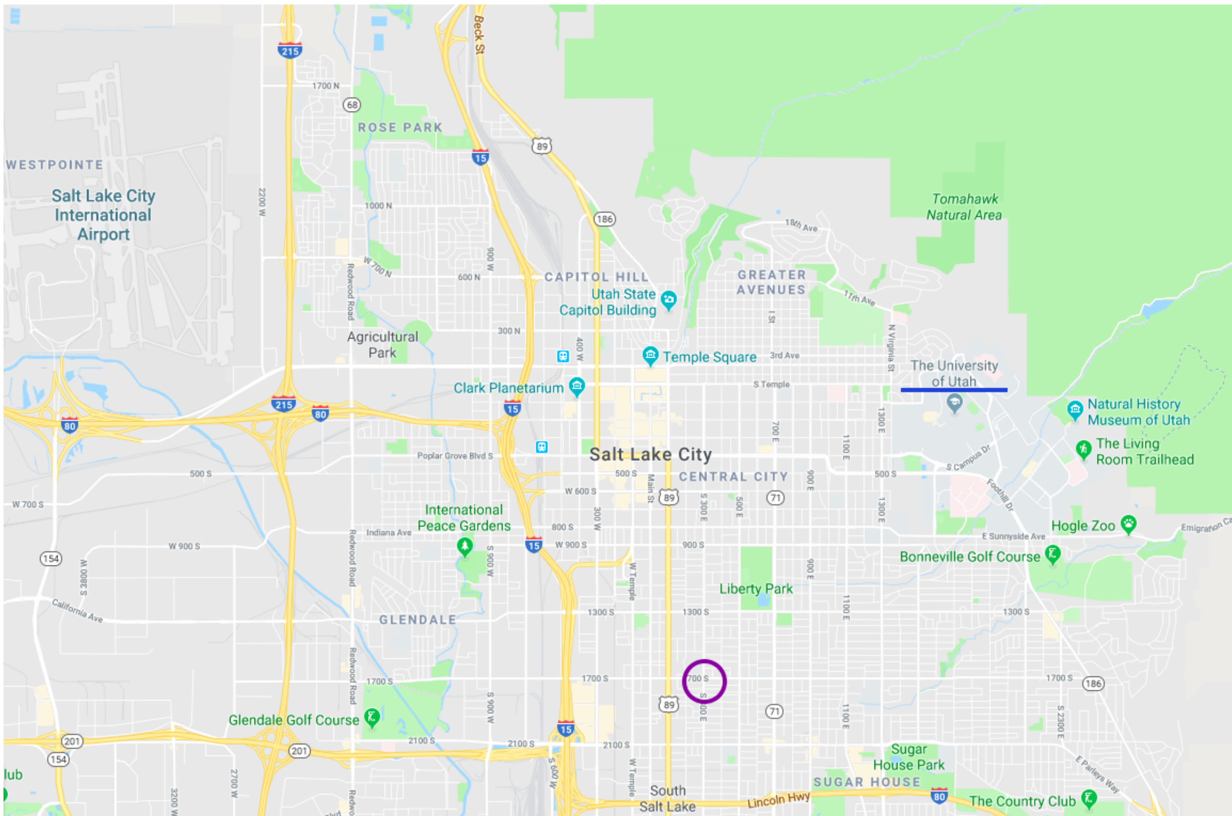
Note.

- Introductory Survey Question 6 will be revisited in Lesson 3 Homework: Constructions
- Introductory Survey Question 7 will be used for discussion of the activity in Lesson 2 and 3

ACTIVITY: CHOOSING A PLACE TO LIVE

Shefali and Vanya have started dating a few months back and things are going pretty well. Vanya isn't too fond of their housemate, and has been advocating for moving in together. Shefali just got a new job, and isn't opposed to the idea in hopes to reduce her commute time.

1. Shefali works at the University of Utah (underlined in blue), while Vanya works southwest of Liberty Park (location circled in purple). They started considering where they should search for a place to live if they wanted their commutes to be roughly equal.



Solution. Solutions will vary depending on the map used. In general, here are some features you should expect from a good solution, as well as what you can anticipate.

- Students may disagree about whether "roughly equal commutes" refers to equal commute *times* or equal commute *distances*. This issue becomes more relevant in the next question.
- Students may initially interpret this question as asking for a location that equalizes *and minimizes* the commute – in other words, they may think the question is asking for the **midpoint**, rather than the perpendicular bisector.
- The fact that (in this map) the University of Utah is labelled at a point in the interior of a region without roads may lead to discussions/disagreements about where, exactly, that endpoint lies, and the degree to which the roads are genuinely a constraint. This issue also becomes more relevant in the next question.
- A reasonable way to solve this problem might be for students to first find the approximate midpoint (in the Euclidean sense) of the segment joining the two points, then map a sequence of points at "corner points" moving diagonally away from that midpoint. There should be some discussion of whether the dots should be connected; this depends on what (if anything) exists "between the gridlines". Are there houses there? Neighborhoods? How grainy is the resolution of this map?

2. Both Shefali and Vanya commute on their bicycles. Turns out Shefali is far better cyclist than Vanya. She can usually ride the same distance in about two thirds the time Vanya takes. What would be a fair place for them to look for the house considering this newly available information?

Solution.

- This question brings the distinction between "equal distance" and "equal time" (which was in the background for the previous problem) to the foreground. The solution set is the locus of points whose distance from Shefali's workplace (the University of Utah) is 3/2 the distance from Vanya's workplace.
- The problem is difficult to answer even in simple Euclidean geometry! If you draw a straight line ℓ through the two workplaces, there are two points (call them P_1 and P_2) on ℓ that satisfy the requirement of having the distances be in the ratio 3 : 2.
- The general solution (in the Euclidean case) is the circle whose diameter is $\overline{P_1P_2}$. Probably the easiest way to show this is to put a coordinate system on the map so that Shefali's workplace is at $A = (0, 0)$ and Vanya's is at $B = (1, 0)$. Then the condition for a point $P = (x, y)$ to satisfy $AP = \frac{3}{2}BP$ can be expressed algebraically by

$$x^2 + y^2 = \frac{9}{4}\left((x-1)^2 + y^2\right)$$

which, after a great deal of simplifying, rearranging, and completing the square, is equivalent to

$$\left(x - \frac{9}{5}\right)^2 + y^2 = \frac{36}{25}$$

This is the equation of a circle centered at $(9/5, 0)$ and with radius $6/5$, which means it passes through the points $P_1 = (3/5, 0)$ and $P_2 = (3, 0)$.
- It is worth noting that exchanging "equal distance" for "equal time" changes a line (the perpendicular bisector) into a circle. So in a certain sense the circles are the "lines" of a geometry based on time, rather than distance.

3. (optional) What isn't necessarily clear from the map is that Shefali's place of work is about 400 feet higher in elevation than Vanya's. What are your recommendations now?

Solution. Without knowing more details about how the elevation varies across the map, it's really impossible to answer this question precisely, but it does seem reasonable to suggest that they should search for a house that is even a bit closer to Shefali's workplace than Vanya's than the previous solution.

Homework: Introduction Survey

1. Your name
2. Email address [1]
   this should be an address you regularly check and one that is associated with your Google Docs and Google Sites account.
3. I have a laptop I can easily bring to class every day
   - Yes, no problem
   - Yes, but I prefer not to bring it every day
   - I do not own one.
4. I have necessary software for the class.
   There is only one correct answer to this question. Visit Geogebra and LaTeX (or Another) to obtain necessary materials. If you are using public computers for this work, you can use Geogebra's web app and save your work to a google drive, or Geogebra account. As for Latex, Overleaf is a lovely web based option.

   > **Note.** You may choose to have your students write the textbook in Word, or as Google Doc for collaboration purposes. In that case you don't need to insist on LaTeX.

   - Yes, I have it!
5. Your thoughts:
   - Any questions or comments?
   - Any concerns?
   - Any requests?
6. Open your newly acquired Geogebra and construct any quadrilateral at all. Make sure to use the *Polygon tool* from the menu, instead of constructing four independent segments connecting at their endpoints. Use the *Midpoint tool* to construct the midpoints of each side, then construct a quadrilateral whose vertices are the consecutive midpoints. Save this file as FirstName_LastName_MidpointQuad. Select the pointer, then drag the vertices of your original quadrilateral around. What do you notice? What do you wonder? Are there any properties of the quadrilateral that stay the same throughout your manipulation of the vertices? Make as many conjectures as you can and be prepared to share with your partners. Is there any way you can be certain of your observations?

   > **Note.** The goal of this problem is to start getting students used to working with Geogebra. Additionally, we are interested in getting to know students and some of their affective traits: how willing are they to play around, how perseverant they are, and what their comfort level with technology may be. We will bring this problem back during this module, but also in the following modules to see a variety of ways in which the problem can be solved.

   > **Solution.** Students are likely to notice that the "inner" quadrilateral appears to always be a parallelogram, no matter how the "outer" quadrilateral is deformed. (This is actually something that is often taught in high school geometry, so for some students it will be less a matter of "noticing" and more a matter of *remembering*). Some other possible observations:
   >
   > - Many students will simply observe that "you always get a quadrilateral".
   > - Some students will claim that the midpoint quadrilateral is always contained in the interior of the original quadrilateral; others may notice that if the original quadrilateral is convex, then the midpoint quadrilateral will leave the interior.
   > - You may get some claims about special cases:
   >   - If the original quadrilateral is a rectangle, then its midpoint quadrilateral is a rhombus.
   >   - If the original quadrilateral is a rhombus, then its midpoint quadrilateral is a rectangle.

[1] I need this for setting up the class website

- ○ If the original quadrilateral is a square, then it is both a rectangle and a rhombus, and by the previous two observations its midpoint quadrilateral is also a square.
  - ○ More generally, if the original quadrilateral has a line of symmetry through two vertices, then the midpoint quadrilateral will have a line of symmetry through two of its sides, and vice versa. This observation implies all of the above, as well as explains what happens when you begin with an isosceles trapezoid or a kite.
- The construction can be iterated; some students may notice that if you begin with a parallelogram, and “dualize” it twice, the double-dual will be similar to the original. (This is easiest to see in the case where we begin with a rectangle or a rhombus.)

7. Before reading further, take a moment to write for yourself what you think geometry is.

The word ”geometry” comes from the Greek for ”Earth Measurement”, so it seems reasonable that Greeks thought that geometry was a true and absolute description of the physical world. Because they were studying physical objects in the real world, the measurements came with a certain degree of imprecision, and the relationships one finds are only as reliable as the measurements one makes. This is always the case when we describe the physical world: our knowledge of the world around us is only as good as the observations or measurements we can make about it.

Despite the origins of its name, geometry is not, in fact, a method for describing the world around us. When studying geometry, we do not try to make claims that can be verified or falsified through measurement; instead, we make claims that can be verified through logical reasoning or falsified through counterexamples. Geometry is not an empirical science, it is a deductive science. We decide whether statements are either true or false GIVEN CERTAIN HYPOTHESES. We give hypotheses and discover their consequences, which we call **theorems**.

Poincaré said:

> If geometry were an experimental science, it would not be an exact science. It would be subject to continual revision. The geometrical axioms are therefore neither synthetic *a priori* intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts, but it remains free, and is only limited by necessity of avoiding every contradiction. What then are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measurements are false, if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another: it can only be more convenient.

What questions do you have after reading these paragraphs? Write at least three you would like to discuss with your colleagues.

Note. We would like to plant the seeds for the conversations that are about to start in our class. students will be introduced to the idea that there isn’t a single geometry, rather we have a system that depends on the hypotheses we choose. They may need help unpacking the meaning of the quote, but try to leave this entire conversation for after they have engaged in the class activity. That work may help them make sense of what this reading was about, and can help them arrive at answers to their own questions.

2 What is Geometry?

Overview

Length

1 Class Meeting, ~ 90 minutes

Summary

In this lesson, we will try to articulate what geometry is. We will attempt to define the objects of study, and try to reconcile some of the issues that the previous lesson brought up. Further, we will start thinking about the role of proof in geometry. The lesson will also serve as a brief historical overview of Euclid and his work.

Goals

- Discuss building blocks of geometry
- Start the conversation about definitions
- Introduce alternatives to Euclidean geometry

Connection to Standards

CCSSM Standard	Connection to Lesson 1
Standard 1	Connection between standard 1 and Lesson 2 will appear here
Standard 2	Connection between standard 2 and Lesson 2 will appear here
Standard 3	Connection between standard 3 and Lesson 2 will appear here

Materials

- Activity: The Building Blocks
- Homework: Shortest paths
- Lénárt spheres, if you have them. Globes, beach balls or tennis balls will work just fine.
- String
- Markers

Description

Throughout the semester we will continue working on understanding the structure of definitions, and how to write them. Ultimately, the point here is for students to understand that some terms cannot be defined, and instead we give them meaning through the assumptions we make about their properties. Before we make that point, we will work on definitions and difference between these different geometries for just a little bit longer. We will first investigate circles and if you choose to do so think little more carefully about distance on a sphere.

You may choose to lead the class with the discussion of homework or simply introducing the activity for the day. Our recommendation is to delay the discussion of reading and the questions students have until after they've engaged with the class activity - it may be helpful to them as they keep their homework in mind. You can be explicit about that: While I am very interested in hearing your impressions and questions that may have been spurred by your reading, I would like to invite you to keep it in the back of your mind as you work on the next activity. Think both about the reading as well as the work you have done in class last time.

Activity: The Building Blocks

> 1. What geometric objects did you work with in Activity: Choosing a place to live? Give a definition for each of the objects.
> 2. Maybe you think there are other objects that are essential to geometry work, but haven't come up in Activity: Choosing a place to live. If so, add them to the list.

1. Have students form small groups to discuss Questions 1 and 2. After group conversations have died down, bring the entire class together for a short class discussion of their responses.
   (a) Invite students to share their definitions. Continue to push their understanding of "line" and "straight" with the goal to challenge their ideas of geometry and to further help them see why some terms ought to stay undefined. Question 3 should further help with this idea.

   > 3. Look back at your definition of a line. Is the flight path of this airplane a line? How was this flight path determined?
   > 4. Take a moment to study definitions given by Euclid: https://goo.gl/axf87H. What do you think about these definitions? Do they communicate precisely what the object is, or do you need to know what they are before the definitions make sense? Do all the lines we've discussed so far fit into Euclid's definition?

3. After having a whole class discussion about Questions 1 and 2, have students get back into their small groups to discuss Question 3.
   (a) Most of students have flown before and will be familiar with the flight path represented in Question 3. Here we have couple of issues: we have a projection of a sphere and the flight path onto the plane. The flight path looks particularly curved and not at all as the shortest path between the two given points; the two towns are at the roughly the same latitude, and some students may want to call that the shortest path. Offer some string and spheres (Lénárt spheres work well, but so do beach balls, or globes) for further investigation. While students are trying to decide what a straight line (or shortest path) is and how to determine whether a line is straight, monitor their small group discussions.

   > **Note.** They will likely pretty quickly decide on shortest path and taut rope idea, which should be the first ones to be discussed as well. You may want to push students toward symmetries. For instance, a straight line has reflectional symmetry both along itself and any line perpendicular to it. We could think of the plane without a straight line as being symmetric across that line as well. Straight lines also have 180° rotational symmetry as well as translational symmetry along a vector parallel to that line. This conversation will give you some insight about how comfortable students are with distance preserving transformations which will be the topic of the second module.

   (b) Once small group discussions have died down, bring students together to have a whole class discussion. Based on your monitoring during small group discussion, choose the order in which groups will share their conversations. Some of the things we would like to accomplish in this discussion are:
   - The shortest path on a sphere are great circles. To get to this point, you may want to turn back to symmetries. If we are to use thinking analogous to that in the Euclidean case, then a shape that we could remove and have the remaining space entirely symmetric would be one that splits the sphere precisely into two "equal" parts. We call this a great circle, a circle whose center is the center of the sphere. You may want to have students play with a Geogebra worksheet https://www.geogebra.org/m/Gh58sVPx

- The will still likely argue that the shortest path is not the flight path, but a path through the Earth. And this is precisely how we differentiate *extrinsic* and *intrinsic* geometry: the plane does not have an option to fly through the Earth. It is constrained by the space it has available.

> **Note.** We established that straight lines are great circles. We needed to define great circles and used extrinsic ideas to do so, but we could also use intrinsic ideas (equally distant from the poles...). The idea of circles having two centers and different radii was brought up. This would lead nicely into the questions of how we measure distances and what triangles might look like on a sphere. If students struggle with conceptualizing extrinsic and intrinsic geometry, I find it helpful to have them read Part I. This World, 1. Of the Nature of Flatland from Flatland: A Romance of Many Dimensions by Edwin A. Abbott as part of homework. I then continue this discussion after that reading. This text can be accessed at: Flatland

(c) Remind students that from the perspective of Shefali and Vanya, the shortest paths were only the ones that were along the roads and they certainly did not looks straight to us and we wouldn't have called them lines. However, if one of our conception of a straight is "shortest", then we can't but not call the zig-zaggy paths lines.

4. Ask students to compare these different *lines* to their definitions. Do they all satisfy those definitions? Does that mean our definitions are wrong? Invite students to consider Euclid's definitions. As they might know, Euclid is considered to be the first person to give a formal development of geometry through his *Elements*. After students had a chance to look at Euclid's definitions, initiate a discussion.

> **Mathematical Note.** I enjoyed very much reading Nathaniel Miller's notes[a] and much of this module stems from his notes:
>
> > In discussing Euclid's definitions, I try to lead students to noticing that they are really of two types. Some of his definitions, such as that of a circle in Definition 15, give specific defining properties that the object defined must have. These definitions are concrete and are sometimes used in his proofs, and are like the mathematical, checkable definitions that we asked students to write for the Straightness on the Sphere assignment. Other definitions, like that of a point given in Definition 1, give more general descriptions of the objects being described. They describe an object, but not in terms of properties that could be concretely used in a proof. A reader who already knows what a point is will probably agree that it "has no part", but a reader who didn't probably wouldn't find find this definition very useful. These are more like dictionary definitions that describe an object in a way that is not unambiguously checkable.
>
> [a] Source

ACTIVITY: THE BUILDING BLOCKS

1. What geometric objects did you work with in Activity: Choosing a place to live? Give a definition for each of the objects:

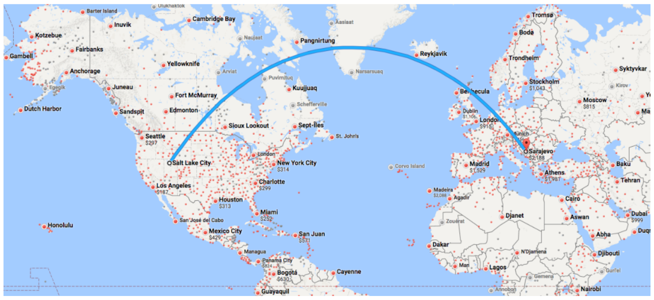
> **Solution.** Answers will, of course, vary, but here are some possibilities, with definitions of varying quality:
>
> - Point
>   - a location in space
>   - the smallest possible geometric object
> - Line
>   - the shortest distance between two points
>   - a straight path
>   - The graph of $y = mx + b$
> - Circle
>   - A set of points that are all equally distant from a center
>   - A curve that looks the same no matter how you rotate it
> - Midpoint
>   - a point exactly halfway between two other points
>   - a point that divides a line segment into two equal parts
> - Perpendcular bisector
>   - a line that cuts through the midpoint of a segment at a right angle
>   - the set of all points that are equally distant from two other points

2. Maybe you think there are other objects that are essential to geometry work, but haven't come up in Activity: Choosing a place to live. If so, add them to the list.

> **Solution.** Some that might get mentioned are: triangle, angle, ray, line segment, quadrilateral, parallelogram, rectangle, rhombus, square, trapezoid. . .

> **Note.** Once you've made your list, compare it with your partners'. See if you can agree on the smallest sensible list to use as a basis for our work.

3. Look back at your definition of a line. Is the flight path of this airplane a line? How was this flight path determined?



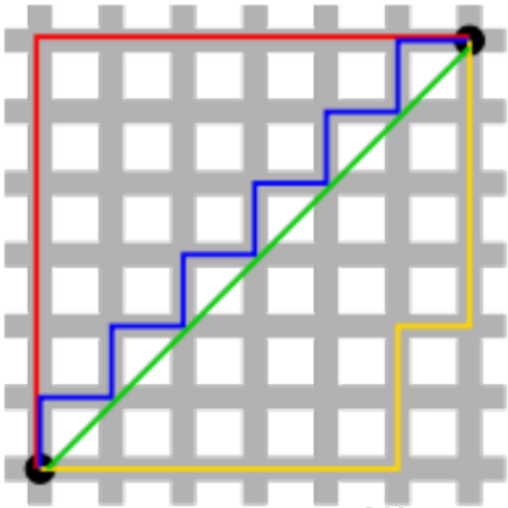
> **Solution.** The obvious answer to this question is "no", and it may be unrealistic to expect any students to figure out why an airplane would follow a path like this. Nevertheless, through class discussion it could be brought out that the path is a portion of a great circle, which is the shortest distance between two points on a sphere. Once this idea has been mentioned, the question should be thrown back to the students: Is this a line? Does it match one or more of the definitions proposed earlier for "line"? Students may also want to discuss the fact that the map itself involves distortions (some of them may know about different projections that mapmakers use, and that all projections involve distorting landmasses in one way or another); there could be discussion about the idea that the distortions that inevitably arise when trying to "flatten" a sphere could end up making a "straight" path look curved, or vice versa.

4. Take a moment to study definitions given by Euclid: https://goo.gl/axf87H. What do you think about these definitions? Do they communicate precisely what the object is, or do you need to know what they are before the definitions make sense? Do all the lines we've discussed so far fit into Euclid's definition?

> **Solution.** Answers will vary. Some things to expect:
>
> - Students will likely be more accepting of Euclid's definition of "line" and "surface" than of "point", because his definition of "point" tells us only what a point is *not*, rather than what it *is*.
> - They may be used to the word "line" always referring to "straight lines", but should note that "line", in Euclid, corresponds to what we would call a "path". For this reason, they may argue that the airplane's flight path is a "line", but not a "straight line".
> - Having said that, students will likely struggle to make sense of Euclid's definition of "straight line".
> - It is not obvious from the definition, but Euclid uses the phrase "straight line" to refer to what we would call a line *segment*. That is, in Euclid all lines are finite (but can be extended indefinitely, per Postulate 2).

HOMEWORK: SHORTEST PATHS

1. You have already gotten to know Taxicab geometry, which is a geometry anyone living in a town built on a grid is very familiar with. In most parts of town, just like a cab, you're confined to traveling along the grid. Let's look at an example:



   (a) How far would you travel from one black point to the other if you took red, blue or yellow path? Are there any paths that are shorter than these? Are there any paths that are longer than these? How many shortest paths are there between the two black points?

   > **Solution.** Assuming the space between the grid lines is 1 unit, the red, blue and yellow paths are all 12 units long. There are many longer paths but there is no path shorter than 12 units long. The number of shortest paths is ${}_{12}C_6$ = 924. The easiest way to see this is that any path of length 12 that begins in the lower-left corner and ends in the upper-right corner will involve 6 steps "up" and 6 steps "right", which can be arranged in any order. So if we let the letters U and R stand for steps up and right, respectively, then the question is equivalent to asking "How many ways can 6 Us and 6 Rs be arranged?", which is in turn equivalent to "How many ways are there to choose 6 positions from a set of 12 options?"

   (b) If you weren't constrained to the grid, let's say this was just an empty field, then the fastest you could get from one point to the other would be along the green line - and you'd be living in Euclidean geometry. Which path is shorter, Euclidean or taxicab?

   > **Solution.** The length of the green path is (by the Pythagorean Theorem) $\sqrt{6^2+6^2} = 6\sqrt{2} \approx 8.48$. Clearly the Euclidean shortest path is shorter than the Taxicab shortest path.

   (c) Use graph paper to make a coordinate system. You are standing at a point $(4, 9)$. Find all the points that are exactly 5 units away from you in taxicab geometry. Find also all the points that are 5 units away in Euclidean geometry. What did you draw in each case?

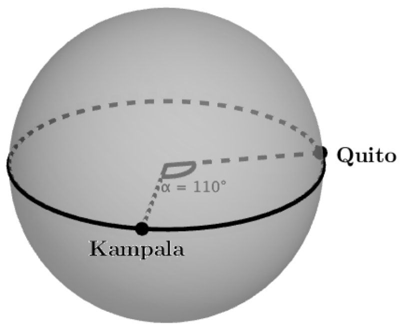
   > **Solution.** In Taxicab geometry, the set of all points that are 5 units away resembles a Euclidean square with vertices at $(9, 9)$, $(4, 14)$, $(-1, 9)$, and $(4, 4)$. It is not *actually* a square, because it only contains the points on the integer lattice, and not the space in between. It should consist of 20 discrete points.
   > In Euclidean geometry, the set of points that are 5 units away is a standard Euclidean circle, centered at $(4, 9)$ and with a radius of 5.

Note. The purpose of this problem is to get students little more comfortable with the taxicab geometry. Note that we are primarily working with the space that is not the entire plane, so arguably students are not getting the full picture of taxicab geometry. Here are are restricted to the actual integer grid, the set $\{(x, y) : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$. The last part of the question brings us back to another object of geometric interest that looks different in different geometries. We are hoping to illuminate for students the role of definition and the need to check whether given conditions are satisfied, rather than relying on the way the figure looks.

2. Consider that the Earth is a sphere with radius 6371 km. What is the shortest distance between each pair of cities listed below?

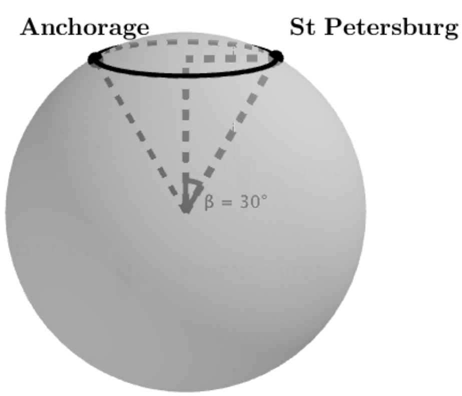
   (a) from Quito, Ecuador (0° N, 78°W) to Kampala, Uganda (0°N, 32°E)?

Solution. The shortest path from Quito to Kampala is an arc along the equator measuring 110° (see figure below). Its length is $2\pi(6371)\frac{110}{360} \approx 12231.4$ km.



   (b) from Saint Petersburg, Russia (60°N, 30°E) to Anchorage, Alaska (60°N, 150°W). Additionally, what is the distance if traveling due east? If you continue traveling due east from Anchorage until you reach St. Petersburg from the west, what path will you have traversed?

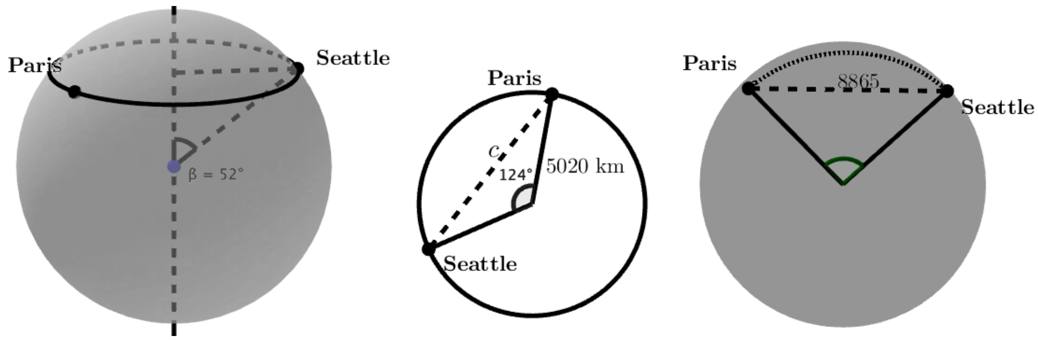
Solution. The most obvious path is *not* the shortest path. If you travel due East from St. Petersburg to Anchorage, you traverse exactly halfway around the circle at 60° latitude. This latitude is a circle whose radius (measured in the extrinsic 3-dimensional geometry) is $\frac{1}{2}(6371) = 3185.5$ (see the figure below, and use 30-60-90 triangles). The distance traveled along that path is therefore $2\pi(3185.5)\frac{1}{2} \approx 10007.5$ km.



However, the *shortest* path is to fly directly over the north pole. In this case the path goes through 60° of latitude, so the path length is $2\pi(6371)\frac{60}{360} \approx 6671.7$ km.

   (c) (optional practice) from Paris (48°N, 2°E) to Seattle (48°N, 122°W).

Solution. Paris and Seattle are on a common latitude of 48° N, which is 52° S of the North-South axis. The latitude is a Euclidean circle whose radius (through the interior of the Earth) is $6371 \cdot \sin 52° \approx 5020$ km. (See first figure below.) Viewed from above, we can see that the chord from Seattle to Paris through the Earth subtends an arc measuring 124° (see second figure below). That chord therefore has a length $c = 2 \cdot 5020 \cdot \sin 62° \approx 8865$ km. Now return to the 3-dimensional picture: the chord joining Seattle to Paris is 8865 km long, and the radius of the Earth is (still) 6371 km. So the angle marked in the third figure below is equal to $2\sin^{-1}\left(\frac{8865}{2 \cdot 6371}\right) \approx 88.18°$. Finally, the length of the curved path following an arc of a great circle from Paris to Seattle is $2\pi(6371) \cdot \frac{88.18}{360} \approx 9805.2$ km.



(d) (extension) from Lincoln, NE (40°N, 96°W) to Sydney, Australia (34°S, 151°E)?

Solution. This problem is considerably more challenging than the previous one, because the two cities do not lie on a common latitude. We use spherical coordinates (see the optional OPTIONAL: Distance on the sphere supplemental activity). For Lincoln, NE we have $\varphi = 50°, \theta = 96°$ and for Sydney, Australia we have $\varphi = 124°, \theta = -151°$. Therefore, the (x, y, z) coordinates of these two points are $L = (-510.15, 4853.7, 4095.2)$ and $S = (-4619.56, -2560.67, -3562.6)$. The dot product of these two vectors, divided by 6371^2, is -0.607, and therefore the arc from Lincoln to Sydney measures $\cos^{-1}(-0.607) \approx 127.4°$. Finally we conclude that the length of that arc is $2\pi(6371) \cdot \frac{127.4}{360} \approx 14166.23$ km.

To help you find the answers to these questions, consider using Geogebra 3D to represent the Earth and visualize the cities under consideration.

Note. In the second part we are again introducing a circle into our discussion. The first part is easily done - as the great circle on which the two towns lie is easily discernible. Quito and Kampala lie on the Equator, so we can use angles given by their longitude to find the distance between them (along the Equator). St. Petersburg and Alaska are on the same latitude, and those aren't the great circles in general. This makes the calculation little more interesting. Your students may or may not be successful with this task, and you can choose whether you want to invest the time to perform the calculations. This could be a reasonable independent project as well. We include the possible approach and discussion in the supplementary materials. One of the interesting results is that the ratio of circumference to the diameter of the circle on a sphere is not constant as it is in the plane.

MODULE II

lightgray Lesson	lightgrayProjected Length	lightgray Geometric Content	lightgray In-Class Activities	lightgray Homework	lightgrayConnections and Notes
1: Introduction to transformations	90 min	Equidistance, perpendicular bisectors, Taxicab Geometry introduction, mathematical modeling	Choosing a Place to Live (CPL)	Introductory Survey (IS)	• CPL will be used to generate discussion in the lesson 2 and 3 activity discussions • IS Question 6 will be revisited in Lesson 3 Homework: Constructions • IS Question 7 will be used for discussion of the activity in Lesson 2 and 3
2: Distance preserving transformations	90 min	Foundations of a Geometry (undefined terms), shortest paths (straightness), Spherical Geometry	The Building Blocks	Shortest Paths (SP)	• SP Question 1c will be used to generate discussion in Lesson 3 • SP Question 2 will be used to generate discussion in Lesson 3
3: Rotations and Reflections	180 min	Circles, attending to precision, definitions, Euclids Postulates, interpretation and model	Euclids Assumptions	Constructions (CS) Distance on a Sphere (optional)	• CS Question 2 will be revisited in Lesson 6
4: Transformations and Congruence	180 min	Incidence Axioms, Hilberts Axioms, proof, Hyperbolic Geometry	Incidence Geometry (IG) Handout: Hilberts Axioms	An Alternative Set of Axioms (ALT) Angle Proofs	• You may choose to assign IG#5 as homework • ALT cannot be assigned until after students have completed IG and started discussion of Hilberts Axioms
5: Fixed points	180 min	Differences between neutral, Euclidean, hyperbolic, and spherical geometries; definitions, right angles, parallel lines, angles formed by transversals	Right Angles Parallel Lines Sum of the Angles in a Triangle	Video Simulation of Practice (VSOP) VanHiele Readings Triangles (optional)	• VSOP can be assigned as soon as Lesson 5 begins • You may choose to assign the Sum of the Angles in a Triangle as homework
6: Triangle congruence	90 min	Summary of different axiomatic systems, examples of axiomatic systems in secondary classrooms	How do students understand axiomatic systems?	Written Simulation of Practice	
7: Culminating Activities				Mid-term Exam	

1 Introduction to Transformations

Overview

Length

2 Class Meetings, ~ 180 minutes

Summary

We have been talking about straight in several different ways, one of which is the shortest path between two points. In the following problems we will attempt to solve some problems whose goal is to find a shortest path, although the constraints given exclude straight lines as a solution.

Incidentally, these problems are much more easily solved if one uses transformations. We are going to use these as a motivation for investigating (rigid) transformations. One of the problems uses reflections, and we focus on that problem in this lesson. As an optional discussion, we include a problem which uses translations. Depending on your time constraints you may choose to do this problem in class, assign it for homework, include it as an assessment, or organize a group experience for your students outside the class.

Goals

- Goal 1
- Goal 2
- Goal 3

Connection to Standards

CCSSM Standard	Connection to Lesson 1
Standard 1	Connection between standard 1 and Lesson 2 will appear here
Standard 2	Connection between standard 2 and Lesson 2 will appear here
Standard 3	Connection between standard 3 and Lesson 2 will appear here

Materials

- Dynamic geometry software
- Compass, ruler, and protractor
- Graph paper
- Activity: Activity: What is the Shortest Way?
- Handout: Handout: CCSS Mathematical Practice 1
- Handout: Handout: Standards for Mathematical Practice
- Homework: Homework: Visualizing Functions

- Homework: Homework: Writing Definitions

Description

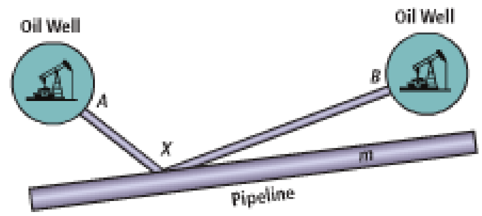
> Sam was on her way to camp from a long hike and notices smoke rising from the general direction of her tent. She figures she better run to the river and get some water to put out a fire. Which path should she take if she wants to save her tent?

1. The students should be placed in groups and allowed time to make headway on the problem. Refrain from jumping in too soon, other than encouraging them to try to check their claims as they make them. The students should be given plenty of time to play and tinker here. You may encourage them to use appropriate tools - some students can construct the problem using straightedge and compass, others might choose to use dynamic geometry software. One of the things our students are frequently unaccustomed to, and often don't provide opportunities for their own students to do, is engaging in a productive struggle. They seem to hold a belief that they should be able to solve all the problems quickly and without much effort thus demonstrating their knowledge and aptitude. We need to provide them opportunities to voice their opinions, bounce ideas off each other, discuss, fail, and try again.

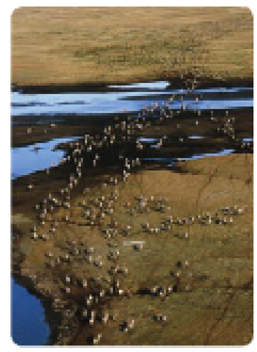
   Classroom Connections A similar problem, Connecting to a Pipeline, is featured in the UCSMP's Geometry textbook[1] which avoids some of the issues we ran into here (should the river be approximated by a straight line?), but introduces others (understanding the problem here is a major hurdle for a 9th grader, but probably not for our students). This problem will appear in one of our Simulations of Practice.

[1] University of Chicago School Mathematics Project, http://ucsmp.uchicago.edu/secondary/curriculum/geometry/

A Problem: Connecting to a Pipeline



The Alaska oil pipeline is a collection of connected cylinders 4 feet (about 120 cm) in diameter and about 800 miles (1300 km) long. Members of the U.S. Congress have debated whether to expand oil drilling into the Arctic National Wildlife Refuge. If the new oil wells are built, they will need to supply oil to the existing pipelines at pumping stations. It is very expensive to build new pumping stations on the existing pipeline, so oil companies try to connect two new wells to the pipeline in one place whenever possible. They need to know where they should tap into the pipeline so that the small pipelines from the two oil wells to the connecting point use the least amount of materials. The least expensive solution to their problem can be found using geometry.



There are 32 distinct herds of caribou in Alaska. This photo shows part of the Porcupine caribou herd migrating through the Arctic National Wildlife Refuge.

Source: Alaska Department of Fish and Game

Simplifying the Situation

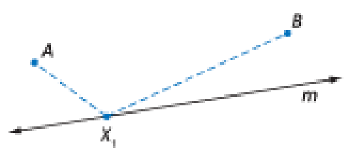
Each oil well takes up some space, and the pipeline is a cylinder. But because the distances are so great, we can think of each oil well as a *point* and the cylinder as a *line*. Now the problem looks like the diagram below. We call the fixed points A and B, and the point on the pipeline whose location is unknown we call X. The line we call m. The distance between A and X is written AX.

Activity

MATERIALS Tracing paper

This activity simulates the idea of finding the best place for the connection. The goal of this activity is to locate a point X on the line m that minimizes the sum $AX + BX$.

Step 1 Trace the picture at the right. X_1 is a possible location of X. Measure to find $AX_1 + BX_1$.



Step 2 Locate a second point X_2, on line m so that $AX_2 + BX_2$ is less than your answer to Step 1.

Step 3 Experiment to locate a point X so that $AX + BX$ is as small as possible.

Step 4 Suppose you wished to communicate your solution to the oil company. How would you describe to them where they should put the pumping station?

While the context of the pipeline does provide a view of mathematics as relevant to solving a problem that could genuinely benefit from a mathematical solution (running to the river to save a burning tent is not an instance in which the mathematical consideration is likely to contribute to the successful outcome of the situation) we object to its usage from the environmental and climate change standpoint. Instead we may choose to pose a problem such as:

> Two small towns were situated fairly close to a major highway connecting them to a large city. As the towns grew, they developed complementary industries and found that their residents, in addition to commuting to the large city, started commuting between towns more and more. The towns decided to collaborate on establishing a supercharger along the highway so that all commuters could benefit from it. Where should the supercharger be built?

We will include the pedagogical and mathematical discussion in the context of the tent problem, but of course the solution is the same for both. You may choose to include either, or possibly both, and discuss advantages and disadvantages of using either context.

Pedagogical Note. We chose not to include a diagram with this problem as we find the discussion involved in figuring out what diagram to draw quite invaluable. Even before they draw a diagram, the students are likely to express opinions that the problem is impossible to solve as they don't have enough information. The common remarks are:

- We don't know where Sam is.
- We don't know where the tent is.
- We don't know what the river looks like.
- We don't know what the terrain is like.

All of these are valid points and students should be encouraged to think about the messiness of the problems in general, the assumptions we frequently make, or need to make. You may want to specifically address the idea that we often need to simplify a problem to even have a chance to solve it. Students are used to "perfect problems" - ones that give all the information necessary to solve the problem and only that information. Excessive or insufficient givens are not often encountered and leave students uncomfortable, declaring the problem impossible. Try to help them see the value of the "What if" questions, and modifying the problems so they can be solved.

Mathematical Note. This problem has two issues: shortest distance and shortest time. The question is phrased ambiguously so we could get at both of those things at the same time.

Distance

This problem is nice as it allows different approaches. Each student can measure a few different paths before settling on their conjecture. In our experience, however, many students do not even consider measurement, and instead they tend to simply declare the "half-point" to be the solution. The students may be prompted to reconsider this "solution" through questions such as:

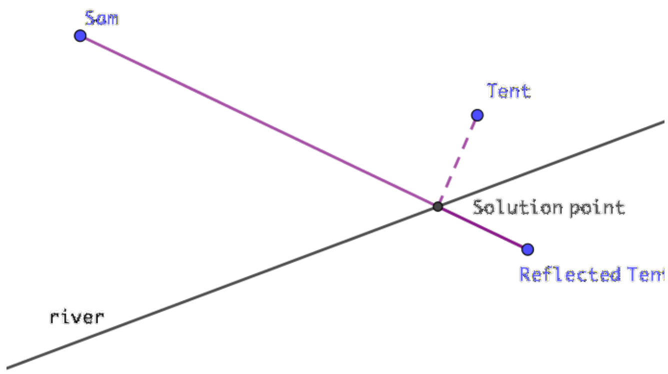
(a) Why do you think this is the right spot?

(b) How do you convince a sceptic?

(c) Is there any evidence you can provide that this spot is the right one?

The "half-point" is a solution in a very specific case: when both Sam and the tent are equally far away from the river, so the natural line of questioning should involve the chosen position of Sam and the tent. The trouble may arise here as the differences between different points on the river might be quite small or even non-existent. Encourage your students to investigate different initial positions. You may need to encourage students to attend to precision, even by using Geogebra, or dynamic geometry software (DGS) of your choice.

Some students may choose to place Sam and the tent on the opposite sides of the river. This strategy is very useful, as it allows you an opportunity to contrast the two approaches. Here it is very clear that most of the time, the midpoint in not the correct solution to the problem. Using these two approaches in a discussion may allow the students to connect the two by reflecting the tent over the river, then simply finding the shortest path between Sam's position and the reflected tent.



We need to prove that this position gives the shortest distance. The proof relies on the fact that reflection preserves the distance which is, of course, to students intuitively clear, but may not be clear how to prove it. Some may be inclined to use coordinates, which is a perfectly sensible way for them to go about, but is not necessary to pursue at the moment. You may choose to have them outline this proof for the moment, and leave the details for later.

Time
This is not generally the direction the students take, but it's worth pointing out that the above geometric solution may not be appropriate. Namely, Sam is likely to be able to run faster when she's not carrying any load, which makes the situation more complicated. We would have to make further assumptions in order to be able to solve this problem.

This problem allows an opportunity to discuss CCSS.MATH.PRACTICE.MP1 Make sense of problems and persevere in solving them as well as modeling standard once again. We recommend having a discussion about students' attempts and struggles with the problem before addressing the solutions and solution strategies. A handout Handout: CCSS Mathematical Practice 1 describing this practice is provided.

2. To wrap up the conversation, you want to ask the students how they think of reflections. What is reflection as an object of mathematical study? We generally think of it as functions whose domain and codomain are a plane. Students often think of it as a motion. We'd like to connect the idea of these motions to the usual representations of functions and to have them start thinking about algebraic representation of isometries. Our goal is to start thinking about transformations, and to realize that ones we ordinarily work with are rather special. We will think about how transformations can be represented, and what the domain and range of these functions are. Our ultimate goal is to give definitions of transformations and isometries.

   You can use this conversation as a lead into the homework, Homework: Visualizing Functions, with which the students are likely to struggle. To set them on the right track, pose a few questions about the assignment for a short discussion at the end of class. You might choose to display one or two of the functions and ask: What do these functions do? How might we visualize them? Students should understand that they are not to produce a graph of the given function, which indeed would be quite hard, although they do attempt it by replacing the given functions with their restrictions to $\mathbb{R} \to \mathbb{R}$. It might be valuable to let them work in groups on this at first since the tools they choose to use might spark a discussion about their value. This activity should be reminiscent of problems that frequently appear in secondary school curriculum: students are given a geometric object, triangle, square, etc., and directed to perform a given transformation, so encourage them to approach the homework in the same way.

3. After the students completed the homework assignment, engage them in a pair-share session in which they discuss their findings. Follow up with the whole class discussion. Some questions to ask: What are the objects you were considering? How did they act? Were there points that remained unchanged? Are there lines that remained unchanged (maybe not point by point, but overall)? What happened to horizontal lines? Vertical lines? Lines with slope 1? Other slopes? Etc.

4. Before proceeding to the discussion below you can inquire about the tools the students chose to utilize. If they chose calculators or other geometry software, there is a possibility that they graphed restrictions of these

functions and basically treated them as real functions. Discuss how tools in this instance may have hindered their progress, while the tools in the Shortest path problems allowed us to note the mistakes we made. This conversation brings back into focus Standards of Mathematical Practice:

> **Note.** CCSS.MATH.PRACTICE.MP5 Use appropriate tools strategically.
>
> Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

5. Continue the conversation about the classification of the given functions. How did the students categorize the functions they were presented with? They're likely to note that some distort figures significantly, and others don't. Some might talk about what happens with distances between points. Encourage them to be as specific as possible and record their findings (or, preferably, let them record their findings). They're less likely to think about collapsing of points into a single point, that is thinking about one-to-one and onto[2], so ask about those explicitly. You may want to note that functions that are too random may not be interesting objects of study, simply due to the amount of possible variance. Instead, we focus our attention to bijective maps, and from there we restrict the work even further, to those transformations that preserve distances.

> **Note.** We include here some vocabulary we would like students to adopt as well as the solution to the homework problems:
>
> **Definition 1.1.** A transformation of the plane is any bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$.
>
> This definition allows us to answer the question: Which of these functions are transformations? A: f, g, h, l, m, n, p, q.
>
> **Definition 1.2.** A transformation, $f : \mathbb{R}^2 \to \mathbb{R}^2$, of the plane is called an isometry (aka rigid transformation, rigid motion, distance preserving transformation, congruence transformation) if it preserves distance between any two given points[a]. For any two points $A, B \in \mathbb{R}^2$
>
> $$d(f(A), f(B)) = d(A, B)$$
>
> Students might claim that n, p, and q are isometries and should be able to provide at least an outline of the algebraic argument (using the distance formula or Pythagorean Theorem). This will be the focus of the next lesson, and further discussion is not necessary at this moment.

[2]The students may require some review of one-to-one, onto, and bijection.

$f(x, y) = (x^3, y^3)$	Here, straight lines through the origin go to straight lines through the origin with a different slope. If students choose to map squares or rectangles with vertical and horizontal sides, they may think that it's simply a dilation. Ask them what happens to the points on the sides of those rectangles. Ask them what happens with the lines with a non-zero y- intercept.
$g(x, y) = (2x, 3y)$	Scaling by a factor of 2 in the horizontal direction and factor 3 in vertical. All vertical and horizontal lines are fixed sets. Do inquire what would happen if the coefficients were the same and remind them of the associated vocabulary: scaling or dilation.
$h(x, y) = (x + 3y, y)$	Students may know this as a shear. In this instance the x-axis is fixed, and each point moves along the horizontal line by a distance proportional to its y-coordinate.
$i(x, y) = (\cos x, \sin x)$	The image of the plane is the circle with radius 1 centered at the origin.
$j(x, y) = (-x, x + 3)$	The image of the plane is a line $y = -x + 3$.
$l(x, y) = (x^3 - x, y)$	It is little harder to see what is going on here as the y-coordinate remains fixed. The horizontal lines are fixed sets, however the order of the point is not preserved. If the students look at lines with non-zero slope, they may notice more interesting behavior: lines are mapped to rotated and reflected graphs of cubic functions.
$m(x, y) = (3y, x + 2)$	We can think of this as a composite of the following functions: translate each point by $< 2, 0 >$, dilation in y direction by a factor of 3, reflection over the line $y = x$.
$n(x, y) = (x + 2, y - 3)$	Translation by $< 2, -3 >$
$p(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$	Rotation by $\cos^{-1} 0.6$

[a]Distance between the points is invariant under f.

Preparation for the next lesson

- Students:
  - Homework: Writing Definitions
- Instructor:
  - Make sure the students turn the Writing Definitions homework in electronically with enough time to use their definitions for class discussion the next class.

Activity: What is the Shortest Way?

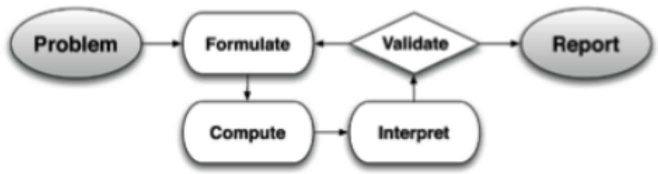
1. Sam was on her way to camp from a long hike and notices smoke rising from the general direction of her tent. She figures she better run to the river and get some water to put out a fire. Which path should she take if she wants to save her tent?

Handout: CCSS Mathematical Practice 1

CCSS.MATH.PRACTICE.MP1 - Make sense of problems and persevere in solving them

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MODELING:



The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them.

HOMEWORK: VISUALIZING FUNCTIONS

We can use coordinates to describe functions on the plane. For each of the given functions, describe their effect on points of the plane, coordinate axes, and characteristics that they leave invariant. Make predictions about what will happen before you "graph" these.[3]

Function	Prediction	Diagram
$f(x,y) = (x^3, y^3)$		
$g(x,y) = (2x, 3y)$		
$h(x,y) = (x + 3y, y)$		
$i(x,y) = (\cos x, \sin x)$		
$j(x,y) = (-x, x + 3)$		
$l(x,y) = (x^3 - x, y)$		

[3] You might consider a discussion concerning the use of the term "graph" in this description.

Function	Prediction	Diagram
$m(x, y) = (3y, x + 2)$		
$n(x, y) = (x + 2, y - 3)$		
$p(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$		

If you were to group these functions into two disjoint groups, what would your classification be:

- Group 1:

- Group 2:

What is your rationale for this classification?

Handout: Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them**

   Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identifying correspondences between different approaches.

2. **Reason abstractly and quantitatively**

   Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize - to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents - and the ability to contextualize - to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand, considering the units involved, attending to the meaning of quantities (not just how to compute them), and knowing and flexibly using different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others**

   Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and - if there is a flaw in an argument - explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics**

   Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically**

   Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision**

   Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure**

   Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. **Look for and express regularity in repeated reasoning**

   Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content.

Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

HOMEWORK: WRITING DEFINITIONS

We defined an isometry as:

Definition 1.3. A transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane is called an isometry if it preserves distance between any two given points: for any A, B

$$d(f(A), f(B)) = d(A, B)$$

1. If you ask a student for examples of these transformations, they are likely to tell you: slide, flip, and turn. What do they mean by that? If you ask them for a definition of these transformations, what might they say?
2. Give precise definitions of those transformations.
3. Use dynamic geometry software and explore these transformations.
4. Through this exploration reflect on your definitions, and modify them if necessary. Explain why you made changes.
5. Look up the definitions in a high school textbook of your choosing and list them here. How are these similar/different from yours?

Contents

Module III

SIMILARITY GEOMETRY MODULE

- **Big Idea** This module is focused on developing students' understanding of dilations and similarity as well as related issues of scaling both distances and areas.
- **Goals for studying the topic**:
  - Develop understanding of similarity transformation.
  - Know how to perform dilations, as well as locate the center of a given dilation and the scale factor.
  - Understand that all circles and parabolas are similar.
  - Know area axioms.
  - Be able to find areas of irregular shapes.
  - Be able to explain formulas for areas of polygons and circles.
  - Be able to explain how dilations impact lengths and areas.
  - Be able to prove Pythagorean theorem in several different ways.
- **Rationale**:
- **Connections to Secondary Mathematics**:
- **Overview of Content**
  - Lesson 1: Transformations Which Arent As Nice
  - Lesson 2: Understanding Dilations
  - Lesson 3: Similarity
  - Lesson 4: Dilations and Slope
  - Lesson 5: Pythagoras
  - Lesson 6: Area
  - Lesson 7: Applications
- **Overview of mathematical and teaching practices** While we hope that all of our lessons will be in line with the Standards of Mathematical Practice as outlined in the CCSS, in this module we will particularly pay attention to:

SMP 1 Make sense of problems and persevere in solving them

SMP 2 Reason abstractly and quantitatively

SMP 3 Construct viable arguments and critique the reasoning of others

SMP 5 Use appropriate tools strategically

SMP 6 Attend to precision

- Summary of what students will do
  - **Overview of Assignments**:
    - Textbook contribution - optional
    - Simulations of Practice - Video and Written Dilations
    - Homework
      - Transforming Graphs
      - Pythagorean Theorem
      - Optional Area Investigations
    - Cumulative Projects
      - Reflective Portfolio
      - Topic Presentation

- **Expectations for assignments**
  * If the instructor so chooses, the students can be tasked with writing a course textbook. After each lesson, a student is asked to write out the accomplishments of the day. The goal is to leave a reference for all students, give everyone an opportunity to practice writing, and provide feedback to each other.
  * Writing Assignments are to be graded and feedback provided. These assignments may be commented on before a final version is submitted for evaluation.
  * Homework Assignments are generally a preparation for class or planning information for the instructor. The instructor may choose whether to review and provide feedback.
  * Instructors may choose to have a class discussion board where questions can be posted and encouraged. The following prompt may serve well to encourage questioning and discussions throughout the course: ”An important part of doing mathematics is to learn to ask one’s own questions. You might not be able to answer them, but you can always learn more through seeking the answer to your questions. Which questions has this work made you ask? Record all the questions that you’d like to investigate on the class website.”

1 Transformations which arent as nice (Length: ˜90 minutes)

Overview

Summary

In this lesson, our goal is to make sense of the definition of similarity and how it is performed. Students will engage in the activity *Make it bigger* and through discussion arrive at a procedure for performing a dilation and its definition.

Materials

- Rubber bands (a variety of sizes but fairly new tightness will aid in precision)
- Activity: Make it Bigger
- Dynamic Geometry Software (Optional)

Description

1. Begin class by posing the following discussion: We worked with distance and angle preserving transformations. Part of doing mathematics is asking "What if...?" So, what if we wanted to consider transformation that preserve only one of those attributes?

   (a) Can you find transformations that preserve distances, but not angles?

   (b) Can you find transformations that preserve angles, but not distances?

   > **Note.** Students can do an exploration in dynamic geometry software, however this is mostly a thinking/imagining experiment. Students could argue that SSS criterion for congruence essentially gives them the proof that distance preserving must be angle preserving. Another venue they might be inclined to explore is using the Pythagorean theorem, and or trigonometry.

   > **Mathematical Note.** There are no transformation that belong to the first class, and the only transformations that belong the second category are similarity transformation. The students are unlikely to think about similarity transformations in general. They are more likely to think about the dilations. If we've done a good job of thinking about compositions they might go to the compositions of dilations and isometries.

2. Allow groups time to engage with the Make it Bigger Activity. They will need time to consider efficient and effective ways of using the rubber band to perform the dilation given the directions.

3. Next, lead a whole class discussion in which students share their work and findings with the whole class. A likely finding will be that the actual procedure they performed made the new object very inaccurate. You can leverage this to move into the following discussion points:

   (a) Ask students to repeat the procedure using more precise tools such as a compass and straightedge, or ruler. Students often have a hard time translating to using different tools, in other words, identifying what was essential in this procedure.

   (b) Allow an opportunity for students to discuss the attributes that stayed the same and those that changed.

   (c) Discuss why the procedure yields a figure twice as large (it's clear why it's twice as far; that is we can tell that the distances along the lines through the special point are doubled). You can ask what is meant by twice as large"? (Here we mean that the linear dimensions have doubled. As an introduction to thinking about the area, you may want to bring it up here.) We can currently answer this question by brute force: use coordinates and some easy algebra to get there. Can it be done geometrically?

> **Mathematical Note.** Here we used algebraic definition of dilation: $(x, y) \to (kx, ky)$. Point out that this restricts us to the origin as the center of dilation. Of course, we can compose with other transformations to obtain dilations with center other than the origin.

4. Close the discussion of this lesson by asking groups to pose a definition for a dilation. After groups have posed a definition, critique and refine their definitions to lead to the following definition of dilation:

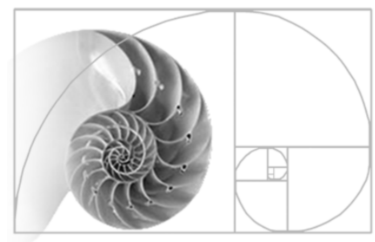
   Definition 1.1. A dilation with center C and factor $k \in \mathbb{R}_+$ is a transformations that maps a point P to point P' such that P' lies on $\overrightarrow{CP}$ and $d(C, P') = kd(C, P)$, and $C' = C$.

Preparation for the next lesson

- Student:
- Instructor:
  - Students may be continuing work on cumulative assignments from Module 1 and that may be sufficient for homework. If you wish, you can assign Find the Center for homework in order to begin thinking for Lesson 2. Find the Center can be found in Lesson 2 documents.

Activity: Make it Bigger

Your goal is to double the picture drawn below.



To save you some trouble, I already googled the solution to this problem for you, and here are the instructions I found:

Take two rubber bands of equal length and tie them together so that there are equally sized bands on either side of the knot. Pick an anchor point somewhere on the paper and pin the end of rubber band to the anchor point with your finger. On the opposite side of the other band, place a pen. Trace a new object while keeping the knot consistently on top of the figure you are trying to enlarge.

Your task is to:

1. Follow directions as accurately as you can.
2. Explain what in this procedure caused the shape to be twice the size of the original one.
3. How can you preform the same process if instead of rubber bands you used a ruler?
4. List as many different things you can notice that
   (a) stayed the same
   (b) changed

   in this process.

To facilitate this procedure, the figure's been drawn on the next page as well.